"Classical" Stereology

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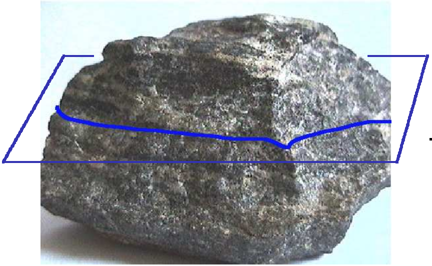
A problem from Geology I

How to determine the quartz content in a block of granite?

A problem from Geology II

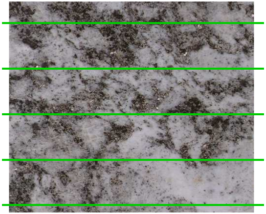
Delesse (1847):



volume fraction in 3D $\approx$ area fraction in a planar section

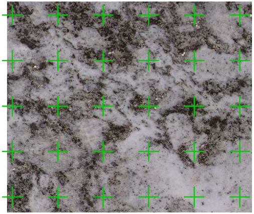
A problem from Geology III

Rosiwal (1898):



volume fraction in 3D $\approx$
length fraction in linear sections

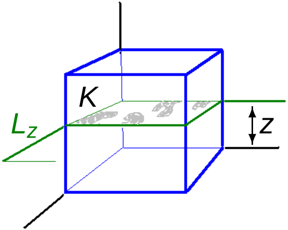
Glagolev (1933):



volume fraction in 3D $\approx$
relative number of points in quartz

Mathematical background

Let K be the phase of interest (quartz)



$$\int_{-\infty}^{\infty} \mathrm{Area}(K \cap L_z)\, dz = \mathrm{Vol}(K)$$

Random sampling: z = uniform random "height" $\xi \in [0, 1]$:

$$\boxed{\mathbb{E}_\xi \, \mathrm{Area}(K \cap L_\xi) = \mathrm{Vol}(K).}$$

expectation w.r.t. ξ

Two basic approaches

- Design based approach:
  The sampling is done in a random, homogeneous way,
  the set K is deterministic.
- Model based approach:
  No assumptions on the sampling procedure
  the set K is “stochastically homogeneous”

  ($\rightsquigarrow$ stochastic geometry: stationary random set)

We will only use the design based approach here!

Stereology: A Definition

Stereology is a sub-area of stochastic geometry dealing with the estimation of geometric characteristics (volume, area, boundary length, particle number,. . .) of structures from samples. Either the structure or the sampling scheme is random.

Sampling schemes can be

- sections with lower dimensional test planes (Delesse, Rosival),
- sections with full-dimensional test windows,
- sections with point lattices (Glagolev).

Digital stereology deals with point lattice samples.

Assumptions on the structure

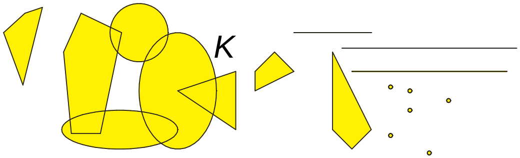
We first specify assumptions on the structure $K \subset \mathbb{R}^d$, $d \geqslant 1$, and then define certain geometric characteristics of K.

$\mathcal{C}:=\{K \subset \mathbb{R}^d \mid K \text{ is compact}\}$.

$\mathcal{K}:=\{K \in \mathcal{C} \mid K \text{ is convex, nonempty}\}$ “convex bodies”.

$\mathcal{R}:=\{K \in \mathcal{C} \mid K \text{ is a finite union of convex bodies}\}$

“convex ring” “polyconvex sets”.

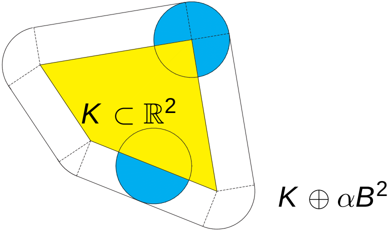


General assumption $K \in \mathcal{C}$, often $K \in \mathcal{R}$.

The Hausdorff metric

We will need Minkowski addition on $\mathcal{C}$: For $K, K' \in \mathcal{C}$ set

$$K \oplus K' := \{x + x' \mid x \in K,\ x' \in K'\}.$$



with $B^d :=$ unit ball in $\mathbb{R}^d$,
and $\alpha \geqslant 0$.

The Hausdorff metric δ on $\mathcal{C}$ is given by

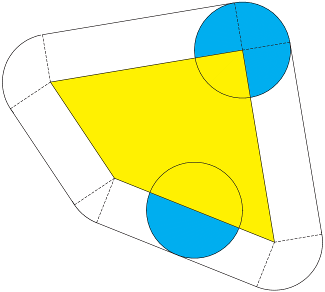
$$\delta(K, K') := \min\{\alpha \geqslant 0 \mid K \subset K' \oplus \alpha B^d, K' \subset K \oplus \alpha B^d\}, \quad K, K' \in \mathcal{C}.$$

Steiner's formula

Let $\mathrm{Vol}(\cdot)$ denote Lebesgue measure on $\mathbb{R}^d$, $\kappa_d := \mathrm{Vol}(B^d)$.

Jakob Steiner [1840]: If K is a convex body, then

$$\boxed{\mathrm{Vol}(K \oplus \varepsilon B^d) = \sum_{j=0}^{d} \kappa_{d-j} V_j(K) \varepsilon^{d-j}, \qquad \varepsilon \geqslant 0.}$$



$V_j(K) =:$ j-th intrinsic volume of K.
(Minkowski functional,
quermass-integral)

e.g. $V_d(K) = \mathrm{Vol}(K)$.

Intrinsic Volumes

Properties of V_j on $\mathcal{K}$:

1. motion-invariant: $V_j\big(\vartheta(K+x)\big) = V_j(K)$,
   translation vector $x \in \mathbb{R}^d$
   rotation $\vartheta \in SO_d$:= rotation group
2. additive: $V_j(K \cup K') = V_j(K) + V_j(K') - V_j(K \cap K')$
   (where K, K' and $K \cup K'$ are convex bodies)
3. homogeneous: $V_j(\alpha K) = \alpha^j V_j(K)$, $\alpha \geqslant 0$
4. monotone: $K \subset K' \Rightarrow V_j(K) \leqslant V_j(K')$
5. continuous: $\delta(K_n, K) \overset{n\to\infty}{\longrightarrow} 0 \;\Rightarrow\; V_j(K_n) \overset{n\to\infty}{\longrightarrow} V_j(K)$

Characterization Theorem

Hadwiger [1957]:

Let $\varphi : \mathcal{K} \to \mathbb{R}$ be motion-invariant and additive.
If φ is monotone or continuous, then
it is a linear combination of $V_0, \dots, V_d$.

Hence: if we want to find $\varphi(K)$, where φ has the above properties, it is enough to determine $V_0(K), \dots, V_d(K)$.

If φ is in addition homogeneous of degree j then $\varphi = cV_j$.

Additive Extension of V_j

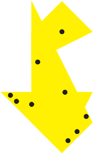
The Inclusion-exclusion formula

$$V_j\left(\bigcup_{i=1}^{n} K_i\right) = \sum_{\emptyset \neq I \subset \{1,\ldots,n\}} (-1)^{|I|+1}\, V_j\left(\bigcap_{i\in I} K_i\right) \tag{1}$$

extends V_j additively to $\mathcal{R}$ (Groemer [1978]).
This extension is again denoted by V_j.

Properties of V_j on $\mathcal{R}$:

1. motion-invariant, homogeneous,
2. additive, and (1) holds for $K_1, \ldots, K_n \in \mathcal{R}$,
3. not monotone for $0 \leqslant j \leqslant d-1$,
4. not continuous for $0 \leqslant j \leqslant d$.



V_j on $\mathcal{R}$

Geometric interpretation:

$$
\begin{aligned}
V_d(K) &= \text{Vol}(X) \text{ is the volume (Lebesgue measure) of } K, \\
2V_{d-1}(K) &= \text{surface area of } K \\
&\quad (= (d-1)\text{-dim. Hausdorff measure } \mathcal{H}^{d-1}(\partial K)), \\
&\;\vdots \\
V_j(K) &= c_{d,j} \int_{\partial K} H_{d-j-1}(x)\, d\mathcal{H}^{d-1}(x) \qquad (\partial K \text{ smooth}) \\
&\;\vdots \\
&\;\vdots \\
V_0(K) &= \text{Euler-Poincaré characteristic of } K.
\end{aligned}
$$

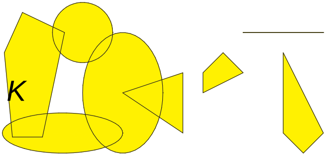
elementary symmetric function
of the principal curvatures at x

The Euler-Poincaré characteristic

For a convex body K: $V_0(K) = 1$.

For $K \in \mathcal{R}$:

- $d = 1$: $V_0(K) = \#$ components (closed intervals) of K.
- $d = 2$: $V_0(K) = \#$ components $- \#$ “holes” of K.



$V_0(K) = 4 - 1 = 3.$

- $d = 3$:
  $V_0(K) = \#$ components $- \#$ “tunnels” $+ \#$ “holes” of K.
- arbitrary d: $V_0(K) =$ alternating sum of Betti numbers of K.

Crofton's formula

Let L be a fixed k-dimensional linear subspace in $\mathbb{R}^d$.

arbitrary movement of L: $\vartheta(L+y)$, $y \in L^{\perp}, \vartheta \in SO_d$.

$\nu :=$ invariant probability measure on SO_d.

Crofton's formula for $K \in \mathcal{R}$:

$$\boxed{\int_{SO_d} \int_{L^{\perp}} V_j\big(K \cap \vartheta(L+y)\big)\, dy\, d\nu(\vartheta) = c_{d,j,k} V_{d+j-k}(K).}$$

In particular $j = k$: "Fubini's theorem"

Proof: For $K \in \mathcal{K}$ apply Hadwiger's theorem to

$$\varphi(K) := \int_{SO_d} \int_{L^{\perp}} V_j\big(K \cap \vartheta(L+y)\big)\, dy\, d\nu(\vartheta).$$

Sampling with lower dimensional planes

As K is bounded, we may exclude planes lying "far out"
(i.e. restrict to $|y| \leqslant M$ for some $M > 0$.)

Intersections with a IUR planes.

$$\boxed{\mathbb{E}_\vartheta \mathbb{E}_{|y| \leqslant M} \, V_j\big(K \cap \vartheta(L + y)\big) = c \cdot V_{d+j-k}(K).}$$

- Unbiased estimator for V_{d+j-k} from a k-dim. IUR section.
- For $j = 0$, $k = 0, \ldots, d-1$, this yields all intrinsic volumes except $V_0(K)$.

Principal kinematic formula

Let $W \in \mathcal{R}$ be a fixed test window.

arbitrary movement of W: $\vartheta(W + x)$, $x \in \mathbb{R}^d, \vartheta \in SO_d$.

Principal kinematic formula for $K, W \in \mathcal{R}$

$$\boxed{\int_{SO_d} \int_{\mathbb{R}^d} V_j\big(K \cap \vartheta(W + x)\big)\, dx\, d\nu(\vartheta) = \sum_{k=j}^{d} c_{d,j,k} V_k(K) V_{d+j-k}(W).}$$

Proof: Hadwiger's theorem.

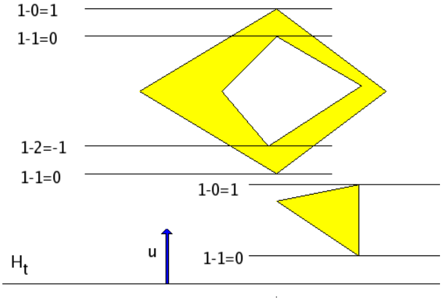
Application: Measuring all intrinsic volumes of K in a randomly moved sampling window allows unbiased estimation of $V_0(K), \ldots, V_d(K)$ (Solve a linear system!).

Hadwiger's inductive definition of V_0 on $\mathcal{R}$:

Choose a unit vector $u \in \mathbb{R}^n$ and set $H_\alpha := u^\perp + \alpha u$. Fix $K \in \mathcal{R}$.

$$\boxed{V_0(K) = \sum_{\alpha\in\mathbb{R}} \left(V_0(K \cap H_\alpha) - \lim_{\beta\to\alpha+} V_0(K \cap H_\beta)\right)}$$

with $V_0(\emptyset) = 0$ and $V_0(K) = 1$ for $\emptyset \neq K \subset \mathbb{R}^1$.



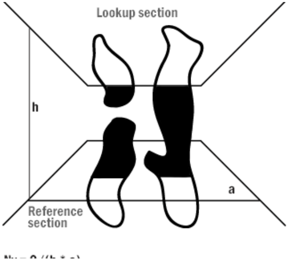
The disector

Assume: $K = \bigcup_{j=1}^{n} K_j$ disjoint union of $K_1, \ldots K_n \in \mathcal{K}$
($\Rightarrow V_0(K) = n$).

Problem: $\#$ particles in planar sections $\not\sim$ $\#$ particles in 3D!

The disector solves this problem. Fix $h > 0$.

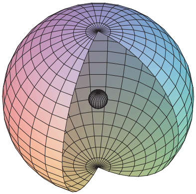
$$\boxed{V_0(K) = \frac{1}{h} \int_{\mathbb{R}} \sum_{j=1}^{n} \big(V_0(K_j \cap H_\alpha)(1 - V_0(K_j \cap H_{\alpha+h}))\big)\, d\alpha}$$



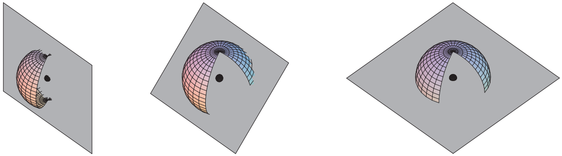
Necessary assumption:
Particles have a height $> h$.

Local sections

Motivation: Microscopy images
of a biological cell K:
K has an identifiable "center" $0 \in K$.



Use this information: Take isotropic planar sections through 0:



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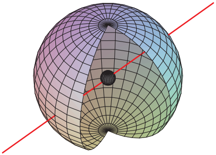
Local volume estimators

The nucleator (Volume of $K \in \mathcal{K}$ from central line sections):

$$\boxed{V_d(K) = \int_{SO_d} F(K \cap \vartheta L)\, d\nu(\vartheta)}$$



with $F(A) = c_d \sum_{x \in \mathrm{vert}(A)} \|x\|^d$.

- proof: polar coordinates,
- can be generalized to k-dimensional central probes,
- can be generalized to $\mathcal{H}^j$ replacing V_d.

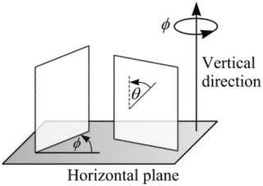
  Proof uses the coarea-formula.

Vertical sections

Example: surface area estimation in $\mathbb{R}^3$

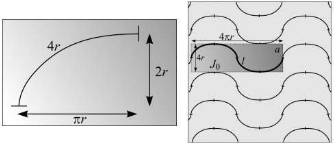
Choose a “vertical” direction u .

Choose an plane E containing direction u, otherwise uniform.



Check intersections with a IUR test line L_θ in E.

$$\boxed{V_2(K) = 2 \int_{\{E \| u\}} \int_{\{L_\theta \subset E\}} V_0(K \cap L_\theta) \sin(\theta) dL_\theta dE .}$$



More variation

Other sampling schemes

- thick planar sections
- projections
- combinations of the above

Other geometric characteristics

- non-additive characteristics
  e.g. $\#$ connected components
- local quantities
  e.g. surface area measures, curvature measures
- ...