Digital Stereology

Digitization and Multigrid Convergence

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Outline of the talk

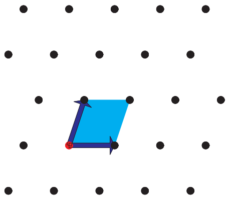
- Digitization of sets
  - Regular lattices
  - Digitization models
- Digitization of geometric characteristics
  - Multigrid convergence
  - A general criterion
- Digitization of the volume
  - Multigrid convergence and digitization error
  - Randomized digitization

Regular lattices in $\mathbb{R}^d$

Definition: $x_1, \ldots, x_d$ basis of $\mathbb{R}^d$.

$$\mathbb{L} = \{n_1 x_1 + \ldots + n_d x_d \mid n_1, \ldots, n_d \in \mathbb{Z}\}$$

is called the regular lattice generated by $x_1, \ldots, x_d$.



- The parallelepiped $C := [0, x_1] \oplus \ldots \oplus [0, x_d]$ is a cell of $\mathbb{L}$.
- Any cell C_0 with minimal diameter $\Delta(\mathbb{L})$ is called fundamental cell.
- $C_0^* = C_0 - (x_1 + \ldots + x_d)/2$: centered fundamental cell.
- $x_1, \ldots, x_d$ standard basis $\Rightarrow$ $\mathbb{L} = \mathbb{Z}^d =$ standard lattice, $C_0 = [0, 1]^d$ standard cube, $\Delta(\mathbb{L}) = \sqrt{d}$, $C_0^* = [-1/2, 1/2]^d$.

Digitizations of a set

A digitization of a set K is a representation of the continuous set K on a discrete lattice $\mathbb{L}$.

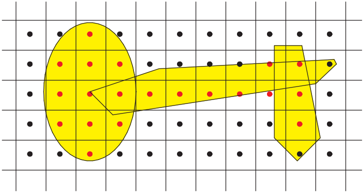
$\mathcal{C} :=$ { compact subsets of $\mathbb{R}^d$ }.
$\mathcal{P}(\mathbb{L}) :=$ power set of $\mathbb{L}$.

Definition: Any mapping from $\mathcal{C}$ to $\mathcal{P}(\mathbb{L})$, $K \mapsto \hat{K}$ is called a digitization on $\mathbb{L}$.

Remark: the notion “digitization” is used in many different ways. Often it refers to a pixel/voxel image of K.
It can also refer to a matrix representation.

Commonly used digitizations

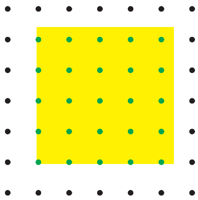
- The hit-or-miss digitization (Gauss digitization):
  $\hat{K} := K \cap \mathbb{L}$.
- The cell covering digitization (outer Jordan digitization):
  $\hat{K} := \{x \in \mathbb{L} \mid (x + C_0^*) \cap K \neq \emptyset\} = (K \oplus C_0^*) \cap \mathbb{L}$
- The volume-threshold digitization with param. $0 < \theta \leqslant 1$:
  $\hat{K} := \{x \in \mathbb{L} \mid \mathrm{Vol}\big((x + C_0^*) \cap K\big) \geqslant \theta \cdot \mathrm{Vol}\big(x + C_0^*\big)\}$.



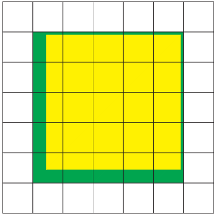
$\theta = 1/2$

Approximation of characteristics from digitizations I

Motivation: Approximate boundary length in $\mathbb{R}^2$.



Hit-or-miss digitization.
From knowledge of $\hat{K}$
find
$\varphi(K) = 2V_1(K) = 4.$



The cell union of $\hat{K}$
$\hat{P} = \bigcup_{x \in \hat{K}} (x + C_0^*)$
Approximation:
$\hat{\varphi}(\hat{K}) := 2V_1(\hat{P}) = 4.2.$

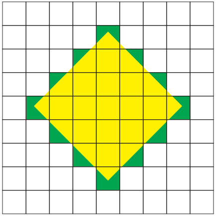
Approximation of characteristics from digitizations II

Improved resolution: replace $\mathbb{L}$ by $t\mathbb{L}$, $0 < t < 1$.

- Digitization $\hat{K}_t := K \cap t\mathbb{L}$ (similar for other digitizations of K)
- Cell union $\hat{P}_t := \bigcup_{x \in \hat{K}_t} (x + tC_0^*)$
- Approximation $\hat{\varphi}_t(\hat{K}_t) := 2V_1(\hat{P}_t)$.

Then we have $\lim_{t \to 0+} \hat{\varphi}_t(\hat{K}_t) = \varphi(K)$ for the unit square

"multigrid convergence".



$$\hat{\varphi}_t(\hat{K}_t) = 2V_1(\hat{P}_t)$$
$$\to 4\sqrt{2} = \sqrt{2}\varphi(K).$$

Deviation of 41 % !!!

Digitization of Characteristics: Definition

Assumptions:

- $\mathcal{M} \subset \mathcal{C}$ is a family of sets,
- $\varphi : \mathcal{M} \to \mathbb{R}$ is a function,
- $\hat{\varphi} : \mathcal{P}(t\mathbb{L}) \to \mathbb{R}$ is a function, a "digitization of φ".

$$\begin{tikzcd} K \in \mathcal{M} \arrow[r, "\varphi"] \arrow[d] & \varphi(K) \\ \hat{K}_t \arrow[r, "\hat{\varphi}"] & \hat{\varphi}(\hat{K}_t) \arrow[u, "\textcolor{red}{?} \text{ as } t \to 0+"'] \end{tikzcd}$$

If $\hat{\varphi}$ satisfies

$$\lim_{t \to 0+} \hat{\varphi}(\hat{K}_t) = \varphi(K), \qquad K \in \mathcal{M},$$

we say that $\hat{\varphi}$ is multigrid convergent to φ on $\mathcal{M}$.

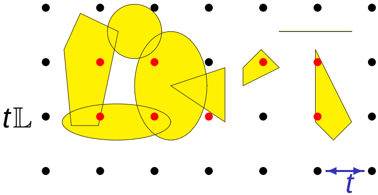
Generalization to set valued characteristics $\varphi : \mathcal{M} \to \mathcal{C}$.

(Serra [1982], Heijmans [1992], Klette & Rosenfeld [2004], K. [2005]).

Digitization of the identity I

Is there a multigrid convergent digitization for the set K?
($\iff$ $\exists \hat{\varphi}_t$ multigrid convergent to $\varphi = i =$identity
on a “large” class $\mathcal{M}$?)

Observation: For the hit-or-miss digitization $\mathcal{M} \neq \mathcal{C}$:



To avoid "lower dimensional parts", we sometimes assume

$$K \in \mathcal{C}_{reg} = \{M \in \mathcal{C} \mid M = \operatorname{cl} \operatorname{int} M\}.$$

Digitization of the identity II

Lemma. There is a digitization $\hat{\iota}_t$ of the identity on

- $\mathcal{M} = \mathcal{C}_{reg}$ for the
  - hit-or-miss digitization, and for the
  - volume-threshold digitization,
- $\mathcal{M} = \mathcal{C}$ for the cell covering digitization, with

$$K \subset \hat{\iota}_t(\hat{K}_t), \qquad K \in \mathcal{C}.$$

In all cases

$$\hat{\iota}_t(\hat{K}_t) = \bigcup_{x \in \hat{K}_t} (x + t\mathrm{C}_0^*)$$

is the cell union.

Digitization of the identity III

Fix a set digitization $K \mapsto \hat{K}$. Let $\mathcal{M} \subset \mathcal{C}$ be such that there is a multigrid convergent digitization $\hat{\imath}_t$ for the identity i on $\mathcal{M}$.

Proposition.

- If $\varphi : \mathcal{M} \to \mathcal{C}$ is continuous,
  there is a multigrid convergent digitization of φ on $\mathcal{M}$.
- If $\varphi : \mathcal{M} \to \mathcal{C}$ is upper semi-continuous, i.e.

$$K_n \searrow K \;\Rightarrow\; \lim_{n\to\infty} \varphi(K_n) \to \varphi(K),$$

  and monotone, and $\hat{\imath}_t : \mathcal{M} \to \mathcal{M}$ satisfies

$$K \subset \hat{\imath}_t(\hat{K}_t), \qquad K \in \mathcal{C},$$

  then $\varphi(\hat{\imath}_t(\cdot))$ is multigrid convergent to φ on $\mathcal{M}$.

Digitization criterion

Theorem. (Heijmans [1992], K. [2005])

- hit-or-miss digitization, and v.-threshold digitization:
  For any continuous $\varphi : \mathcal{M} \to \mathcal{C}$, there is a multigrid convergent digitization, if $\mathcal{M} \subset \mathcal{C}_{reg}$.
- cell covering digitization:
  For any upper semi-continuous, monotone $\varphi : \mathcal{C} \to \mathcal{C}$, there is a multigrid convergent digitization.

Application (Serra [1982]):

For the cell covering digitization, morphological dilation ($K \mapsto K \oplus M$, $M \in \mathcal{C}$ fixed), erosion, opening and closing have multigrid convergent digitizations on $\mathcal{C}$.

Intrinsic volumes!?

Recall: The intrinsic volumes are continuous on $\mathcal{K}$.

- hit-or-miss digitization, and v.-threshold digitization:
  There is a multigrid convergent digitization of V_j on $\mathcal{K}_{reg} = \{K \in \mathcal{K} \mid K = \operatorname{cl}\operatorname{int} K\}$.

- cell covering digitization:
  There is a multigrid convergent digitization of V_j on $\mathcal{K}$.

In both cases, $\hat{V}_j(\hat{K}) = V_j\big(\text{convex hull}(\hat{K})\big)$.

Attention: On $\mathcal{R}$, V_j is

not continuous for $0 \leqslant j \leqslant d$ and

not monotone for $0 \leqslant j \leqslant d-1$.

Digitization of the volume

We have

- hit-or-miss digitization, and v.-threshold digitization:
  There is a multigrid convergent digitization of Vol on $\mathcal{R}$.
- cell covering digitization:
  There is a multigrid convergent digitization of Vol on $\mathcal{C}$.

For both results, we used the cell union

$$\widehat{\mathrm{Vol}}_t(\hat{K}_t) = \mathrm{Vol}\big(\bigcup_{x\in\hat{K}_t}(x + tC_0^*)\big).$$

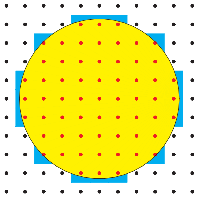
For the hit-or-miss digitization with $\mathbb{L} = \mathbb{Z}^d$ this is

$$\widehat{\mathrm{Vol}}_t(\hat{K}_t) = t^d \cdot \#(K \cap t\mathbb{Z}^d).$$

Quality of the digitization

Consider the simple case of the hit-or-miss digitization with $\mathbb{L} = t\mathbb{Z}^2$, $t \leqslant 1$, K being the unit disk. Set $\hat{A} = t^2 \#(K \cap t\mathbb{Z}^2)$.

Bounds for the deviation from the true value?



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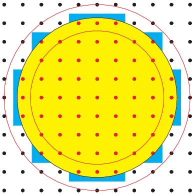


$$(1 - 1/\sqrt{2}t)^2 \pi \leqslant \hat{A} \leqslant (1 + 1/\sqrt{2}t)^2 \pi$$

Hence (Gauss): $|\hat{A} - \pi| \leqslant c \cdot t$.
(Error decreases linear)

There are better values for γ in $|\hat{A} - \pi| \leqslant c \cdot t^{\gamma}$.

Gauss' Circle problem

Find largest γ in $|\hat{A} - \pi| \leqslant c \cdot t^{\gamma}$!

Gauss	$\gamma \geqslant 1$
Voronoi & Sierpinski [1903]	$\gamma \geqslant 4/3 \approx 1.333$
Littlewood and Walfisz [1924]	$\gamma \geqslant 75/56 \approx 1.340$
Chen [1963]	$\gamma \geqslant 50/37 \approx 1.351$
Vinogradov	$\gamma \geqslant 72/53 \approx 1.358$
Huxley [1990]	$\gamma \geqslant 100/73 \approx 1.370$
Hardy & Landau [1915]	$\gamma < 3/2$

Gauss' circle problem:

Is there, for any $\varepsilon > 0$, a constant c with

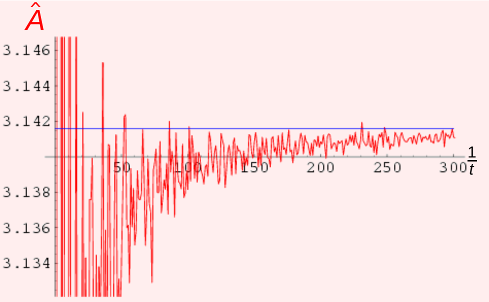
$$|\hat{A} - \pi| \leqslant c \cdot t^{3/2-\varepsilon}?$$

THIELE CENTRE

Illustration

Problem: Very strong fluctuation of the measurements.

$\hat{A} = t^2 \#(K \cap t\mathbb{Z}^2)$ with K = unit disk in $\mathbb{R}^2$.



Random digitization

We randomize the sampling scheme

- randomly translated lattice:
  Choose ξ uniformly in C_0^* and consider $t(\xi + \mathbb{L})$.
- randomly rotated lattice:
  Choose ϑ uniformly in SO_d and consider $\vartheta(t\mathbb{L})$.

In both cases: $\hat{K}_t$ becomes a (finite) random closed set.
Write $\tilde{K}_t$ for $\hat{K}_t$, whenever the randomized lattice is used.

We will only work with randomly translated lattices here, these are stationary random closed sets.

Unbiased digitization of the volume

Let $\tilde{K}_t$ be a random hit-or-miss digitization of $K \in \mathcal{C}$ in $\mathbb{R}^d$,

$$\tilde{V}_t := \mathrm{Vol}(t\mathrm{C}_0^*)\ \#\tilde{K}_t.$$

Important observation: $\boxed{\mathbb{E}\tilde{V}_t = \mathrm{Vol}(K).}$

Proof: $(t = 1)$

$$\mathrm{Vol}(\mathrm{C}_0^*)\ \mathbb{E}\#\tilde{K}_1 = \int_{\mathrm{C}_0^*} \#\big(K \cap (x + \mathbb{L})\big)\, dx = \sum_{y \in \mathbb{L}} \int_{\mathrm{C}_0^*} \mathbf{1}_K(x + y)\, dx$$

$$= \sum_{y \in \mathbb{L}} \mathrm{Vol}(K \cap (y + \mathrm{C}_0^*)) = \mathrm{Vol}(K).$$

THIELE CENTRE

Variance of unbiased volume digitization

What is the variance of $\tilde{V}_t$?

- Huxley's result implies for sufficiently smooth $K \subset \mathbb{R}^2$,

$$\mathbb{V}\mathrm{ar}\tilde{V}_t = \mathbb{E}\big(\tilde{V}_t - \mathrm{Vol}(K)\big)^2 \leqslant c \cdot t^{200/73}, \quad 0 < t \leqslant 1.$$

- Kiěu & Mora [2004]:

$$\mathbb{V}\mathrm{ar}\tilde{V}_t = \mathbb{E}\big(\tilde{V}_t - \mathrm{Vol}(K)\big)^2 \leqslant cV_1(K) \cdot t^3, \quad 0 < t \leqslant 1,$$

  for sufficiently smooth and “randomized” $K \subset \mathbb{R}^2$.

- Hlawka [1950] showed for general regular $\mathbb{L}$ in $\mathbb{R}^d$:

$$\boxed{\mathbb{E}\big(\tilde{V}_t - \mathrm{Vol}(K)\big)^2 \leqslant c \cdot t^{4d/(d+1)}, \quad 0 < t \leqslant 1.}$$

  where ∂K is smooth and has positive curvature everywhere. (cf. Kendall [1948], $d = 2$.)

Hlawka's result

The main steps of the proof ($\mathbb{L} = \mathbb{Z}^d$):

- Express $\mathbb{E}\tilde{V}_t^2$ with the geometric covariogram

$$\mathbb{E}\tilde{V}_t^2 = t^d \sum_{x \in t\mathbb{Z}^d} C_K(x)$$

  with $C_K(x) = \mathrm{Vol}\big(K \cap (K - x)\big)$.

- Poisson's formula $t^d \sum_{x \in t\mathbb{Z}^d} f(x) = \sum_{x \in \frac{1}{t}\mathbb{Z}^d} \hat{f}(x)$,

  ($\hat{f}$ = Fourier transform of f) implies

$$\mathbb{V}\mathrm{ar}(\tilde{V}_t) = \sum_{x \in 1/t\mathbb{Z}^d \setminus \{0\}} \hat{C}_K(x)$$

- Use that $\hat{C}_K(x) = |\widehat{\mathbf{1}}_K(x)|^2$ (power spectral density) can be estimated for large x.