# Digital Stereology Digitization of Intrinsic Volumes 

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## Reminder: Digitization of intrinsic volumes

Recall:

- hit-or-miss digitization, and $v$. threshold digitization:

There is a multigrid convergent digitization of $V_{j}$ on $\mathcal{K}_{\text {reg }}=\{K \in \mathcal{K} \mid K$ topological regular $\}$.

- cell covering digitization: There is a multigrid convergent digitization of $V_{j}$ on $\mathcal{K}$.

For the volume $(j=d)$, there is a multigrid convergent digitization on $\mathcal{R}=\{$ finite unions of convex bodies $\}$.

Question: Are there better digitizations for $j<d$ ?
For $d=2$, we have to consider $V_{0}$ (Euler-Poincaré characteristic) and $2 V_{1}$ (boundary length).

## Outline of the talk

Digitization of the Euler characteristic in the plane
Graph theoretic approach
Polygonal approximations
Discretization of continuous approaches

Digitization of boundary length in the plane
Common approaches
Configuration theory

Extensions to higher dimensions

## Neighbourhood graphs

In the following write $\chi=V_{0}=\#$ components $-\#$ holes.

Motivation:
-•••• . red points= $\hat{K}$

Define a neighbourhood graph on $\mathbb{L}$ :


## Adjacency graph

Choose a neighbourhood graph $N$ on $\mathbb{L}$.
Define a planar adjacency graph $G_{N}(\hat{K})$ with nodes $\hat{K}$ and all edges in the neighbourhood graph on $\mathbb{L}$ with endpoints in $\hat{K}$ (For the 8-neighbourhood omit all diagonals in cells).


For the 4-connected graph For the 6-connected and the 8 -connected graph
The number of components of $K$ is approximated by the \# components of $G_{N}(\hat{K})$.

## Adjacency graph for the complement

Idea: do the same for the complement $\mathbb{L} \backslash \hat{K}$ and get an approximation for \# components $\left(\mathbb{R}^{2} \backslash K\right)$.

Problem: Jordan's curve theorem in a digital setting?!


Foreground and background are both supplied with the 8 -neigbourhood.

Solution:
Provide the background with another neigbourhood graph.
The pairs $(4,8),(6,6)$ and $(8,4)$ are Jordan pairs, i.e. a digital version of Jordan's curve theorem holds.

## Graph theoretic Euler characteristic I

Digitization $\hat{\chi}$ of the Euler charcteristic:
Choose a Jordan pair $(N, \bar{N})$ and set

$$
\begin{aligned}
\hat{\chi}(\hat{K}) & :=\# \operatorname{components}\left(G_{N}(\hat{K})\right) \\
& -\# \operatorname{components}\left(G_{\bar{N}}(\mathbb{L} \backslash \hat{K})\right)+1 .
\end{aligned}
$$

Euler's relation $\Rightarrow$ \#components $\left(G_{N}(\hat{K})\right)=v-e+f-1$ with $v=\#$ vertices, $e=\#$ edges, $f=\#$ faces of $\hat{G}_{N}(\hat{K})$.
Hence $\hat{\chi}(\hat{K})=v-e+c$
with $c=\#$ cells of $\hat{G}_{N}(\hat{K})$.

## Graph theoretic Euler characteristic II

$$
\hat{\chi}(\hat{K})=v-e+c
$$

Count configurations:


Example: \#( $\left.\begin{array}{ll}\bullet & \cdot \\ \bullet & \bullet\end{array}\right)$

- 4-neighbourhood

$$
\begin{aligned}
& \hat{\chi}(\hat{K})=\#\left(\begin{array}{ll}
\bullet & \cdot \\
\cdot & \cdot
\end{array}\right)-\left(\#\left(\begin{array}{ll}
\bullet & \bullet \\
\cdot & \cdot
\end{array}\right)+\#\left(\begin{array}{ll}
\bullet & \cdot \\
\bullet & \cdot
\end{array}\right)\right)+\#\left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right)= \\
& \#\left(\begin{array}{ll}
\bullet & \cdot \\
\bullet & \bullet
\end{array}\right)-\#\left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right) .
\end{aligned}
$$

## Graph theoretic Euler characteristic III

$$
\hat{\chi}(\hat{K})=v-e+c
$$

- 4-neighbourhood

$$
\hat{\chi}_{4}(\hat{K})=\#\left(\begin{array}{ll}
\bullet & \cdot \\
\bullet & \bullet
\end{array}\right)-\#\left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right) .
$$

- 6-neighbourhood

$$
\hat{\chi}_{6}(\hat{K})=\ldots=\#\left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right)-\#\left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right) .
$$

- 8-neighbourhood

$$
\hat{\chi}_{8}(\hat{K})=\ldots=\#\left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right)-\#\left(\begin{array}{ll}
\bullet & \cdot \\
\bullet & \bullet
\end{array}\right) .
$$

## Definition of polygonal approximations I

Idea:

- Approximate $K$ by a polygon $P$ only depending on $\hat{K}$,
- condsider $\chi(P)$ as an approximation for $\chi(K)$.

Definition of $P$ :

- Cell covering $P_{8}$ :


$$
\hat{K}
$$



$$
\longrightarrow \quad P_{8}:=\bigcup_{x \in \hat{K}}\left(x+\mathrm{C}_{0}^{*}\right)
$$

## Definition of polygonal approximations II

- Square digitization $P_{4}$ :

Define "elementary polygons" $\mathcal{E}=\{\cdot,-, \mid, \square\}$ (and all their lattice translations).


- More genereal: define $P$ using other sets $\mathcal{E}$.


## Properties of the polygonal approach

- Additivity of $\chi \Rightarrow \chi\left(P_{8}\right)=\hat{\chi}_{8}(\hat{K}), \quad \chi\left(P_{4}\right)=\hat{\chi}_{4}(\hat{K})$, similar simple formulas for other polygonal approximations.
- Efficient implementations use e.g. quadtrees (Dyer [1980]), induction on dimension (Bieri \& Nef [1984]), use of variuos filter masks (Ohser \& Mücklich [2000]).


## Multigrid convergence I

$K$ may not have "too small structures".
Let $B$ be a compact set.

- Minkowski addition:

$$
K \oplus(-B)=\{x-b \mid x \in K, b \in B\}=\{x \mid(x+B) \cap K \neq \emptyset\}
$$ "dilation with $B$ ",

- Minkowski substraction: $K \ominus B=\{x \mid(x+B) \subset K\}$,
- morphological opening: $K \circ B:=(K \ominus(-B)) \oplus B$,
- morphological closing: $K \bullet B:=(K \oplus(-B)) \ominus B$.
$K$ is morphologically open and closed wrt. $B$ $\qquad$

$$
K=K \circ B=K \bullet B .
$$

## Multigrid convergence II

Let $\hat{K}$ be the hit-or-miss digitization of $K$.
$\mathcal{R}_{o c}(\varepsilon):=$ the class of all $K \in \mathcal{R}$, which are morphologically open and closed w.r.t. any set in $\varepsilon\{-, \mid, /, \backslash\}$.
("horizontal", "vertical" and "diagonals" of the fundamental cell).
Ohser \& Nagel [1996] $\Rightarrow$
Theorem.
The digital algorithms $\hat{\chi}_{4}$ and $\hat{\chi}_{6}$ are multigrid convergent to $\chi$ on the class $\mathcal{R}_{o c}(\varepsilon)$ for any $\varepsilon>0$.

## A variant of the polygonal approach I

The problem of isolated points:

Idea to estimate \# components( $K$ ):
Let $\hat{N}:=$ \#components of $P_{8}$
that contain four neighbouring cells.


## A variant of the polygonal approach II

Approximate \#holes(K):

- Interchange background and foreground, to obtain $\hat{N}_{c}$,
- approximate \#components $\left(\mathbb{R}^{2} \backslash K\right)$ by $\hat{N}_{c}-1$.

Theorem. (K. [2006])
The digital algorithm $\hat{N}-\left(\hat{N}_{c}-1\right)$ is multigrid convergent to $\chi$ on the class $\tilde{\mathcal{R}}$.
$\tilde{\mathcal{R}}=$ family of all $K \in \mathcal{R}$ with a representation $K=\bigcup_{i=1}^{n} K_{i}$,
$K_{i} \in \mathcal{K}$, such that

- for any $I \subset\{1, \ldots, n\}$ the set $\bigcap_{i \in I} K_{i}$ is empty or has interior points.
- $i \neq j \Rightarrow \partial K_{i} \cap \partial K_{j}$ is finite.


## Discretization of continuous approaches

- Discrete intrinsic volumes
(Bieri \& Nef [1984], Voss [1993]), ( $\left.\rightsquigarrow \hat{\chi}_{4}\right)$,
- Integral geometry on the lattice (cf. K. Mecke [1993]),
- Discretization of Hadwiger's definition of $\chi$ (Ohser, Nagel [1996]), ( $\left.\rightsquigarrow \hat{\chi}_{4}\right)$,
- Discretization of Schneider's index function
(Guderlei et al. [2007]).


## Digitization of $\chi$ in $\mathbb{R}^{2}$

## Summary:

- For the hit-or-miss digitization, good multigrid convergent discretizations of $\chi$ are known in $\mathbb{R}^{2}$,
- for other digitizations, known results are rather weak, "Serra's regular model" (Serra [1988])
- it is not known whether a randomization of $\mathbb{L}$ can improve properties of digitizations.


## Boundary length: common digitizations

Digitizations of the boundary length $L(K)=2 V_{1}(K)$.

- Local approaches

Essentially: use $L(P)$, where $P$ is a polygonal approximation
(the union of suitable translates of polygons in $\mathcal{E}$ ).
Not even multigrid convergent on $\mathcal{K}_{\text {reg }}$.

- Global approaches (cf. Klette \& Rosenfeld [2004])
- DSS based (digital straight line segment)
- Tangent based (estimate tangent vector)

Multigrid convergent at least on $\mathcal{K}_{\text {reg }}$.

## The length measure

Define a local counterpart of $L(K)=2 V_{1}(K)$ :
The length measure $L(K, \cdot)$ of $K \in \mathcal{R}$ is the image measure of $\mathcal{H}^{1}$ on $\partial K$ under the spherical image map.


## Configuration theory

General assumptions:
$\xi+\mathbb{Z}^{d}$ stationary random lattice ( $\xi$ uniform in the fundamental cell $\mathrm{C}_{0}^{*}=[0,1]^{d}$ ). $\tilde{K}=$ hit-or-miss digitization in $\xi+\mathbb{Z}^{d}$.

Motivation: Assume
$K=$ halfplane with outer unit normal $u$,
$\mathbf{C}=$ configuration,
e.g. $\mathbf{C}=\left(\begin{array}{ll}\bullet & \bullet \\ \bullet & \bullet\end{array}\right)$


## Asymptotic result for configuration counts I

General setting: $K \in \mathcal{R}_{\text {reg }}$ arbitrary.
Set $\# C_{t}:=\left\{x \in t\left(\xi+\mathbb{Z}^{d}\right) \mid x+t R \subset \tilde{K}_{t},(x+t B) \cap \tilde{K}_{t}=\emptyset\right\}$ (number of observed occurrencies of $t \mathbf{C}$ in $\tilde{K}_{t}$ ).

Theorem. (Jensen, K. [2003])

$$
t \mathbb{E} \# \mathbf{C}_{t} \longrightarrow \int_{S^{1}} h_{\mathbf{c}}(-u) d L(K, u), \quad t \rightarrow 0+
$$

Write $\mathbf{C}$ as $\mathbf{C}=(R, B)$ with ("red" and "black") $R, B \subset \mathbb{Z}^{d}$

$$
h_{\mathbf{c}}(-u)=\left(\min _{r \in R} r \cdot u-\max _{b \in B} b \cdot u\right)^{+},
$$

with $a^{+}=\max \{a, 0\}$.

## Asymptotic result for configuration counts II

$$
t \mathbb{E} \# \mathbf{C}_{t} \longrightarrow \int_{S^{1}} h_{\mathbf{C}}(-u) d L(K, u), \quad t \rightarrow 0+
$$

Far reaching generalizations (K. \& Rataj [2007])

- also holds for cell covering and v . threshold digitizations,
- extends to $\mathbb{R}^{d}, d \geqslant 2$,
- extends to much more general set classes containing $\mathcal{R}_{\text {reg }}$,
- is based on a local Steiner-type formula for closed sets (Hug et al. [2004]).

Observation: Asymptotically, we obtain directional information on the boundary.

## Excursion: $L(K, \cdot)$ estimation I

All $2 \times 2$-configurations that yield non-vanishing integrals:

$$
\begin{array}{|lll|}
\hline 1 & \left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right) & - \\
\hline 2 & \left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right) & \left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right) \\
\hline 3 & \left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right) & - \\
\hline 4 & \left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right) & \left(\begin{array}{ll}
\bullet & \bullet \\
\bullet & \bullet
\end{array}\right) \\
\hline
\end{array}
$$

| 5 | $\left(\begin{array}{ll}\bullet & \bullet \\ \bullet & \bullet\end{array}\right)$ | - |
| :--- | :--- | :--- | :--- |
| 6 | $\left(\begin{array}{lll}\bullet & \bullet \\ \bullet & \bullet\end{array}\right)$ | $\left(\begin{array}{lll}\bullet & \bullet \\ \bullet & \bullet\end{array}\right)$ |
| 7 | $\left(\begin{array}{lll}\bullet & \bullet \\ \bullet & \bullet\end{array}\right)$ | - |
| 8 | $\left(\begin{array}{lll}\bullet & \bullet \\ \bullet & \bullet\end{array}\right)$ | $\left(\begin{array}{lll}\bullet & \bullet \\ \bullet & \bullet\end{array}\right)$ |

## Excursion: $L(K, \cdot)$ estimation II

Application: For sufficientely small $t>0$ the counts

$$
\# \mathbf{C}_{t} \approx t^{-1} \int_{S^{1}} h_{\mathbf{C}}(-u) d L(K, u)
$$

lead to (estimates of) 8 different integrals of $L(K, \cdot)$.
Model: $\hat{L}(K, \cdot)=\sum_{i=1}^{8} \alpha_{i} \delta_{u_{i}}$ with $\alpha_{1}, \ldots, \alpha_{8} \geqslant 0$.


Approach: Determine $\alpha_{1}, \ldots, \alpha_{8} \geqslant 0$ in such a way that $t \int_{S^{1}} h_{\mathbf{C}}(-u) d \hat{L}(K, u)$ is "close to" $\# \mathbf{C}_{t}$.

## Application example: Rolled Steel



The digital image of a rolled steel (black phase = K).


The estimated masses of $L(K, \cdot)$ from
$2 \times 2$-configurations.

The total mass of this estimator also yields an estimator for $L(K)=2 V_{1}(K)$.
(cf. Jensen \& K. [2003], K. \& Jensen [2003])

## Digitization of the total projection

$$
t \mathbb{E} \# \mathbf{C}_{t} \longrightarrow \int_{S^{1}} h_{\mathbf{C}}(-u) d L(K, u), \quad t \rightarrow 0+.
$$

$\mathbf{C}_{t}=(R, B)=: \mathbf{C}_{t}(x)$ with singletons $R=\{x\}, B:=\{0\}$, $x \in t \mathbb{Z}^{d}$.

$$
t \mathbb{E} \# \mathbf{C}_{t}(x) \rightarrow \int_{\mathcal{S}^{1}}(x \cdot u)^{+} L(K, d u)
$$

"total projection of $K$ in direction $x /\|x\|$ "
Application in $\mathbb{R}^{2}$ :
Choose $x_{1}, \ldots, x_{k} \in \mathbb{Z}^{2}$ and weights $\alpha\left(x_{i}\right) \geqslant 0, i=1, \ldots, k$, and consider

$$
\hat{L}_{t}:=t \sum_{i=1}^{k} \alpha\left(x_{i}\right)\left\|x_{i}\right\|^{-1} \# \mathbf{C}_{t}\left(x_{i}\right) .
$$

## Digitization of boundary length I

$$
\mathbb{E} \hat{L}_{t} \rightarrow \int_{S^{1}} \underbrace{\sum_{i=1}^{k} \alpha\left(x_{i}\right)\left(\frac{x_{i}}{\left\|x_{i}\right\|} \cdot u\right)^{+}}_{\text {should } \approx 1} L(K, d u), \quad t \rightarrow 0+.
$$

Common choice of the weights $\alpha\left(x_{1}\right), \ldots, \alpha\left(x_{k}\right)$ :

$\alpha\left(x_{i}\right):=\left(\right.$ length of the spherical Voronoi cell of $\left.x_{i} /\left\|x_{i}\right\|\right) / 2$.

## Digitization of boundary length II

Refinement: Choose $\rho_{t}>0$ with $\rho_{t} \rightarrow \infty$ and $t \rho_{t} \rightarrow 0$, as $t \rightarrow 0$.

$$
\hat{L}_{t}:=t \sum_{x \in \mathbb{L},\|x\| \leqslant \rho_{t}} \alpha(x)\|x\|^{-1} \# \mathbf{C}_{t}(x) .
$$

Theorem.
If $K \in \mathcal{R}_{r e g}$ is digitized with the randomized hit-or-miss, cell covering or volume-threshold digitization $\tilde{K}_{t}$, then

$$
\lim _{t \rightarrow 0+} \mathbb{E} \hat{L}_{t}=L(K)
$$

i.e. $\hat{L}$ is multigrid convergent in mean to $L$ on $\mathcal{R}_{\text {reg }}$.

Note: $\hat{L}_{t}$ is not local!

## Extensions to higher dimensions

- Discretization of Hadwiger's definition of $\chi$ (Nagel et al. [2000])
- Adjacency systems as generalization of graph theoretic and polygonal approach.
- Discretization of Crofton's formula (cf. Ohser \& Mücklich [2000])
- Configuration theory in $\mathbb{R}^{d}$ to treat surface area. ( $d=3$ : Gutkowski et al. [2004])

