Random sets

Distributions, capacities and their applications

Ilya Molchanov

University of Bern, Switzerland

Definitions

- $\mathbb{E}$ locally compact Hausdorff second countable space (usually $\mathbb{E} = \mathbb{R}^d$)

- $\mathcal{F}$ family of closed subsets of $\mathbb{E}$

- $(\Omega, \mathfrak{F}, \mathbf{P})$ probability space

- $X : \Omega \mapsto \mathcal{F}$ is a random closed set if $\{\omega : \ X(\omega) \cap K \neq \emptyset\} \in \mathfrak{F}$ for all $K \in \mathcal{K}$ (compact sets in $\mathbb{E}$).

- In other words, X is measurable with respect to the σ-algebra on $\mathcal{F}$ generated by families of sets

$$\{F \in \mathcal{F} : \ F \cap K \neq \emptyset\} \text{ for } K \in \mathcal{K}.$$

Simple examples

- $X = \{\xi\}$ random singleton
- $X = B_\eta(\xi)$ random ball. Check measurability

$$\{X \cap K \neq \emptyset\} = \{\rho(K,\xi) \leq \eta\}$$

and use the fact that $\rho(K,\xi)$ is a random variable

- $X = \triangle_{\xi_1,\xi_2,\xi_3}$ random triangle
- $X = \{x \in \mathbb{R}^d : \langle x, \xi\rangle \leq t\}$ random half-space
- $X = \{t \geq 0 : \xi_t = 0\}$ for a continuous stochastic process ξ_t

Random variables associated with random closed sets

- The norm $\|X\| = \sup\{\|x\| : x \in X\}$ is a random variable (with possibly infinite values).
- For every $x \in \mathbb{E}$ the indicator $\mathbb{1}_X(x)$ is a random variable.
- If μ is a locally finite Borel measure on $\mathbb{E}$, then $\mu(X)$ is a random variable. This follows directly from Fubini's theorem since $\mu(X) = \int \mathbb{1}_X(x)\mu(dx)$.

Capacity functional

- If X is a random closed set, then

$$T_X(K) = \mathbf{P}\{X \cap K \neq \emptyset\}$$

is called the capacity functional of X. Also write $T(K)$ instead of $T_X(K)$.

- Usually K belongs to the family $\mathcal{K}$ of compact sets, but T_X can be extended for more general arguments K.

Simple examples

- If $X = \{\xi\}$ random singleton, then

$$T_X(K) = \mathbf{P}\{\xi \in K\}$$

is the probability distribution of ξ. This is the only case of an additive T_X.

- If $X = B_r(\xi)$ (ball of radius r centred at ξ), then

$$T_X(K) = \mathbf{P}\{\xi \in K^r\},$$

where K^r is the r-neighbourhood of K. Note that K_1^r and K_2^r are not necessarily disjoint even if K_1 and K_2 are.

- If $X = [\xi, \infty)$ on $\mathbb{R}^1$, then

$$T_X(K) = \mathbf{P}\{\xi \le \sup K\}$$

Capacity functional (properties)

- Monotonic $T(K_1) \leq T(K_2)$ if $K_1 \subseteq K_2$.
- Subadditive $T(K_1 \cup K_2) \leq T(K_1) + T(K_2)$.
- Concave $T(K_1 \cap K_2) + T(K_1 \cup K_2) \leq T(K_1) + T(K_2)$.
- Semicontinuous $T(K_n) \downarrow T(K)$ if $K_n \downarrow K$.

Complete alternation

Succesive differences for $T = T_X$ are defined as

$$\Delta_{K_1} T(K) = T(K) - T(K \cup K_1)\,,$$

$$\begin{aligned}\Delta_{K_n} \cdots \Delta_{K_1} T(K) &= \Delta_{K_{n-1}} \cdots \Delta_{K_1} T(K) \\ &\quad - \Delta_{K_{n-1}} \cdots \Delta_{K_1} T(K \cup K_n)\,, \quad n \geq 2\,.\end{aligned}$$

Then $\Delta_{K_1} T(K) = \mathbf{P}\{X \cap K = \emptyset,\ X \cap K_1 \neq \emptyset\}$ and

$$\begin{aligned}&- \Delta_{K_n} \cdots \Delta_{K_1} T(K) \\ &\qquad = \mathbf{P}\{X \cap K = \emptyset,\ X \cap K_i \neq \emptyset,\ i = 1, \dots, n\} \geq 0\,.\end{aligned}$$

T is said to be completely alternating

The Choquet theorem

Let $\mathbb{E}$ be a locally compact Hausdorff second countable space.

Theorem: A functional $T\colon \mathcal{K} \mapsto [0,1]$ such that $T(\emptyset) = 0$ is the capacity functional of a (necessarily unique) random closed set in $\mathbb{E}$ if and only if T is upper semicontinuous and completely alternating.

- Reason: $\chi_K(F) = \mathbb{1}_{F \cap K = \emptyset}$ satisfies

$\chi_K(F_1 \cup F_2) = \chi_K(F_1)\chi_K(F_2)$ (compare $e^{it(x+y)} = e^{itx}e^{ity}$).

Then $\mathbf{E}\,\chi_K(X) = 1 - T_X(K)$ is the Laplace transform of the distribution of X.

Complete alternation corresponds to the positive definiteness property.

Point processes

- X in $\mathbb{R}^d$ is locally finite if $X \cap K$ is finite for each bounded set K

- A locally finite random closed set is a point process

- Then $N(K) = \mathrm{card}(X \cap K)$ is a random variable (counting random measure)

- The Choquet theorem implies that the distribution of a simple (no multiple points) point process is uniquely determined by the avoidance probabilities $\mathbf{P}\{X \cap K = \emptyset\} = \mathbf{P}\{N(K) = 0\}$.

Special capacities

- If $T(K) = \sup\{f(x) :\ x \in K\}$, then

$$X = \{x :\ f(x) \geq \alpha\}$$

  with α uniformly distributed on $[0, 1]$.

- If $T(K) = 1 - e^{-\Lambda(K)}$, then X is a Poisson point process with intensity measure Λ.
- If $T(K) = 1 - e^{-C(K)}$, where $C(K)$ is the Newton capacity. Then X is related to graphs of Wiener processes.

Random sets vs fuzzy sets

A fuzzy set is a function $f : \mathbb{E} \mapsto [0,1]$, so that $f(x)$ is the "degree of membership" for a point x.

If f is upper semicontinuous (usually assumed), then it is possible to write

$$f(x) = \mathbf{P}\{x \in X\}$$

for

$$X = \{x : \; f(x) \geq \alpha\}.$$

However, the distribution of X contains more information than its one-point covering probabilities.

Extension problems

- $T(K)$ is defined on some family of compact sets K.

Aim: find if it is possible to extend it to a capacity functional on $\mathcal{K}$ and find (all possible) extensions.

- Trivial case: $T(\{x\}) = f(x)$ defined for all singletons K. Then

define $X = \{x : \; f(x) \geq \alpha\}$, i.e. define the extension by
$T(K) = \sup\{f(x) : \; x \in K\}$.

- Open problem: $T(\{x, y\}) = f(x, y)$. Characterise all possible

functions f such that T is extendable to a capacity functional.
If X is stationary, then $T(\{x, y\}) = f(x - y)$.

Measurability issues

The fundamental measurability theorem implies that the following are equivalent (in case $\mathbb{E}$ is Polish and the probability space is complete)

- $\{X \cap B \neq \emptyset\}$ is measurable event for all Borel B;
- $\{X \cap F \neq \emptyset\}$ is measurable event for all closed F;
- $\{X \cap F \neq \emptyset\}$ is measurable event for all open G;
- the distance function $\rho(y, X)$ is a random variable for all y;
- there exists a sequence of random singletons $\{\xi_n\}$ such that X is the closure of $\{\xi_n,\ n \geq 1\}$;
- the graph $\{(\omega, x);\ x \in X(\omega)\}$ is measurable set in the product space $\Omega \times \mathbb{E}$.

Notes to measurability issues

- If $\mathbb{E}$ is locally compact, all these statements are equivalent to the fact that X is a random closed set.

- A random element ξ satisfying $\xi \in X$ a.s. is called a selection of X.

- $X : \Omega \mapsto \mathcal{F}$ is called a multifunction, and

$$X^{-}(B) = \{\omega : \ X(\omega) \cap B \neq \emptyset\}$$

is its inverse.

Capacity functionals and properties of random sets

- T_X determines the distribution of X and so all features of X.

- E.g. if μ is a σ-finite measure, then

$$\mathbf{E}\,\mu(X) = \int T_X(\{x\})\mu(dx) = \int \mathbf{P}\{x \in X\}\mu(dx)$$

(Robbins' theorem).

- In general, the situation is more complicated, e.g. if μ is a Hausdorff measure.

Convex random sets

X is said to be convex if its realisations are a.s. convex (check that the family of convex closed sets is measurable!).

Theorem. X is convex if and only if

$$T_X(K_1 \cap K_2) + T_X(K_1 \cup K_2) = T_X(K_1) + T_X(K_2)$$

for all convex compact sets K_1, K_2 such that $K_1 \cup K_2$ is also convex.

Reason: If X is convex, then

$$0 = \mathbf{P}\{X \cap K = \emptyset,\ X \cap K_1 \neq \emptyset,\ X \cap K_2 \neq \emptyset\} = -\Delta_{K_2}\Delta_{K_1}T_X(K)\,.$$

In the reverse direction take $K_1 = [x, z]$ and $K_2 = [z, y]$.

Weak convergence

- $X_n \xrightarrow{d} X$ (weakly converges) iff the corresponding probability measures weakly converge.

- This is equivalent to

$$T_{X_n}(K) \to T_X(K)$$

for all compact sets K such that $T_X(K) = T_X(\mathrm{Int}\, K)$ (continuity sets).

- Each sequence of random closed sets in a locally compact space possesses a weak convergent subsequence.

Capacities elsewhere

- $\varphi(A)$, $A \subset \mathbb{E}$, is a non-additive measure if $\varphi(\emptyset) = 0$, $\varphi(\mathbb{E}) = 1$ and φ is monotonic. Such non-additive measures (capacities) build a topological space, it is is possible to study their random variants, weak convergence etc.

- Game theory: elements of $\mathbb{E}$ are players, subsets of $\mathbb{E}$ are coalitions, φ is a game. Dual $\tilde{\varphi}(A) = 1 - \varphi(A^c)$.

- If $\varphi(A) = \mathbf{P}\{X \subset A\}$ (containment functional of X), then $\tilde{\varphi}(A) = T(A) = \mathbf{P}\{X \cap A \neq \emptyset\}$ is the capacity functional.

Core of non-additive measure

- The core of φ is the family of finite-additive measures μ such that $\mu(A) \geq \varphi(A)$ for all A.
In terms of random sets $\mathbf{P}\{X \subset A\} \leq \mu(A) \leq \mathbf{P}\{X \cap A \neq \emptyset\}$.

- If φ is convex, i.e.

$$\varphi(A \cap B) + \varphi(A \cup B) \geq \varphi(A) + \varphi(B),$$

then the core is non-empty.

- If φ is the containment functional of a.s. non-empty random set X, then all σ-additive measures from the core correspond to selections of X, i.e. μ is the distribution of ξ such that $\xi \in X$ a.s.

Selections and comparison

- If $\xi \in X$, then $\mathbf{P}\{\xi \in K\} \leq T_X(K)$ for all $K \in \mathcal{K}$. This is also sufficient condition for ξ being a selection of X.

- If $Y \subset X$ a.s., then $T_Y(K) \leq T_X(K)$ for all K.

- However the inequality between the capacity functionals does not suffice to deduce that $Y \subset X$. One needs inequalities between successive differences $\Delta(\cdots)$ of all orders.

Choquet integral

Probability measure μ	Non-additive measure φ
Expectation $\int \xi(\omega)\mu(d\omega)$	Choquet integral

Choquet integral is defined for a non-negative function f as

$$\int f d\varphi = \int_0^\infty \varphi(\{x : \ f(x) \geq t\})dt \,.$$

Compare: if $\mu = \mathbf{P}$ is a probability measure and $\xi \geq 0$ a.s., then

$$\mathbf{E}\,\xi = \int_0^\infty \mathbf{P}\{\xi \geq t\}dt \,.$$

It is easy to see that

$$\int f dT_X = \mathbf{E} \sup\{f(x) : \ x \in X\} \,.$$

Choquet integral and pricing

- The Lebesgue integral (expectation of a random variable X) $\Leftrightarrow$ linear pricing functional.

- Choquet integral $\Leftrightarrow$ sub-additive and comonotone additive, i.e. $\int(f+g)d\varphi = \int f d\varphi + \int g d\varphi$ if f and g are comonotonic.

Random variables X and Y are comonotonic if $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$ a.s. $\omega, \omega' \in \Omega$

Note that $f \mapsto \mathbf{E} \sup f(X)$ is comonotonic with respect to f (but not necessarily additive)

Risk measures

$\xi \in L^\infty$ represents a financial gain. Functional $\rho : L^\infty \mapsto \mathbb{R}$.

Translation invariance. For all $\xi \in L^\infty$ and $t \in \mathbb{R}$: $\rho(\xi + t) = \rho(\xi) - t$.

Subadditivity. For all $\xi_1, \xi_2 \in L^\infty$: $\rho(\xi_1 + \xi_2) \le \rho(\xi_1) + \rho(\xi_2)$.

Positive homogeneity. For all $c \ge 0$ and $\xi \in L^\infty$: $\rho(c\xi) = c\rho(\xi)$.

Monotonicity. If $\xi_1 \le \xi_2$ a.s., then $\rho(\xi_1) \ge \rho(\xi_2)$.

Then ρ is called a coherent risk measure.

Representation of coherent risk measures

If ρ is coherent and lower semicontinuous in probability, then there exists a convex set $\mathcal{A}$ of probability measures, such that

$$\rho(\xi) = \sup\{\mathbf{E}_\mu \xi : \mu \in \mathcal{A}\}.$$

Particularly important families of risk measures are given by Choquet integrals

$$\rho(\xi) = \int \xi d\varphi .$$

If φ is the capacity functional T_X with $X \subset \Omega$, then

$$\rho(\xi) = \mathbf{E} \sup\{\xi(\omega) : \omega \in X\}.$$

Although too simple in the univariate case, this representation is useful for multivariate risk measures.

Belief and plausibility functions

Containment functional $\mathbf{P}\{X \subset K\}$	Belief function
Capacity functional T_X	Plausibility functional

Example: Likelihood plausibility $X = \{x : \; L(x) \geq \alpha\}$, where L is the normalised likelihood.

Aim: Update of belief functions.

Open problem. For a given belief function find a probability measure that dominates it and has the maximal entropy. In other words, find a selection (of a random set) with the maximal entropy.

Capacities in robust statistics

"Contaminated" families of probability measures, e.g.

$$(1-\varepsilon)\mathbf{P}+\varepsilon\mathbf{Q}$$

for all probability measures $\mathbf{Q}$ (contaminations).

The corresponding upper probability is a capacity functional

$$T_X(K)=\varepsilon+(1-\varepsilon)\mathbf{P}(K)\,,\quad K\neq\emptyset\,,$$

so that X equals the whole space with probability ε and otherwise is the singleton with distribution $\mathbf{P}$.

Statistical issues for random sets themselves

- Reference measure (like Lebesgue or Gaussian) is not available.
- Likelihood-based methods are not available (so far).
- Models are scarce.

Random set models

- Point-based models $X = \{\xi_1, \dots, \xi_n\}$.

- Convex hull based models $X = \mathrm{co}\,(\xi_1, \dots, \xi_n)$.

E.g., random segments or triangles.

- More general setting: $M : \mathbb{R}^m \to \mathcal{F}$ is a set-valued measurable function on an auxiliary space. Then $X = M(\zeta)$ becomes a random closed set.

Stationary random sets

- Similar setting $X = M(\zeta)$, but ζ now is a point process.

- If $\Psi = \{x_1, x_2, \dots\}$ is a point process and $X_1, X_2, \dots$ i.i.d. random compact sets, then

$$\bigcup_{x_i \in \Psi} (x_i + X_i)$$

is a germ-grain model. Called Boolean model if the point process is Poisson.

Random fractal sets

- Example. Cantor set construction.

- At each step we delete the mid-parts of the remaining intervals (starting from $[0,1]$), so that the left remaining part occupies the C_1-share and the right remaining part C_2-share of the whole length.

- The usual Cantor set appears if $C_1 = C_2 = 1/3$.
If C_1 and C_2 are random and i.i.d. at every step and every interval, then the Hausdorff dimension s of the obtained random fractal set satisfies

$$\mathbf{E}(C_1^s + C_2^s) = 1 \,.$$

Sets related to stochastic processes

ζ_t, $t \geq 0$, is a stochastic process.

- Level set, e.g.

$$X = \{t \geq 0 : \ \zeta_t = 0\}$$

is a random closed set if ζ is sample continuous.

- Epigraph

$$X = \{(t,x) : \ x \geq \zeta_t, \ t \geq 0, \ x \in \mathbb{R}\} = \operatorname{epi} \zeta$$

is a random closed set if ζ is a.s. lower semicontinuous.

References

[1] Ph. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Math. Finance*, 9:203–228, 1999.

[2] F. Delbaen. Coherent risk measures on general probability spaces. In K. Sandmann and P. J. Schönbucher, editors, *Advances in Finance and Stochastics*, pages 1–37. Springer, Berlin, 2002.

[3] I. Molchanov. *Theory of Random Sets*. Springer, London, 2005.

[4] T. Norberg. On the existence of ordered couplings of random sets — with applications. *Israel J. Math.*, 77:241–264, 1992.

[5] W. Vervaat. Random upper semicontinuous functions and extremal processes. In W. Vervaat and H. Holwerda, editors, *Probability and Lattices*, volume 110 of *CWI Tracts*, pages 1–56. CWI, Amsterdam, 1997.