Limit theorems for functionals of stationary random fields

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Random field = Set of random variables indexed by $\mathbb{R}^d$

**Aim:**

Examine the asymptotic behavior of random variables

$$\int_{W_n} f(X(t)) \, dt,$$

where $\{X(t)\}_{t \in \mathbb{R}^d}$ is a stationary random field and $f : \mathbb{R} \to \mathbb{R}$ is a deterministic function

as $W_n$ tends to the whole space $\mathbb{R}^d$. 
Overview

- Motivation
- Excursion sets of random fields and integral geometric functionals
- LTs: state of art
- CLT for the volume of excursion sets of stationary random fields
  - Second order quasi-associated fields
  - **Examples**: Shot noise, Gaussian case
  - **PA-** or **NA-** fields (possibly not second order!)
  - **Examples**: infinitely divisible, max- and \( \alpha \)-stable fields
  - Multivariate CLT with a Gaussianity test
- FCLT
- Open problems
Motivation

Paper surface
(Voith Paper, Heidenheim)

Simulated Gaussian field
\[ EX(t) = 126 \]
\[ r(t) = 491 \exp \left( -\frac{\|t\|_2}{56} \right) \]

▶ Is the paper surface Gaussian?
Excursion sets

Let $X$ be a measurable real-valued random field on $\mathbb{R}^d$, $d \geq 1$ and let $W \subset \mathbb{R}^d$ be a measurable subset. Then for $u \in \mathbb{R}$

$$A_u(X, W) := \{ t \in W : X(t) \geq u \}$$

is called the excursion set of $X$ in $W$ over the level $u$.

Volume of excursion set:

$$|A_u(X, W)| = \int_W f(X(t)) \, dt, \quad f(t) = 1\{ t \geq u \},$$

de$|$ \cdot \rvert$ is the Lebesgue measure in $\mathbb{R}^d$ (cardinality of a set in $\mathbb{Z}^d$).
Centered Gaussian random field on $[0, 1]^2$, 

$$r(t) = \exp(-\|t\|_2 / 0.3),$$

Levels: $u = -1.0, 0.0, 1.0$
LTs for integral geometric functionals of random fields

- Gaussian random fields
  - CLTs:
    - Stationary processes, $d = 1$: Belyaev & Nosko (1969); Cuzick (1976); Elizarov (1988); Kratz (2006)
    - Volume, $d \geq 2$: Ivanov & Leonenko (1989)
    - FCLT: Meschenmoser & Shashkin (2011)

- Non-Gaussian random fields
  - Volume of excursion sets: Bulinski, S. & Timmermann (2012); Karcher (2012); Leonenko & Olenko (2014); Demichev (2013), Demichev & Olszewski (2015);
Growing sequence of observation windows

A sequence of compact Borel sets \((W_n)_{n \in \mathbb{N}}\) is called a Van Hove sequence (VH) if \(W_n \uparrow \mathbb{R}^d\) with

\[
\lim_{n \to \infty} |W_n| = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{|\partial W_n \oplus B_r(0)|}{|W_n|} = 0, \quad r > 0.
\]
Theorem (Bulinskii & Zhurbenko, 1976)
Let \( \{X(t)\}_{t \in \mathbb{R}^d} \) be a stationary, measurable random field fulfilling some \( \alpha \)-mixing assumptions.
Let \( f : \mathbb{R} \to \mathbb{R} \) be some measurable map such that \( \{f(X(t))\}_{t \in \mathbb{R}^d} \) fulfills integrability assumptions.
Let \( \{W_n\}_{n \in \mathbb{N}} \) be a VH-growing sequence of compact sets of \( \mathbb{R}^d \).
Then
\[
\frac{\int_{W_n} f(X(t)) \, dt - |W_n| \cdot \mathbb{E}[f(X(0))]}{\sqrt{|W_n|}} \xrightarrow{d} \mathcal{N}(0, \sigma^2),
\]
as \( n \to \infty \), where
\[
\sigma^2 = \int_{\mathbb{R}^d} \text{Cov} \left(f(X(0)), f(X(t))\right) \, dt.
\]
Theorem (CLT for the volume of $A_u$ at a fixed level $u \in \mathbb{R}$)

Let $X$ be a strictly stationary random field satisfying some additional conditions and $u \in \mathbb{R}$ fixed. Then, for any sequence of $VH$-growing sets $W_n \subset \mathbb{R}^d$, one has

$$\frac{|A_u(X, W_n)| - P(X(0) \geq u) \cdot |W_n|}{\sqrt{|W_n|}} \xrightarrow{d} N \left( 0, \sigma^2(u) \right)$$

as $n \to \infty$. Here

$$\sigma^2(u) = \int_{\mathbb{R}^d} \text{cov} \left( 1\{X(0) \geq u\}, 1\{X(t) \geq u\} \right) \, dt.$$
Second order quasi-associated random fields

Let $X = \{X(t), \ t \in \mathbb{R}^d\}$ have the following properties:

- square-integrable
- has a continuous covariance function $r(t) = \text{Cov}(X(o), X(t)), \ t \in \mathbb{R}^d$
- $|r(t)| = \mathcal{O}\left(\|t\|^{-\alpha}_2\right)$ for some $\alpha > 3d$ as $\|t\|_2 \to \infty$
- $X(0)$ has a bounded density
- quasi-associated.

Then $\sigma^2(u) \in (0, \infty)$ (Bulinski, S., Timmermann (2012)).
Quasi-association

A random field $X = \{X(t), t \in \mathbb{R}^d\}$ with finite second moments is called quasi-associated if

$$|\text{cov}(f(X_I), g(X_J))| \leq \text{Lip}(f) \text{Lip}(g) \sum_{i \in I} \sum_{j \in J} |\text{cov}(X(i), X(j))|$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any Lipschitz functions $f : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\text{card}(J)} \to \mathbb{R}$ where $X_I = \{X(t), t \in I\}$, $X_J = \{X(t), t \in J\}$.

Idea of the proof of the Theorem: apply a CLT for $(BL, \theta)$-dependent stationary centered square-integrable random fields on $\mathbb{Z}^d$ (Bulinski & Shashkin, 2007).
(BL, \theta)-dependence

A real-valued random field \( X = \{ X(t), t \in \mathbb{Z}^d \} \) is called (\( BL, \theta \))-dependent, if there exists a sequence \( \theta = \{ \theta_r \}_{r \in \mathbb{R}^+} \), \( \theta_r \downarrow 0 \) as \( r \to \infty \) such that for any finite disjoint sets \( I, J \subset \mathbb{Z}^d \) with \( \text{dist}(I, J) = r \), and any functions \( f \in BL(|I|), g \in BL(|J|) \), one has

\[
|\text{cov}(f(X_I), g(X_J))| \leq \min\{|I|, |J|\} \text{Lip}(f) \text{Lip}(g) \theta_r.
\]

Possible choice of \( \theta_r \):

\[
\theta_r = \sup_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d: \|j-i\|_{\infty} \geq r} |\text{cov}(X(i), X(j))|.
\]
CLT for \((BL, \theta)-dependent\) stationary random fields

Theorem (Bulinski & Shashkin, 2007)

Let \(Z = \{Z(j), j \in \mathbb{Z}^d\}\) be a \((BL, \theta)-dependent\) strictly stationary centered square-integrable random field. Then, for any sequence of regularly growing sets \(U_n \subset \mathbb{Z}^d\), one has

\[
\sum_{j \in U_n} Z(j) / \sqrt{|U_n|} \xrightarrow{d} \mathcal{N}(0, \sigma^2)
\]

as \(n \to \infty\), with

\[
\sigma^2 = \sum_{j \in \mathbb{Z}^d} \text{cov}(Z(0), Z(j)).
\]
Special case - Shot noise random fields

The above CLT holds for a stationary shot noise random field

\[ X = \{ X(t), t \in \mathbb{R}^d \} \]

given by

\[ X(t) = \sum_{i \in \mathbb{N}} \xi_i \varphi(t - x_i) \]

where

\[ \{ x_i \} \text{ is a homogeneous Poisson point process in } \mathbb{R}^d \text{ with intensity } \lambda \in (0, \infty) \]

\[ \{ \xi_i \} \text{ is a family of i.i.d. non–negative random variables with } \mathbb{E} \xi_i^2 < \infty \]

\[ \text{and the characteristic function } \varphi_\xi \]

\[ \{ \xi_i \}, \{ x_i \} \text{ are independent} \]

\[ \varphi : \mathbb{R}^d \to \mathbb{R}_+ \text{ is a bounded and uniformly continuous Borel function with} \]

\[ \varphi(t) \leq \varphi_0(\|t\|_2) = O(\|t\|_2^{-\alpha}) \text{ as } \|t\|_2 \to \infty \]

for a function \( \varphi_0 : \mathbb{R}_+ \to \mathbb{R}_+, \alpha > 3d, \) and

\[ \int_{\mathbb{R}^d} \exp \left\{ \lambda \int_{\mathbb{R}^d} (\varphi_\xi(s\varphi(t)) - 1) \, dt \right\} \, ds < \infty. \]
Special case - Gaussian random fields

Consider a stationary Gaussian random field $X = \{X(t), t \in \mathbb{R}^d\}$ with the following properties:

- $X(0) \sim \mathcal{N}(a, \tau^2)$
- has a continuous covariance function $r(\cdot)$
- $\exists \alpha > d : |r(t)| = O(\|t\|_2^{-\alpha})$ as $\|t\|_2 \to \infty$
Special case - Gaussian random fields

Let \( X \) be the above Gaussian random field and \( u \in \mathbb{R} \). Then,

\[
\sigma^2(u) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^1 \rho(t) \frac{1}{\sqrt{1 - r^2}} e^{-\frac{(u-a)^2}{\tau^2(1+r)}} \, dr \, dt,
\]

where \( \rho(t) = \text{corr}(X(0), X(t)) \). In particular, for \( u = a \)

\[
\sigma^2(a) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \arcsin(\rho(t)) \, dt.
\]
Positively or negatively associated random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ have the following properties:

- stochastically continuous (evtl. not second order!)
- $\sigma^2(u) \in (0, \infty)$
- $P(X(0) = u) = 0$ for the chosen level $u \in \mathbb{R}$
- positively (PA) or negatively (NA) associated.

Then the above CLT holds (Karcher (2012)).
Association

A random field $X = \{X(t), t \in \mathbb{R}^d\}$ is called positively (PA) or negatively (NA) associated if

$$\text{cov}(f(X_I), g(X_J)) \geq 0 \quad (\leq 0, \text{ resp.})$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any bounded coordinatewise non-decreasing functions $f: \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$, $g: \mathbb{R}^{\text{card}(J)} \to \mathbb{R}$ where $X_I = \{X(t), t \in I\}$, $X_J = \{X(t), t \in J\}$. 
Special cases

Subclasses of PA or NA

- infinitely divisible
- max-stable
- $\alpha$-stable

random fields
Special cases: Max-stable random fields

Let \( X = \{X(t), t \in \mathbb{R}^d\} \) be a stationary max-stable random field with spectral representation

\[
X(t) = \max_{i \in \mathbb{N}} \xi_i f_t(y_i), \quad t \in \mathbb{R}^d,
\]

where \( f_t : E \to \mathbb{R}_+ \) is a measurable function defined on the measurable space \((E, \mu)\) for all \( t \in \mathbb{R}^d \) with

\[
\int_E f_t(y) \, \mu(dy) = 1, \quad t \in \mathbb{R}^d,
\]

and \( \{(\xi_i, y_i)\}_{i \in \mathbb{N}} \) is the Poisson point process on \((0, \infty) \times E\) with intensity measure \( \xi^{-2} d\xi \times \mu(dy) \). Assume that

\[
\int_{\mathbb{R}^d} \int_E \min\{f_0(y), f_t(y)\} \, \mu(dy) \, dt < \infty
\]

and \( \|f_s - f_t\|_{L^1} \to 0 \) as \( s \to t \).
Special cases: $\alpha$-stable random fields

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary $\alpha$-stable random field ($\alpha \in (0, 2)$, for simplicity $\alpha \neq 1$) with spectral representation

$$X(t) = \int_E f_t(x) \Lambda(dx), \quad t \in \mathbb{R}^d,$$

where $\Lambda$ is a centered independently scattered $\alpha$-stable random measure on space $E$ with control measure $m$ and skewness intensity $\beta : E \rightarrow [-1, 1]$, $f_t : E \rightarrow \mathbb{R}_+$ is a measurable function on $(E, m)$ for all $t \in \mathbb{R}^d$ with

$$\int_{\mathbb{R}^d} \left( \int_E \min\{|f_0(x)|^\alpha, |f_t(x)|^\alpha\} m(dx) \right)^{1/(1+\alpha)} dt < \infty$$

and $\int_E |f_s(x) - f_t(x)|^\alpha m(dx) \rightarrow 0$ as $s \rightarrow t$. 
Multi-dimensional CLT

\[ S_{\bar{u}}(W_n) = (|A_{u_1}(X, W_n)|, \ldots, |A_{u_r}(X, W_n)|) \top \]

\[ \Psi(\bar{u}) = (\Psi((u_1 - a)/\tau), \ldots, \Psi((u_r - a)/\tau)) \top \]
Theorem (Multi-dimensional CLT)

Let $X$ be the above Gaussian random field and $u_k \in \mathbb{R}$, $k = 1, \ldots, r$. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has

$$|W_n|^{-1/2} \left( S_{\vec{u}}(W_n) - \Psi(\vec{u}) |W_n| \right) \xrightarrow{d} \mathcal{N}(0, \Sigma(\vec{u}))$$

as $n \to \infty$. Here, $\Sigma(\vec{u}) = (\sigma_{lm}(\vec{u}))_{l,m=1}^r$ with

$$\sigma_{lm}(\vec{u}) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-r^2}} \exp \left\{ -\frac{(u_l-a)^2 - 2r(u_l-a)(u_m-a) + (u_m-a)^2}{2\tau^2(1-r^2)} \right\} \, dr \, dt.$$
Theorem (Statistical version of the CLT)

Let $X$ be the above Gaussian random field, $u_k \in \mathbb{R}$, $k = 1, \ldots, r$ and $(W_n)_{n \in \mathbb{N}}$ be a sequence of VH-growing sets. Let $\hat{C}_n = (\hat{c}_{nlm})_{l,m=1}^r$ be statistical estimates for the nondegenerate asymptotic covariance matrix $\Sigma(\vec{u})$, such that for any $l, m = 1, \ldots, r$

$$\hat{c}_{nlm} \xrightarrow{p} \sigma_{lm}(\vec{u}) \text{ as } n \to \infty.$$  

Then

$$\hat{C}_n^{-1/2} |W_n|^{-1/2} (S_{\vec{u}}(W_n) - \Psi(\vec{u}) |W_n|) \xrightarrow{d} \mathcal{N}(0, I).$$
Hypothesis testing

\(H_0 : X\) Gaussian vs. \(H_1 : X\) Non-Gaussian

Test statistic:

\[ T = \left| W_n \right|^{-1} (S_{\tilde{u}}(W_n) - \psi(\tilde{u}) \mid W_n \mid) ^\top \hat{C}_n^{-1} (S_{\tilde{u}}(W_n) - \psi(\tilde{u}) \mid W_n \mid) \]

We know \( T \xrightarrow{d} \chi^2_r \). Reject null-hypothesis if \( T > \chi^2_{r,1-\nu} \).
## Numerical results

<table>
<thead>
<tr>
<th>Series</th>
<th>Resolution</th>
<th>Realizations</th>
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<th>FTR6.6</th>
<th>Sim. Gaussian</th>
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Further results

- **CLT for the volume as level** $u \to \infty$

- **FCLT (variable $u \in \mathbb{R}$):**
  - Volume for second order $A$-random fields with a.s. continuous paths and bounded density in Skorokhod space: Meschenmoser & Shashkin (2011)
  - Volume for random fields with a.s. continuous paths and bounded density in Skorokhod space (evtl. not second order!): Karcher (2012)
FCLT for excursion sets (Meschenmoser & Shashkin, 2011)

Let \( \{X(t)\}_{t \in \mathbb{R}^d} \) be a stationary, measurable, associated random field fulfilling some integrability and regularity assumptions. Let \( \{W_n\}_{n \in \mathbb{N}} \) be a VH-growing sequence of compact sets of \( \mathbb{R}^d \). Then the sequence of stochastic processes defined by

\[
Y_n(u) := \frac{\int_{W_n} 1_{[u,\infty)}(X(t)) \, dt - |W_n| \cdot \mathbb{E}[1_{[u,\infty)}(X(0))]}{\sqrt{|W_n|}}
\]

converges in distribution to a centered Gaussian process \( Y \) with covariance function

\[
\text{Cov}(Y(u_1), Y(u_2)) = \int_{\mathbb{R}^d} \mathbb{P}(X(0) > u_1, X(t) > u_2) - \mathbb{P}(X(0) > u_1) \cdot \mathbb{P}(X(t) > u_2) \, dt
\]

as \( n \to \infty \) in the Skorokhod topology.
Replace indicator functions by more general functions? Consider the space $V$ of Lipschitz continuous functions with norm

$$
\|f\|_{\text{Lip}} := \text{Lip} f + |f(0)|.
$$

Theorem (Kampf & S. (2015))
Let $\{X(t)\}_{t \in \mathbb{R}^d}$ be a stationary and measurable random field. Assume there $n \in \mathbb{N}$, $\delta > 4$ and $C, l > 0$ with $n/d > \max\{l + \delta/(\delta - 2), \delta/(\delta - 4)\}$ such that

$$
\alpha_{\gamma}(r) \leq Cr^{-n\gamma/l} \text{ for all } \gamma \geq 2\kappa_d, \ r > 0,
$$

and

$$
\mathbb{E}X(0)^{\delta} < \infty.
$$
Let \( \{ W_n \}_{n \in \mathbb{N}} \) be a VH-growing sequence of compact sets of \( \mathbb{R}^d \). Then the sequence of stochastic processes defined by

\[
\Phi_n(f) := \frac{\int_{W_n} f(X(t)) \, dt - |W_n| \cdot \mathbb{E}[f(X(0))]}{\sqrt{|W_n|}}, \quad f \in \mathcal{V},
\]

converges in distribution to a centered Gaussian process \( \Phi \) with covariance function

\[
\text{Cov}(\Phi(f), \Phi(g)) = \int_{\mathbb{R}^d} \text{Cov} \left( f(X(0)), g(X(t)) \right) \, dt
\]

as \( n \to \infty \) in the weak topology.
Sketch of the proof:

According to Oppel (1973) it suffices to show:

1. The finite-dimensional distributions converge appropriately.
2. The processes $\Phi_n$ have linear and continuous paths.
3. The process $\Phi$ exists and has a version with linear and continuous paths.

How we show it?

1. Employ Cramér-Wold-technique
2. Trivial
Show 3. (Continuity of limiting process):
Employ theory of GB- and GC-sets

What is this theory about?
For a Hilbert space $H$ with scalar product $\langle \cdot, \cdot \rangle$ the isonormal process is the centered Gaussian process $\Phi$ with

$$\text{Cov}(\Phi(f), \Phi(g)) = \langle f, g \rangle.$$ 

Sets, on which a version of the isonormal process has bounded / continuous paths, are called GB / GC -sets.
How to apply this theory to our problem?

Define scalar product such that $\Phi$ becomes the isonormal process:

$$\langle f, g \rangle = \text{Cov}(\Phi(f), \Phi(g))$$

$$= \int_{\mathbb{R}^d} \text{Cov} \left( f(X(0)), g(X(t)) \right) dt$$

This is a symmetric, non-negative definite, bilinear form. Passing to equivalence classes and completion we obtain a Hilbert space.

We have to show that (the projection of)

$$B := \{ f \in V \mid \| f \|_{\text{Lip}} \leq 1 \}$$

is a GB-set.
Let \( N(\epsilon) \) be the minimal number of elements of an \( \epsilon \)-net on \( B \). An \( \epsilon \)-net on \( B \) are elements \( f_1, \ldots, f_n \in B \) such that

\[
\forall g \in B : \exists i \in \{1, \ldots, n\} : \| f_i - g \|_{\langle \cdot, \cdot \rangle} < \epsilon.
\]

It is well known (see e.g. Dudley, 1999) that it suffices to show

\[
\int_0^1 (\log N(\epsilon))^{1/2} \, d\epsilon < \infty. \tag{1}
\]
For $m \in \mathbb{N}$ and $c > 0$ consider the Lipschitz functions $f$ with

- $f(0) = 0$
- On each interval $[(k - 1)c, kc]$, $k = -m + 1, \ldots, m$, the function $f$ is either increasing with constant slope 1 or decreasing with constant slope $-1$.
- On $(-\infty, -mc]$ and $[mc, \infty)$, the function $f$ is constant.

Under appropriate inequalities on $\epsilon$, $c$ and $m$, these functions form an $\epsilon$-net with (1).
Does the asymptotic variance $\text{Var}(\Phi(f)), f \in V$, vanish?

For Gaussian random fields with non-negative covariance function:

$$\text{Var}(\Phi(f)) = 0 \iff \text{Var}(X(0)) = 0$$

or $f$ is constant
Construction of a field $X(t)_{t \in \mathbb{R}^d}$ that fulfills all assumptions of our theorem, but $\text{Var}(\Phi(f)) = 0$ for all $f \in V$:

Consider a Poisson-Voronoi-mosaic, i.e. a random partition of $\mathbb{R}^d$ into convex polytopes.

$$X(t) := \frac{|\{v \in C(t) \mid \|v - \xi_t\| \leq \|t - \xi_t\|\}|}{|C(t)|}, \quad t \in \mathbb{R}^d,$$

where $C(t)$ is the cell in which $t$ lies and $\xi_t$ is its nucleus.
\[
\int_C f(X(t)) \, dt = \int_0^1 f(x) \, dx \cdot |C|
\]
for each (random) cell \(C\).

\[
\int_W f(X(t)) \, dt \approx \int_0^1 f(x) \, dx \cdot |W|
\]
for each (large, deterministic) compact set \(W \subseteq \mathbb{R}^d\), strengthened by mixing properties of Poisson-Voronoi-mosaic.

\[\Rightarrow\] The variance vanishes asymptotically.

Further issue: Orthogonality in \((V, \langle \cdot, \cdot \rangle)\)

For a certain class of random fields constructed with help of Lévy-Meixner-systems we obtained explicit ONBs.
Open problems

- Proof of the last theorem (FCLT) for a space larger than $V$ and under association instead of mixing
- LTs for stationary long range dependent random fields
References

- A. Bulinski, E. Spodarev, F. Timmermann: „Central limit theorems for the excursion sets volumes of weakly dependent random fields“, Bernoulli (2012) 18, 100-118.
References


