Random sets

Random sets and stochastic processes

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Definitions

- $\mathcal{F}$ family of closed subsets in locally compact Hausdorff second countable space $\mathbb{E}$ (usually $\mathbb{E} = \mathbb{R}^d$).

- X is a random closed set in $\mathbb{E}$.

- $\zeta(t)$, $t \geq 0$ or $t \in \mathbb{E}$, is a stochastic process.

Content of the lecture

- Random sets X_t depending on parameter $t \geq 0$ (time).
  - set-valued martingales
  - applications in quantitative finance
- Random sets that appear in relation to graphs and level sets of stochastic processes
  - Markov sets
  - epigraphs and stochastic optimisation

Set-valued martingales

- $X_n, n \geq 1$, an adapted sequence of integrable random sets, i.e. X_n is $\mathfrak{F}_n$-measurable

- X_n is called a martingale if

$$\mathbf{E}(X_{n+1}|\mathfrak{F}_n) = X_n \quad \text{for all } n \geq 1.$$

- The conditional expectation is understood as random closed set Y such that $h(Y,u) = \mathbf{E}(h(X_{n+1},u)|\mathfrak{F}_n)$ for all $u \in \mathbb{S}^{d-1}$.

Sub- and supermartingales

- X_n is supermartingale if $\mathbf{E}(X_{n+1}|\mathfrak{F}_n) \subset X_n$
- X_n is submartingale if $\mathbf{E}(X_{n+1}|\mathfrak{F}_n) \supset X_n$
- If X_n is singleton all these definitions leads to the usual martingale concept.
- It is not possible to turn a submartingale into supermartingale by changing its sign.

Main tools

- Support functions, $h(X_n, u)$ becomes a martingale if X_n is a set-valued martingale.

- Martingale selections, i.e. a sequence ξ_n of selections $\xi_n \in X_n$, which forms a martingale.
Any set-valued martingale admits at least one martingale selection and, moreover, it is possible to find a dense family of such selections, i.e. to “fill” X_n by its martingale selections.

Convergence theorem

Theorem (Hiai–Umegaki). If X_n is integrably bounded for each $n \geq 1$ and $\|X_n\|$ is uniformly integrable for a set-valued martingale X_n, then there exists a unique integrably bounded random convex closed set X_∞ such that $X_n = \mathbf{E}(X_\infty | \mathfrak{F}_n)$ and $\rho_{\mathrm{H}}(X_n, X_\infty) \to 0$.

Generalisations exist for supermartingales and martingales with possibly unbounded values.

Optional sampling

If X_n is a set-valued martingale and τ_n is an increasing sequence of bounded stopping times, then X_{τ_n} is a set-valued martingale too.

Application: transaction costs

Bid price $\leq$ Ask price

- Price is interval $[z', z]$.

- Price regions for multiple assets can be described as convex sets.

- A combination $u = (u_1, \dots, u_d)$ (amounts of all assets) costs $p(u)$.

Then. $p(u + u') \leq p(u) + p(u')$ and $p(\lambda u) = \lambda p(u)$ for $\lambda > 0$ and $u \in \mathbb{R}^d$, i.e. $p(u)$ is the support function of a convex set in $\mathbb{R}^d$

$$p(u) = h(Z, u) = \sup\{\langle z, u \rangle : \; z \in Z\} = \sup\langle Z, u \rangle \, .$$

Z is a price set (that may depend on time)

Set-valued price process

- $Z(t), \quad t = 0, 1, \dots, T$ satisfies a sort of no-arbitrage condition if and only if $Z(t)$ possesses at least one strict martingale selection with respect to a martingale measure $\mathbf{P}^*$, i.e. a martingale $\xi(t)$, such that $\xi(t)$ a.s. belongs to the relative interior of $Z(t)$ for all t.

- The set $Z^*(t), t = 0, 1, \dots, T$, of all martingale selections determines the prices of claims. If C is a claim (a random vector that gives amounts of assets), then its no-arbitrage price $p(C)$ satisfies

$$-\mathbf{E}^* \, h(Z^*(T), -C) \le p(C) \le \mathbf{E}^* \, h(Z^*(T), C),$$

where $\mathbf{E}^*$ is the martingale measure.

- Important issue. Find ways to identify the largest set-valued martingale contained in an adapted set-valued process.

Level sets of stochastic processes

$$X = \{t \geq 0 : \ \zeta_t = 0\}$$

Two main issues

- properties of X arising from particular stochastic process ζ (or family of processes)
- intrinsic characterisation of level sets for stochastic processes (i.e. conditions on $X \subset [0, \infty)$ that imply that X is the level set of a stochastic process from a certain family)

Strong Markov sets

$X \subset [0, \infty)$, $X^\flat$ is the set of isolated or right-limit points of X

$$\theta_t(X) = X \cap [t, \infty) - t$$

Definition. X adapted to filtration $\mathfrak{F}_t$, $t \geq 0$, is said to be homogeneous strong Markov if $0 \in X$ a.s. and, for every stopping time τ with $\tau \in X^\flat$ on $\{\tau < \infty\}$

(i) $\theta_\tau(X)$ and $X \cap [0, \tau]$ are conditionally independent given $\{\tau < \infty\}$;

(ii) conditional distribution of $\theta_t(X)$ given $\{\tau < \infty\}$ coincides with the distribution of X.

Origin: recurrent events introduced by W.Feller (discrete time), later studied by J.F.C.Kingman as regenerative phenomena (continuous time)

$$\mathbf{P}\{\{t_1, \ldots, t_n\} \subset X\} = p(t_1)p(t_2 - t_1) \cdots p(t_n - t_{n-1})$$

where $p(t) = \mathbf{P}\{t \in X\}$ is called the p-function of X.

Level sets of strong Markov processes

strong Markov property = Markov property at stopping times

Theorem. (Krylov–Yushkevich & Hoffman-Jørgensen) X is strong Markov iff there exists a right-continuous real valued strong Markov process ξ_t such that $X = \{t : \xi_t = 0\}$ and $\xi_0 = 0$ a.s.

Most difficult part: If X is strong Markov, show that the backward recurrence process $x_t^- = t - \sup X \cap [0,t]$ is a strong Markov process, so that one can set $\xi_t = x_t^-$.

Subordinators

Every strong Markov set can be obtained as the range of a subordinator (non-decreasing process with independent increments), i.e.

$$X = \{\zeta_t : \ t \geq 0\}$$

E.g. if $X = \{t : \ w_t = 0\}$ is zero set for the Wiener process, then $\mathbf{E}\, e^{-t\zeta_s} = e^{-s\Phi(t)}$ with $\Phi(t) = \sqrt{t}$.

It is possible to express the hitting probability

$$\mathbf{P}\{X \cap (a,b] \neq \emptyset\} = \mathbf{P}\{\zeta_t \in (a,b] \text{ for some } t\}$$

in terms of the characteristics of the subordinator ζ.

Intrinsic characterisations of level sets

Open problems is to characterise level sets of

- Gaussian continuous processes
- Markov processes (not strong Markov)
- diffusions

etc.

Epigraphs

$$f : \mathbb{E} \mapsto \mathbb{R}$$

$$\operatorname{epi} f = \{(x, t) \in \mathbb{E} \times \mathbb{R} : \; t \geq f(x)\}$$

- f is lower semicontinuous, i.e.

$$\liminf_{x \to a} f(x) \geq f(a) \quad \text{for all } a\,,$$

if and only if $\operatorname{epi} f$ is a closed set

- Idea: treat functions as closed sets (their epigraphs).

Epiconvergence

- f_n epiconverges to f if $\operatorname{epi} f_n \to \operatorname{epi} f$ as closed sets.

- Recall that $F_n \to F$ means
  - $F \cap G \neq \emptyset$ for an open G implies $F_n \cap G \neq \emptyset$ for all large n
  - $F \cap K = \emptyset$ for compact K implies that $F_n \cap K = \emptyset$ for all large n.

- Epiconvergence is the weakest convergence type that implies convergence of minima, i.e. $f_n \xrightarrow{\text{epi}} f$ implies that

$$\limsup_{n\to\infty} (\inf f_n) \leq \inf f$$

with the equality if f_n has a relatively compact sequence of ε-optimal points (e.g. if $\mathbb{E}$ is compact).

Normal integrands

- $\zeta(x) = \zeta(x, \omega)$ is normal integrand if $\operatorname{epi} \zeta(\cdot, \omega)$ is a random closed set in $\mathbb{E} \times \mathbb{R}$.

- Possible to treat non-separable stochastic processes, e.g. if

$$\zeta(x) = \begin{cases} 0, & x = \xi \\ 1, & \text{otherwise} \end{cases}$$

for a random point ξ, then $\operatorname{epi} \zeta$ is a non-trivial random closed set, although $\zeta(x) = 0$ a.s. for all x if ξ has a continuous distribution.

- Weak epiconvergence of normal integrands.

- An integrand is sharp if $\partial \operatorname{epi} \zeta = \operatorname{epi} \zeta$ and the family of points $(x, t) \in \operatorname{epi} \zeta$ such that $(x, s) \notin \operatorname{epi} \zeta$ for all $s < t$ is locally finite.

Stochastic optimisation

Theorem. If ζ_n weakly epiconverges to ζ, then

$$\mathbf{P}\{\inf \zeta < t\} \leq \liminf_{n\to\infty} \mathbf{P}\{\inf \zeta_n < t\}, \quad t \in \mathbb{R}\,.$$

- $\inf \zeta_n$ converges in distribution to $\inf \zeta$ if for every $t \in \mathbb{R}$ there exists a compact set K such that $\{\zeta_n \leq t\} \subset K$ a.s. for all $n \geq 1$.

- This condition always holds if $\mathbb{E}$ is compact itself.

Convergence of averages

- ζ normal integrand, $\mathbf{E}\,\zeta(x)$ is well defined (perhaps, infinite).

- ζ_n are i.i.d copies of ζ, then

$$\eta_n(x) = \frac{1}{n}\sum_{i=1}^{n} \zeta_i(x)$$

estimates $\mathbf{E}\,\zeta(x)$ for all x.

Theorem (Artstein–Wets). Assume that each $x_0 \in \mathbb{E}$ has an open neighbourhood G such that $\zeta(x) \geq \alpha$ for all $x \in G$ and an integrable random variable α. Then ζ_n epiconverges to $\mathbf{E}\,\zeta$ a.s. as $n \to \infty$.

Maximum likelihood estimators

Maximum likelihood estimator $\hat{\theta}_n$ appears as the maximiser of the log-likelihood function

$$\sum_{i=1}^{n} \log f_\theta(x_i)$$

Under certain technical conditions, one can derive consistency of the maximum likelihood estimator.

Polyhedral approximations

- F strictly convex in $\mathbb{R}^d$ with twice differentiable boundary
- $\xi_1, \dots, \xi_n$ i.i.d. on F with density f and $\Xi_n = \mathrm{co}\,(\xi_1, \dots, \xi_n)$
- Define $\eta_n(u) = h(F, u) - h(\Xi_n, u)$.
- Then $H_n = \mathrm{epi}\,\eta_n = X_1 \cup \dots \cup X_n$ where

$$X_i = \{(u, t) : \ \|u\| = 1, t \geq 0, \langle u, \xi_i \rangle \geq h(F, u) - t\}$$

It is possible to use limit theorems for unions of random closed sets.

Theorem. If f does not vanish identically on ∂F, then $n\eta_n$ weakly epiconverges to a sharp integrand generated by the Poisson process with intensity measure

$$\Lambda(K) = \int_{F_K} f(x)\mathcal{H}^{d-1}(dx)dt\,, \quad K \subset \mathbb{S}^{d-1} \times [0,\infty)\,,$$

where

$$F_K = \bigcup_{(u,t)\in K} \{(x,s):\ x \in \partial F,\ \mathbf{n}(x) = u,\ s \in [0,t]\}\,.$$

- If $f(x) = 0$ for all $x \in \partial F$ and $\langle f'(x), \mathbf{n}(x)\rangle$ does not vanish identically, then $\sqrt{n}\eta_n$ weakly epiconverges.

Graphical convergence

Non-traditional convergence

- $f_n(t) = nt\,\mathbb{I}_{0\le t\le 1/n} + \mathbb{I}_{t>1/n}$ converges pointwisely to $f(t) = \mathbb{I}_{t>0}$ but not uniformly
- $f_n(t) = nt$, $0 \le t \le 1$, “converges” to the vertical line
- $f_n(t) = \sin nt$, $0 \le t \le 1$, “fills” the rectangle $[0,1] \times [-1,1]$

❑ Formalise as convergence of graphs (also for possible set-valued functions F_n)

$$\{(t,x) :\ t \in \mathbb{E},\ x \in F_n(t)\}$$

In the single-valued case $F_n(t) = \{f_n(t)\}$.

Random step-functions

$\alpha_n, n \geq 0$, i.i.d. random variables

Step-function

$$\zeta_n(t) = a_n^{-1} \alpha_{[nt]}\,, \quad 0 \leq t \leq 1\,.$$

The graphs of ζ_n weakly converge as random closed sets iff

$$n\mathbf{P}\{\alpha_0 \in a_n[x,y]\} \to \nu([x,y])$$

for a measure ν. The limiting random set has the capacity functional

$$T_X(K) = 1 - \exp\{-(\mathrm{mes}_1 \otimes \nu)(K)\}\,, \quad K \subset [0,1] \times \mathbb{R}\,.$$

Random broken lines

The linearly interpolated step-function ζ_n graphically converges to random closed set Z with the capacity functional

$$T_Z(K) = 1 - \exp\{-(\mathrm{mes}_1 \otimes \nu)(\mathrm{co}\,(K \cup K'))\}\,, \quad K \subset [0,1] \times \mathbb{R}\,,$$

where K' is the projection of K onto $[0,1]$.

Excursion sets

f is lower semicontinuous function

Then

$$F_t = \{f \le t\} = \{x :\ f(x) \le t\}, \quad t \in \mathbb{R}\,,$$

becomes an increasing set-valued process.

- Consider various convergence concepts for the process F_t, in particular, the Skorohod convergence.

- Define operations with normal integrands by performing set-operations with their excursion sets and then stacking them together to obtain the resulting function.

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