THREE LECTURES ON GEOMETRIC MEASURE THEORY

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ABSTRACT. The aim of the series of three lectures is to give an introduction and overview of some basic results of geometric measure theory which can be applied in stochastic geometry.

The first lecture will start with the definition of the Hasdorff measures in \mathbb{R}^d , Hausdorff dimension and (Hausdorff) k-dimensional density of a set. Then, a survey of basic properties of Lipschitz mappings and the area and coarea theorems will be presented. Finally, Hausdorff rectifiable sets will be introduced and a general area-coarea formula presented, together with examples.

The second lectures will be devoted to currents. First, the basic multilinear algebra (multivectors and their wedge products) will be introduced. Then, the general definition of a current will follow, with emphasis to rectifiable currents, together with certain forms of convergence of currents and a few chosen general results, accompanied by illustrating examples.

The aim of the third lecture is to introduce curvature measures for general convex bodies and sets with positive reach. This will be done by means of the unit normal bundle and the associated normal cycle (current). Some integralgeometric formula will be presented and possible extensions to more general sets will be discussed.

1. Area, coarea and rectifiability

1.1. Hausdorff measure. The Hausdorff measures are defined as outer measure on \mathbb{R}^d . An outer measure on a set X is a set function $\mu: \mathcal{P}(X) \to [0,\infty]$ defined on all subsets of X with the properties:

- (i) $\mu(\emptyset) = 0$,
- (ii) $A \subseteq B \Longrightarrow \mu(A) \le \mu(B),$ (iii) $\mu(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu(A_i).$

To an outer measure μ , the system of μ -measurable sets is assigned:

$$\mathcal{A}_{\mu} := \{ A \subseteq X : \, \mu(T) = \mu(T \cap A) + \mu(T \setminus A) \, \forall T \subseteq X \}.$$

Theorem 1.1. \mathcal{A}_{μ} is a σ -algebra and the restriction of μ to \mathcal{A}_{μ} is a (σ -additive) measure.

There is a plenty of outer measures whose σ -algebra is very poor or even trivial. There exists, however, a simple criterion assuring that the σ -algebra is rich enough. We say that an outer measure μ on a metric space (X, ρ) is metric if $\mu(A \cup B) =$ $\mu(A) + \mu(B)$ whenever

dist
$$(A, B) := \inf \{ \rho(a, b) : a \in A, b \in B \} > 0.$$

Theorem 1.2. If μ is a metric outer measure on a metric space (X, ρ) then \mathcal{A}_{μ} contains all Borel sets. Moreover, μ is Borel regular, i.e., to any $A \subseteq X$ there exists a Borel set $B \supseteq A$ such that $\mu(A) = \mu(B)$.

Let $s \ge 0$. Denote

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$$

(note that if s is an integer then ω_s is the volume of the unit ball in \mathbb{R}^s). We define the s-dimensional Hausdorff measure in \mathbb{R}^d as

$$\mathcal{H}^{s}(A) = \lim_{\delta \to 0_{+}} \inf_{\substack{A \subseteq \bigcup_{i \in G_{i} \\ \text{diam } G_{i} \leq \delta}}} \sum_{i} \omega_{s} \left(\frac{\operatorname{diam} G_{i}}{2}\right)^{s}.$$

The infimum above is taken over all finite or countable coverings of A with (arbitrary) subsets G_1, G_2, \ldots of \mathbb{R}^d of diameters at most δ .

Proposition 1.3. (1) \mathcal{H}^s is a metric outer measure on \mathbb{R}^d for any $0 \le s \le d$. (Hence, \mathcal{H}^s is Borel regular.)

- (2) The measures \mathcal{H}^s are translation and rotation invariant.
- (3) \mathcal{H}^0 is the counting measure.
- (4) \mathcal{H}^d is the Lebesgue measure $(\mathcal{H}^d = \lambda^d)$.
- (5) $\mathcal{H}^s = 0$ if s > d.

Note that the definition of \mathcal{H}^s would not change if coverings by only say open, or closed, or even compact convex, sets would be considered. Covering by balls would, however, produce another measure (though its values on "nice" sets would be the same).

Let A be a subset of \mathbb{R}^d . The Hausdorff dimension of A is defined as

$$\dim_H A := \inf\{s \ge 0 : \mathcal{H}^s(A) < \infty\}.$$

The Hausdorff dimension has the following meaning.

Proposition 1.4. If $s < \dim_H A$ then $\mathcal{H}^s(A) = \infty$. If $s > \dim_H A$ then $\mathcal{H}^s(A) = 0$.

Examples: Any nonempty open set in \mathbb{R}^d has Hausdorff dimension d. A k-dimensional C^1 -submanifold has Hausdorff dimension k. Any countable set has Hausdorff dimension 0. The Cantor set in \mathbb{R}^1 has Hausdorff dimension $\frac{\log 2}{\log 3}$. The trajectory of Brownian motion in \mathbb{R}^d has almost surely Hausdorff dimension 2 (nevertheless, its two-dimensional Hausdorff measure vanishes).

1.2. **Densities of sets.** Given $a \in \mathbb{R}^d$ and r > 0, let B(a, r) denote the closed ball with centre a and radius r. Let A be a subset of \mathbb{R}^d and $a \in \mathbb{R}^d$ a point. Let s > 0. Define

$$\begin{split} \Theta^{*s}(A,a) &= \limsup_{r \to 0_+} \frac{\mathcal{H}^s(A \cap B(a,r))}{\omega_s r^s}, \\ \Theta^s_*(A,a) &= \limsup_{r \to 0_+} \frac{\mathcal{H}^s(A \cap B(a,r))}{\omega_s r^s}, \end{split}$$

the upper and lower s-dimensional density of A at a. If both the upper and lower densities agree we call the common value s-dimensional density of A at a and denote it by $\Theta^{s}(A, a)$.

- **Theorem 1.5.** (1) If $A \subseteq \mathbb{R}^d$ is Lebesgue measurable then $\Theta^d(A, \cdot)$ equals 1 λ^d -almost everywhere on A and equals 0 λ^d -almost everywhere on the complement of A (Lebesgue Density Theorem).
 - (2) If $\mathcal{H}^{s}(A) < \infty$ then $\Theta^{*s}(A, \cdot) \leq 1 \mathcal{H}^{s}$ -almost everywhere on A.

1.3. Lipschitz mappings. A mapping $f : A \to \mathbb{R}^n$ defined on a set $A \subseteq \mathbb{R}^d$ is Lipschitz if there exists a number $L \ge 0$ such that

 $|f(y) - f(x)| \le L|y - x| \quad \text{for all } x, y \in A.$

The infimum of all constants L with the above property as called the *Lipschitz* constant of f and denoted L = Lip f. Of course, any Lipschitz mapping is continuous, but not vice versa. The following result says that we can mostly work with Lipschitz mappings defined on the whole space.

Theorem 1.6 (Kirszbraun). Any Lipschitz mapping from a subset of \mathbb{R}^d to \mathbb{R}^n can be extended to a Lipschitz mapping defined on the whole \mathbb{R}^d , with the same Lipschitz constant.

Lipschitz mapping are used in geometric measure theory in place of C^1 smooth mappings from the classical calculus. The following two results make this possible.

Theorem 1.7 (Rademacher). A Lipschitz mapping $f : \mathbb{R}^d \to \mathbb{R}^n$ is differentiable λ^d -almost everywhere.

Theorem 1.8 (Whitney). Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be Lipschitz and let $\varepsilon > 0$. Then there exists a C^1 mapping $g : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$\lambda^d \{ x \in \mathbb{R}^d : f(x) \neq g(x) \} < \varepsilon.$$

The basic connection of Hausdorff measure and Lipschitz mappings is given in the following simple proposition.

Proposition 1.9. If $A \subseteq \mathbb{R}^d$ and $f : A \to \mathbb{R}^n$ is Lipschitz then $\mathcal{H}^s(f(A)) \leq (\operatorname{Lip} f)^s \mathcal{H}^s(A), s \geq 0.$

Let a function f with values in \mathbb{R}^n be differentiable at a point $a \in \mathbb{R}^d$ (with differential Df(a)) and let $0 \leq k \leq d$ be an integer. The *k*-dimensional Jacobian of f at a is defined as

 $J_k f(a) = \sup \{ \mathcal{H}^k(Df(a)(C)) : C \text{ is a } k \text{-dimensional unit cube in } \mathbb{R}^d \}.$

Particular cases:

(1) If k = d = n then $J_d f(a) = |\det Df(a)|$.

- (2) If k = d < n then $J_d f(a) = \sqrt{\det(Df(a))^{\mathsf{T}}(Df(a))}$. Moreover, $J_d f(a) = \mathcal{H}^d(Df(a)(C))$ for any unit cube in \mathbb{R}^d , or $J_d f(a) = \mathcal{H}^d(Df(a)(A))/\lambda^d(A)$ for any measurable subset $A \subseteq \mathbb{R}^d$ of positive finite Lebesgue measure.
- (3) If k = n < d then $J_n f(a) = \sqrt{\det(Df(a))(Df(a))^{\mathsf{T}}}$. If the rank of Df(a) is less than n then $J_n f(a) = 0$. If the rank of Df(a) equals n then $J_n f(a) = \mathcal{H}^n(Df(a)(C))$ for any unit n-cube in $(\ker Df(a))^{\perp}$, the orthogonal complement of the kernel of Df(a) $(\ker Df(a) = \{u \in \mathbb{R}^d : Df(a)u = o\}$, or $J_n f(a) = \mathcal{H}^n(Df(a)(A))/\mathcal{H}^n(A)$ for any measurable subset $A \subseteq (\ker Df(a))^{\perp}$ of positive finite n-dimensional Hausdorff measure.

Theorem 1.10 (Area formula). Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be Lipschitz, $d \leq n$, and let $A \subseteq \mathbb{R}^d$ be Lebesgue measurable. Then

$$\int_{A} J_d f \, d\lambda^d = \int_{\mathbb{R}^n} \operatorname{card}(A \cap f^{-1}\{z\}) \, \mathcal{H}^d(dz).$$

If, moreover, $h \ge 0$ is a λ^d -measurable function on A then

$$\int_{A} h(x) J_d f(x) \lambda^d(dx) = \int_{\mathbb{R}^n} \sum_{x \in A \cap f^{-1}\{z\}} h(x) \mathcal{H}^d(dz).$$

Theorem 1.11 (Coarea formula). Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be Lipschitz, $d \ge n$, and let $A \subseteq \mathbb{R}^d$ be Lebesgue measurable. Then

$$\int_{A} J_n f \, d\lambda^d = \int_{\mathbb{R}^n} \mathcal{H}^{d-n}(A \cap f^{-1}\{z\}) \, \mathcal{H}^n(dz).$$

If, moreover, $h \ge 0$ is a λ^d -measurable function on A then

$$\int_A h(x) J_n f(x) \lambda^d(dx) = \int_{\mathbb{R}^n} \int_{A \cap f^{-1}\{z\}} h(x) \mathcal{H}^{d-n}(dx) \mathcal{H}^n(dz).$$

1.4. Tangent cones. If $A \subseteq \mathbb{R}^d$ and $a \in \mathbb{R}^d$, the tangent cone of A at a is defined as

$$\operatorname{Tan}(A,a) = \{ u \in \mathbb{R}^d : \exists (a_n) \subseteq A \setminus \{a\}, (r_n) \subseteq (0,\infty), a_n \to a, r_n(a_n-a) \to u \}.$$

Another description is that a nonzero vector u belongs to $\operatorname{Tan}(A, a)$ if and only if there exist points $a_n \neq a$ from A such that $\frac{a_n-a}{|a_n-a|} \rightarrow \frac{u}{|u|}$. Tan(A, a) is always a closed cone with vertex at the origin.

Given an integer $0 \le k \le d$, the cone of (\mathcal{H}^k, k) -approximate tangent vectors of A at a is defined as

$$\operatorname{Tan}^{k}(A,a) = \bigcap \{ \operatorname{Tan}(E,a) : E \subseteq A, \Theta^{k}(A \setminus E, a) = 0 \}.$$

Clearly, $\operatorname{Tan}^{k}(A, a)$ is a closed subcone of $\operatorname{Tan}(A, a)$.

1.5. **Approximate differential.** A function $f : A \to \mathbb{R}^n$ $(A \subseteq \mathbb{R}^d)$ is said to be (\mathcal{H}^k, k) -approximatively differentiable at $a \in A$ if there exists a mapping $g : \mathbb{R}^d \to \mathbb{R}^n$ differentiable at a and such that

$$\Theta^k(\{x \in A : f(x) \neq g(x)\}, a) = 0.$$

The mapping

$$(\mathcal{H}^k, k)$$
ap $Df(a) := Dg(a) \mid \operatorname{Tan}^k(A, a)$

(restriction of Dg(a) to $\operatorname{Tan}^{k}(A, a)$) is called the (\mathcal{H}^{k}, k) -approximate differential of f at a. We often write only ap Df(a) for brevity.

It can be shown that ap Df(a) does not depend on the choice of the function g.

Assume that f is (\mathcal{H}^k, k) -approximatively differentiable at $a \in A$ and that $\operatorname{Tan}^k(A, a)$ is a k-dimensional subspace of \mathbb{R}^d . For an integer $0 \leq m \leq k$, we define the *m*-dimensional approximate Jacobian of f at a as

$$\operatorname{ap} J_m f(a) = \sup \{ \mathcal{H}^m(\operatorname{ap} Df(a)(C)) : C \text{ is a unit } m \text{-cube in } \operatorname{Tan}^k(A, a) \}.$$

1.6. Rectifiable sets. Let $k \in [0,d]$ be an integer. A set $A \subseteq \mathbb{R}^d$ is called krectifiable if it is a Lipschitz image of a bounded subset of \mathbb{R}^k . A is called (\mathcal{H}^k, k) rectifiable if $\mathcal{H}^k(A) < \infty$ and there exist k-rectifiable sets W_1, W_2, \ldots such that $\mathcal{H}^k(A \setminus \bigcup_i W_i) = 0$ (see [2]). Finally, we call A k-dimensional rectifiable if it is (\mathcal{H}^k, k) -rectifiable and \mathcal{H}^k -measurable. (The last terminology is taken from Morgan [6]; it should be noticed that the (Hausdorff) dimension of A can be less than k.) Examples:

- (1) A k-dimensional C^1 -submanifold M of \mathbb{R}^d is locally k-rectifiable (i.e., to any $a \in M$ there exists a neighbourhood U of a in \mathbb{R}^d such that $M \cap U$ is k-rectifiable).
- (2) The graph of a Lipschitz function from a bounded subset of \mathbb{R}^{d-1} to \mathbb{R} is (d-1)-rectifiable.
- (3) If $A \subseteq \mathbb{R}^d$ is k-dimensional rectifiable and $f: A \to \mathbb{R}^n$ Lipschitz then f(A)is k-dimensional rectifiable as well.

Theorem 1.12 (Federer). Let $A \subseteq \mathbb{R}^d$ be k-dimensional recitiable and $\gamma > 1$. Then

(1) there exist C^1 -diffeomorphisms g_1, g_2, \ldots from \mathbb{R}^k to \mathbb{R}^d with $\operatorname{Lip} g_i \leq \gamma$ and Lip $g_i^{-1} \leq \gamma$ and compact subsets K_1, K_2, \ldots of \mathbb{R}^d such that the images $g_i(K_i) \cap g_j(K_j) = \emptyset$ for $i \neq j$ and

$$\mathcal{H}^k\left(A\setminus \bigcup_i g_i(K_i)\right)=0;$$

(2) for \mathcal{H}^k -almost all $a \in A$, $\Theta^k(A, a) = 1$ and $\operatorname{Tan}^k(A, a)$ is a k-dimensional subspace of \mathbb{R}^d .

Proposition 1.13. Let $A \subseteq \mathbb{R}^d$ be k-dimensional recitiable and $f : A \to \mathbb{R}^n$ Lipschitz. Then for \mathcal{H}^k -almost all $a \in A$, $\operatorname{Tan}^k(A, a)$ is a k-dimensional subspace and f is (\mathcal{H}^k, k) -approximatly differentiable at a.

Theorem 1.14 (General Area-coarea Formula). Let $A \subseteq \mathbb{R}^d$ be k-dimensional rectifiable and $Z \subseteq \mathbb{R}^n$ m-dimensional recifiable, $k \geq m$, and let $f : A \to Z$ be Lipschitz. Then

- (1) $f^{-1}\{z\}$ is (k-m)-dimensional rectifiable for \mathcal{H}^m -almost all $z \in Z$, (2) $\int_A \operatorname{ap} J_m f \, d\mathcal{H}^k = \int_Z \mathcal{H}^{k-m}(f^{-1}\{z\}) \, \mathcal{H}^m(dz)$,
- (3) for any nonegative \mathcal{H}^k -measurable function h on A,

$$\int_A \operatorname{ap} J_m f(x) h(x) \,\mathcal{H}^k(dx) = \int_Z \int_{f^{-1}\{z\}} h(x) \,\mathcal{H}^{k-m}(dx) \,\mathcal{H}^m(dz).$$

Example 1.1. Given two subspaces L_p, L_q of \mathbb{R}^d , of dimension p, q, respectively, we define

$$J(L_p, L_q) = J_r(p_{L_q} \mid L_p),$$

the r-dimensional Jacobian of the orthogonal projection to L_q defined on L_p , where $r = \min\{p, q\}.$

Let $A \subseteq \mathbb{R}^d$ be k-dimensional rectifiable, let L be a j-dimensional subspace of \mathbb{R}^d and set $r = \min\{j,k\}$. If $f = p_L \mid A : A \to L$ is the orthogonal projection p_L from \mathbb{R}^d to L restricted to A, then

$$J_r f(a) = J(\operatorname{Tan}(A, a), L)$$

for \mathcal{H}^k -almost all $a \in A$. The area-coarea theorem thus yields

$$\int_{A} J(\operatorname{Tan}(A, a), L) \,\mathcal{H}^{k}(da) = \int_{L} \mathcal{H}^{k-j}(A \cap p_{L}^{-1}\{z\}) \,\mathcal{H}^{j}(dz)$$

if $k \geq j$ and

$$\int_{A} J(\operatorname{Tan}(A, a), L) \,\mathcal{H}^{k}(da) = \int_{p_{L}(A)} \mathcal{H}^{j-k}(A \cap p_{L}^{-1}\{z\}) \,\mathcal{H}^{k}(dz)$$

if $k \leq j$. Assume now that k = j; integrating the last formula with respect to the invariant probability measure ν_k^d over the Grassmannian G(d, k) all k-subspaces, we get the Crofton formula

$$\mathcal{H}^{k}(A) = c(d,k) \int_{G(d,k)} \int_{L} \operatorname{card}(A \cap p_{L}^{-1}\{z\}) \mathcal{H}^{k}(dz) \nu_{k}^{d}(dL)$$

with

$$c(d,k) = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{d-k+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

1.7. **Purely unrectifiable sets.** A set $E \subseteq \mathbb{R}^d$ is called *purely k-unrectifiable* if it contains no k-rectifiable subset of positive \mathcal{H}^k -measure.

Proposition 1.15. Any \mathcal{H}^k -measurable set $W \subseteq \mathbb{R}^d$ with $\mathcal{H}^k(W) < \infty$ can be written as disjoint union $W = A \cup E$ of a k-dimensional rectifiable set A and purely k-unrectifiable set E.

Theorem 1.16 (Structure Theorem). If E is purely k-unrectifiable then $\mathcal{H}^k(p_L(E)) = 0$ for almost all k-subspaces E of \mathbb{R}^d . Consequently, the Crofton formula fails for purely unrectifiable sets with positive \mathcal{H}^k -measure.

Exercises:

- (1) Compute $\Theta^{*1}(A, 0)$ and $\Theta^{1}_{*}(A, 0)$ for the set $A = [\frac{1}{2}, 1] \cup [\frac{1}{8}, \frac{1}{4}] \cup [\frac{1}{32}, \frac{1}{16}] \cup \cdots$ in \mathbb{R} .
- (2) Show that the distance function $d_A : x \mapsto \text{dist}(x, A)$ to a nonempty set is Lipschitz, find $\text{Lip} d_A$, and compute $J_1 d_A(x)$ for $x \notin \text{cl} A$.
- (3) Prove Proposition 1.9.
- (4) Calculate $J_1 f(x)$ for the mapping $f : x \mapsto |x|$ defined in \mathbb{R}^d and use the coarea formula to show that for any cone C with vertex at the origin and any R > 0,

$$\lambda^d(C \cap B(o, R)) = \frac{R^d}{d} \mathcal{H}^{d-1}(C \cap S^{d-1}).$$

- (5) Let *L* be a *j*-subspace of \mathbb{R}^d and let $\pi_L : \mathbb{R}^d \setminus L^{\perp} \to L \cap S^{d-1}$ be defined by $\pi_L(x) = \frac{p_L x}{|p_L(x)|}$ (p_L is the orthogonal projection to *L*). Calculate $J_{j-1}\pi_L(x)$.
- (6) Find $\operatorname{Tan}(A, a)$, $\operatorname{Tan}^2(A, a)$ and $\operatorname{Tan}^1(A, a)$ for the planar set

$$A = \{x \le 0, y \le 0\} \cup \{x = 0\} \cup \{(\frac{1}{n}, 0), n \in \mathbb{N}\}.$$

- (7) Show that the approximate differential ap Df(x) is uniquely determined.
- (8) Show that the boundary of a convex body in \mathbb{R}^d is (d-1)-rectifiable.
- (9) (*) If $\mathcal{H}^k(A) = 0$ then $\mathcal{H}^{k+1}(A \times [0,1]) = 0$.

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- (10) If A is k-rectifiable then $A \times [0, 1]$ is (k+1)-rectifiable. If A is k-dimensional rectifiable then $A \times [0, 1]$ is (k + 1)-dimensional rectifiable.
- (11) Let $A \subseteq \{(x_1, \ldots, x_d \in \mathbb{R}^d : x_d = 1\}$ be k-dimensional rectifiable. Show that $W = \{ta : a \in A, 0 \le t \le 1\}$ is (k+1)-dimensional rectifiable, and compute $H^{k+1}(W)$.

2. Differential forms and currents

2.1. Multilinear algebra. Let V be a finite (d-)dimensional vector space over \mathbb{R} (usually $V = \mathbb{R}^d$). If $k \ge 0$ is an integer, we denote by $\bigotimes^k V$ the linear space of all k-linear functions $f: V^k \to \mathbb{R}$. Elements of $\bigotimes^k V$ are called *covariant k-tensors* and dim $\bigotimes^k V = d^k$. We say that a covariant k-tensor f is antisymmetric if for any permutation $\sigma \in \Sigma(k)$ of $\{1, \ldots, k\}$ and for all vectors $v_1, \ldots, v_k \in V$,

$$f(v_{\sigma(1)},\ldots,v_{\sigma(k)}) = (\operatorname{sgn} \sigma)f(v_1,\ldots,v_k).$$

The set of all antisymmetric covariant k-tensors will be denoted by $\bigwedge^k V$. It is a linear subspace of $\bigotimes^k V$. Elements of $\bigwedge^k V$ are called *k*-covectors or multi-covectors. It follows from the antisymmetry that $f(v_1, \ldots, v_k) = 0$ whenever the vectors v_1, \ldots, v_k are linearly dependent. Thus, if k > d then $\bigwedge^k V$ is trivial. Given two multi-covectors $f \in \bigwedge^k V, g \in \bigwedge^m V$, we define their *exterior* (or *wedge*) *product* bv

$$(f \wedge g)(v_1, \dots, v_{k+m}) = \sum_{\sigma \in \Sigma(k,m)} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+m)}),$$

where

$$\Sigma(k,m) = \{ \sigma \in \Sigma(k+m) : \sigma(1) < \dots < \sigma(k), \sigma(k+1) < \dots < \sigma(k+m) \}.$$

It is easy to verify that $f \wedge g \in \bigwedge^{k+m} V$. k-covectors of the form $v_1^* \wedge \cdots \wedge v_k^*$, where $v_1^*, \ldots, v_k^* \in V^* \cong \bigwedge^1 V$, are called *simple*. Any k-covector can be written as a linear combination of simple k-covectors.

The dual space to $\bigwedge^k V$ will denoted by $\bigwedge_k V$ and its elements called *k*-vectors or multivectors. *k*-vectors correspond naturally to *k*-covectors of the dual space, thus $\bigwedge_k V \cong \bigwedge^k V^*$. Hence, the exterior product is defined on multivectors as well. To any vectors v_1, \ldots, v_k from V, a k-vector can be naturally assigned:

$$F_{v_1,\ldots,v_k}: f \mapsto f(v_1,\ldots,v_k), \quad f \in \bigwedge^{\kappa} V.$$

The mapping $(v_1, \ldots, v_k) \mapsto F_{v_1, \ldots, v_k}$ is k-linear and antisymmetric. We shall denote the k-vector F_{v_1,\ldots,v_k} simply by $v_1 \wedge \cdots \wedge v_k$ and we remark that this notation is consistent with the definition of wedge product. k-vectors of this type are again called *simple*.

Note that if $\{e_1, \ldots, e_d\}$ is a basis of V then

$$\{e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \le i_1 < \dots < i_k \le d\}$$

is a basis of $\bigwedge_k V$ and

$$\{e_{i_1}^* \land \dots \land e_{i_k}^* : 1 \le i_1 < \dots < i_k \le d\}$$

is a basis of $\bigwedge^k V$, where e_i^* are the dual forms to e_i . These linear forms are often denoted by $dx_i = e_i^*$. It follows that the dimension of $\bigwedge_k V$ and $\bigwedge^k V$ is $\binom{d}{k}$.

(1) If $u = \alpha_1 e_1 + \dots + \alpha_d e_d$ and $v = \beta_1 e_1 + \dots + \beta_d e_d$ then Example 2.1.

$$u \wedge v = \sum_{i < j} (\alpha_i \beta_j - \alpha_j \beta_i) (e_i \wedge e_j).$$

(2) If
$$u_1, \ldots, u_d \in V$$
, $u_i = \sum_{j=1}^d u_i^j e_j$, then

$$u_1 \wedge \cdots \wedge u_d = \det \left(u_i^j \right)_{i,j=1}^d (e_1 \wedge \cdots \wedge e_d).$$

(3) There exists multivectos which are not simple if dim $V \ge 4$, e.g., $e_1 \wedge e_2 + e_2 + e_3 \wedge e_4$ $e_3 \wedge e_4$ is not simple.

If $\phi \in \bigwedge^k V$ is a k-covector and $\xi \in \bigwedge_k V$ a k-vector, we shall write $\langle \xi, \phi \rangle$ instead of $\phi(\xi)$ or $\xi(\phi)$ is the sequel.

The operations of *interior multiplication* are defined as follows. Let $\xi \in \bigwedge_k V$ and $\phi \in \bigwedge^m V$.

- (1) If $k \leq m$ we define $\xi \lrcorner \phi \in \bigwedge^{m-k} V$ by $\langle \alpha, \xi \lrcorner \phi \rangle = \langle \alpha \land \xi, \phi \rangle, \ \alpha \in \bigwedge_{m-k} V$. (2) If $k \geq m$ we define $\xi \lrcorner \phi \in \bigwedge_{k-m} V$ by $\langle \xi \lrcorner \phi, \psi \rangle = \langle \xi, \phi \land \psi \rangle, \ \psi \in \bigwedge^{k-m} V$.

Let now $V = \mathbb{R}^d$ and let $\{e_1, \ldots, e_d\}$ be its canonical basis. We define a scalar product on $\bigwedge_k \mathbb{R}^d$ as follows. Assume that $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ and set

$$(e_{i_1} \wedge \dots \wedge e_{i_k}) \cdot (e_{j_1} \wedge \dots \wedge e_{j_k}) = \begin{cases} 1 & \text{if } i_1 = j_1, \dots i_k = j_k, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding norm on $\bigwedge_k \mathbb{R}^d$ will be denoted by $|\cdot|$. On the dual space $\bigwedge^k \mathbb{R}^d$, besides of the dual norm $|\cdot|$, we define another norm (called *comass*)

$$\|\varphi\| = \sup\{|\langle \xi, \varphi \rangle : \xi \in \bigwedge_k \mathbb{R}^d \text{ simple }, |\xi| = 1\}, \quad \varphi \in \bigwedge^k \mathbb{R}^d.$$

Example 2.2. We have

$$(u_1 \wedge \dots \wedge u_k) \cdot (v_1 \wedge \dots \wedge v_k) = \det (u_i \cdot v_j)_{i,j=1}^k$$

If $L: V \to W$ is a linear mapping between two finite-dimensional vector spaces over \mathbb{R} and if $k \leq \max\{\dim V, \dim W\}$, we define the linear mapping

$$\bigwedge_k L : \bigwedge_k V \to \bigwedge_k W$$

by

$$(\bigwedge_k L) (v_1 \wedge \dots \wedge v_k) = (Lv_1) \wedge \dots \wedge (Lv_k)$$

2.2. The Grassmannian. Let G(d, k) denote the set of all k-dimensional linear subspaces of \mathbb{R}^d . With any k-vector $\xi \in \bigwedge_k \mathbb{R}^d$ we associate the subspace

$$L(\xi) = \{ u \in \mathbb{R}^d : \xi \land u = o \}.$$

(1) $o \neq \xi \in \bigwedge_k \mathbb{R}^d$ is simple if and only if dim $L(\xi) = k$. Proposition 2.1. (2) If ξ, ζ are two simple k-vectors with $L(\xi) = L(\zeta)$ then $\xi = c\zeta$ for some $c \in \mathbb{R}$.

Consequently, we can represent G(d, k) as the submanifold

$$\{\xi \in \bigwedge_k \mathbb{R}^d \text{ simple}, |\xi| = 1\}$$

of $\bigwedge_k \mathbb{R}^d$ modulo the change of sign. We have dim G(d,k) = k(d-k).

2.3. Vectorfields and differential forms. Let U be an open subset of \mathbb{R}^d . A *k*-vectorfield on U is a mapping $\xi : U \to \bigwedge_k \mathbb{R}^d$. A differential form of power k (or briefly a *k*-form) on U is a C^{∞} differentiable mapping $\phi : U \to \bigwedge^k \mathbb{R}^d$.

Example 2.3. If f_1, \dots, f_d are C^{∞} -functions then

$$\phi(x) = f_1(x)dx_1 + \dots + f_d(x)dx_n$$

is a 1-form on \mathbb{R}^d . A k-form can always be expressed in the form

$$\phi(x) = \sum_{1 \le i_1 < \dots < i_k \le d} \phi_{i_1,\dots,i_k}(x) (dx_{i_1} \land \dots \land dx_{i_k})$$

with C^{∞} functions ϕ_{i_1,\ldots,i_k} (called coefficients of ϕ).

The *pull-back* of a *k*-form ϕ on $U \subseteq \mathbb{R}^d$ with a differentiable mapping $f : G \subseteq \mathbb{R}^n \to U$ is a *k*-form on *G* denoted by $f^{\#}\phi$ and defined by

$$\langle \xi, f^{\#}\phi(x) \rangle = \langle (\bigwedge_k Df(x))\xi, \phi(f(x)) \rangle$$

The exterior derivative $d\phi$ od a k-form ϕ on U is a (k + 1)-form on U defined by

$$\langle v_1 \wedge \dots \wedge v_{k+1}, d\phi(x) \rangle = \sum_{i=1}^{k+1} (-1)^{i-1} \langle v^{(i)}, D\phi(x)v_i \rangle$$

where $v^{(i)} = v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_{k+1}$ and $D\phi$ is the k-form on U obtained by differentiating the coefficients of ϕ . Using coordinates, one can write

$$d\phi(x) = \sum_{i=1}^{d} dx_i \wedge \frac{\partial}{\partial x_i} \phi(x).$$

The differential operator d is additive and has the property $d^2 = 0$.

2.4. **Currents.** Let U be an open set in \mathbb{R}^d , let $\mathcal{E}^k(U)$ denote the linear space of all k-forms on U and let $\mathcal{D}^k(U)$ be the subspace of k-forms with compact support. The dual space to $\mathcal{D}^k(U)$ is denoted by $\mathcal{D}_k(U)$ and its elements are called k-dimensional currents on U. The support of T is defined as

$$\operatorname{spt} T = U \setminus \bigcup \{ V \subseteq U \text{ open, } \operatorname{spt} \phi \subseteq V \implies T(\phi) = 0 \}.$$

Example 2.4. Let $A \subseteq U$ be a k-dimensional (locally) rectifiable set in \mathbb{R}^d contained in an open set U. We say that $\xi : A \to \bigwedge_k \mathbb{R}^d$ is its unit orienting k-vectorfield if ξ is \mathcal{H}^k -measurable and for \mathcal{H}^k -almost all $a \in A$, $\xi(a)$ is a unit simple k-vector with associated k-subspace $\operatorname{Tan}^k(A, a)$. Then, the mapping

$$T:\phi\mapsto \int_A \langle \xi(a),\phi(a)\rangle\,\mathcal{H}^k(da),\quad \phi\in\mathcal{D}^k(U),$$

is a k-dimensional current in U and is denoted by

$$T = (\mathcal{H}^k \llcorner A) \land \xi.$$

We can generalize this notion by allowing $\xi(x)$ to carry positive multiplicities (i.e., $\xi(x)$ is again a simple k-vector associated with $\operatorname{Tan}^{k}(A, a)$, but $|\xi(x)|$ is an integer. Such currents are called (*locally*) rectifiable.

To any current $T \in \mathcal{D}^k(U)$ we attach a functional ||T|| on the space of continuous nonegative functions f on U with bounded support

$$||T||(f) = \sup\{T(\phi) : \phi \in \mathcal{D}^{k}(U), ||\phi(x)|| \le f(x) \,\forall x\}.$$

Theorem 2.2. Assume that $||T||(f) < \infty$ for any nonnegative continuous function f on U with bounded support. Then ||T|| defines a Radon measure on \mathbb{R}^d and there exists a ||T||-measurable k-vectorfield $\xi : U \to \bigwedge_k \mathbb{R}^d$ such that

$$T(\phi) = \int_G \langle \xi(x), \phi(x) \rangle \, \|T\|(dx), \quad \phi \in \mathcal{D}^k(U).$$

Currents T fulfilling the assumption of the above theorem are called *representable* by integration. For such currents, $T(\phi)$ is defined even for nonsmooth k-forms ϕ , namely whenever $\phi : U \to \bigwedge^k \mathbb{R}^d$ is ||T||-measurable with $\int ||\phi|| d||T|| < \infty$. Of course, rectifiable currents are representable by integration.

Let $T \in \mathcal{D}_k(U)$ be a k-dimensional current and $\phi \in \mathcal{D}^m(U)$ an m-form, $m \leq k$. Then $T_{\perp}\phi$ is a (k-m)-dimensional current on U defined by

$$(T \llcorner \phi)(\psi) = T(\phi \land \psi), \quad \psi \in \mathcal{D}^{k-m}(U).$$

If ξ is a C^∞ $m\text{-vector$ field on <math display="inline">U, we define the $(k+m)\text{-dimensional current}\ T\wedge\xi$ on U by

$$(T \wedge \xi)(\phi) = T(\xi \lrcorner \phi), \quad \phi \in \mathcal{D}^{k+m}(U).$$

If T is representable by integration then $T \wedge \xi$ and $T \lrcorner \phi$ are defined even whenever ξ and ϕ are ||T||-integrable.

If T is a k-dimensional current on U $(k \ge 1)$, its boundary ∂T is a (k-1)-dimensional current on U given by

$$\partial T(\phi) = T(d\phi), \quad \phi \in \mathcal{D}^k(U).$$

Example 2.5 (Gauss-Green formula). Let $E^d = \lambda^d \wedge (e_1 \wedge \cdots \wedge e_d)$ denote the *d*-dimensional current in \mathbb{R}^d defined by Lebesgue integration, with the canonical orientation of \mathbb{R}^d . If U is an open simply connected subset of \mathbb{R}^d then $T = E^d \sqcup U$ is the *d*-dimensional current given by integration over U and its boundary fulfills spt $(\partial T) \subseteq \partial U$. If, moreover, ∂U is smooth and n(x) is the unit outer normal to U at $x \in \partial U$, then

$$\partial(E^d \llcorner U) = (\partial U) \land \eta.$$

where $\eta(x) = (e_1 \wedge \cdots \wedge e_d) \sqcup dn(x)$ (η is the unit simple (d-1)-vectorfield orienting ∂U with orientation given by $\eta(x) \wedge n(x) = 1$).

Example 2.6. Let $\xi : \mathbb{R}^d \to \bigwedge_k \mathbb{R}^d$ be a C^1 smooth k-vectorfield in \mathbb{R}^d $(k \ge 1)$ such that $\int_K \|\xi\| \, d\lambda^d < \infty$ for any compact set K. Then $\lambda^d \wedge \xi \in \mathcal{D}_k(\mathbb{R}^d)$ and

$$\partial(\lambda^d \wedge \xi) = -\lambda^d \wedge \operatorname{div} \xi$$

where the divergence of ξ , div $\xi : \mathbb{R}^d \to \bigwedge_{k-1} \mathbb{R}^d$, is defined through

div
$$\xi(x) = \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i} \xi(x) \, dx_i \right).$$

Theorem 2.3 (Stokes Theorem). Let M be an oriented k-dimensional C^1 submanifold of \mathbb{R}^d with boundary ∂M , let ξ be the smooth orienting k-vectorfield over Mand set $T = (\mathcal{H}^k \sqcup M) \land \xi$. Then

$$\partial T = (\mathcal{H}^{k-1} \sqcup \partial M) \land \eta$$

where η is a unit smooth (k-1)-vectorfield orienting ∂M .

The *push-forward* of a current $T \in \mathcal{D}_k(U)$ by a smooth proper mapping $F : U \to V$ is a k-dimensional current on V defined by

$$F_{\#}T(\phi) = T(F^{\#}\phi), \quad \phi \in \mathcal{D}^k(V).$$

(F is proper if the preimage of any compact set is compact.) If T is (locally) rectifiable then the definition can be extended even for (locally) Lipschitz proper mappings F.

Theorem 2.4 (Area formula for rectifiable currents). Let $T = (\mathcal{H}^k \sqcup A) \land \xi$ be a locally rectifiable k-dimensional current in an open set $U \subseteq \mathbb{R}^d$ and let $F : U \to \mathbb{R}^n$ be proper and Lipschitz. Denote $f = F \mid A$. Then

$$F_{\#}T = (\mathcal{H}^k \llcorner f(A)) \land \eta$$

with

$$\eta(y) = \sum_{x \in f^{-1}\{y\}} \frac{(\bigwedge_k \operatorname{ap} Df(x))\xi(x)}{\operatorname{ap} J_k g(x)} = \sum_{x \in f^{-1}\{y\}} \frac{(\bigwedge_k \operatorname{ap} Df(x))\xi(x)}{|(\bigwedge_k \operatorname{ap} Df(x))\xi(x)|} |\xi(x)|.$$

Example 2.7. Let $A \subseteq \mathbb{R}^d$ be bounded and Lebesgue measurable and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz. Then

$$f_{\#}(E^d \llcorner A) = E^d \llcorner (\deg f \mid A),$$

where

$$(\deg f \mid A)(y) = \sum_{x \in A \cap f^{-1}\{y\}} \operatorname{sgn} \det Df(x).$$

deg $f \mid A$ is zero outside f(A) and constant on any connected component of $\mathbb{R}^d \setminus f(\partial A)$.

Theorem 2.5 (Constancy Theorem). Let T be a d-dimensional current in an open set $U \subseteq \mathbb{R}^d$, let G be a connected open subset of U and assume that $\operatorname{spt} \partial T \subseteq U \setminus G$. Then there exists a real number c such that

$$\operatorname{spt}\left(T - c(E^d \llcorner U)\right) \subseteq U \setminus G.$$

The *comass* of a k-form ϕ on U is defined as

$$M(\phi) = \sup\{\|\phi(x)\| : x \in U\}$$

and the mass of a k-dimensional current T on U as

$$M(T) = \sup\{T(\phi) : \phi \in \mathcal{D}^k(U), M(\phi) \le 1\}.$$

If, in particular, T is representable by integration then M(T) = ||T||(G). If $k \ge 1$ we define the norm

$$N(T) = M(T) + M(\partial T).$$

Given a compact set $K \subseteq \mathbb{R}^d$, we define the *flat seminorm* of T

$$F_K(T) = \sup\{T(\phi) : \phi \in \mathcal{D}^k(U), \|\phi(x)\| \le 1, \|d\phi(x)\| \le 1 \,\forall x \in K\}.$$

Theorem 2.6 (Compactness Theorem). Let K be a full-dimensional convex body in \mathbb{R}^d , $k \in [0, d]$ an integer and C > 0. Then the set

$${T \in \mathcal{D}_k(\mathbb{R}^d) : \operatorname{spt} T \subseteq K, N(T) \le C}$$

is F_K -compact.

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The flat convergence of currents T_i to T is defined by $F_K(T_i - T) \to 0, i \to \infty$, for all compact sets K. If T_i, T are rectifiable currents with $\partial T_i, \partial T = 0$ then flat convergence $T_i \to T$ is equivalent to weak convergence $(T_i(\phi) \to T(\phi))$ for any differential form ϕ).

Exercises:

- (1) Show that $e_1 \wedge \cdots \wedge e_d = u_1 \wedge \cdots \wedge u_d$ with $u_k = e_1 + \cdots + e_k, k = 1, \dots, d$.
- (2) Let $L \in G(d, k)$ and let $\{u_1, \ldots, u_d\}$ be an orthonormal basis of \mathbb{R}^d such that $u_1, \ldots, u_k \in L$. Show that the k-vectors

 $u_1 \wedge \dots \wedge u_{i-1} \wedge u_i \wedge u_{i+1} \wedge \dots \wedge u_k, \quad 1 \le i \le k, \, k+1 \le j \le d,$

form an orthonormal basis of $\operatorname{Tan}(G(d, k), L)$.

- (3) Let u be a fixed unit vector and consider the mappings $f : L \mapsto p_L u$, $g: L \mapsto \frac{p_L u}{|p_L u|}$ defined on $G(d,k), \{L \in G(d,k) : u \not\perp L\}$, respectively, where $1 \leq k \leq d-1$. Compute $J_k f(L)$ and $J_{k-1}g(L)$. (4) Evaluate $T(\phi)$, where $T = (\mathcal{H}^2 \sqcup A) \land (e_1 \land e_2), A = [0,1]^2 \times \{o\}$ and

 $\phi(x, y, z) = x \sin(xy) dx dy + e^{x+y+z} dx dz + (y-z)^3 dy dz.$

- (5) Evaluate T(fdx + gdy) and $\partial T(f)$ for $T = (\mathcal{H}^1 \sqcup A) \land \sqrt{2}(e_1 + e_2), A =$ $\{(x,x): 0 \le x \le 1\} \subseteq \mathbb{R}^2$ and smooth real functions f, g in \mathbb{R}^2 .
- (6) Find a 2-dimensional rectifiable current T in \mathbb{R}^3 such that $\partial T = 0$ and spt $T = \mathbb{R}^3$. (Hint: Let T be carried by a countable union of circles.)
- (7) Let $T_n = E^d B(o, 1 + \frac{1}{n}), n \in \mathbb{N}, T = E^d B(o, 1)$. Show that: (i) $M(T_n T) \to 0$, (ii) $M(\partial T_n \partial T) \neq 0$, (iii) $F_{B(o,2)}(\partial T_n \partial T) \to 0, n \to \infty$.

3. Curvature measures and normal cycles

3.1. Sets with positive reach. Given a set $X \subseteq \mathbb{R}^d$, we denote by Unp X the set of all $z \in \mathbb{R}^d$ for which there exists a unique nearest point $\Pi_A(z) \in X$. The mapping Π_X : Unp $X \to X$ is called *metric projection to* X.

The reach of a set $X \subseteq \mathbb{R}^d$ is the supremum of all $r \ge 0$ such that Unp X contains the open r-neighbourhood of X. Whenever reach X > 0 we say that X has positive reach. Any set with positive reach is closed. Any closed convex set has infinite reach.

Federer [1] introduced curvature measures for sets with positive reach by the Steiner formula. Let X_r denote the closed *r*-neighbourhood of a set X, i.e.,

$$X_r = \{ z \in \mathbb{R}^d : \operatorname{dist}(z, X) \le r \}.$$

If reach X > 0 and $B \subseteq \mathbb{R}^d$ is bounded then the local Steiner formula

$$\lambda^{d}(X_{r} \cap \Pi_{X}^{-1}(B)) = \sum_{k=0}^{d} \omega_{d-k} r^{d-k} \bar{C}_{k}(X;B), \quad 0 < r < \operatorname{reach} X,$$

defines the curvature measures $\bar{C}_k(X; \cdot)$ of X, of orders $k = 0, 1, \ldots, d$. $\bar{C}_k(X; \cdot)$ are signed Radon measures on \mathbb{R}^d , they are positive for k = d - 1, d (in fact, $\overline{C}_d(X; \cdot) = \lambda^d(X \cap \cdot)$). The suport of $\overline{C}_k(X; \cdot)$ is contained in ∂X whenever k < d. If ∂X is bounded then the total values $V_k(X) = \overline{C}_k(X; \mathbb{R}^d)$ are defined and called intrinsic volumes of X. Let χ denote the Euler-Poincaré characteristic (defined through the simplicial cohomology).

Theorem 3.1 (Gauss-Bonnet Theorem, [1]). If reach X > 0 and X is bounded then $V_0(X) = \chi(X)$.

Let \mathcal{G}_d denote the space of all Euclidean motions equipped with the invariant measure ϑ_d (normalized so that it corresponds to the product measure of λ^d (motions) and invariant probability measure on SO(d) (rotations)).

Theorem 3.2 (Principal Kinematic Formula, [1]). If reach X > 0, reach Y > 0and A, B are bounded Borel sets then

$$\int_{\mathcal{G}_d} \bar{C}_k(X \cap gY; A \cap gB) \,\vartheta_d(dg) = \sum_{r+s=d+k} \gamma_{d,r,s} \bar{C}_r(X; A) \bar{C}_s(Y, B),$$

where

$$\gamma_{d,r,s} = \frac{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{r+s-d+1}{2}\right)}$$

3.2. Integral and current representation of curvature measures. If $X \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we define the *normal cone* of X at x as the dual cone to the tangent cone Tan(X, x), i.e.,

$$Nor(X, x) = \{ v \in \mathbb{R}^d : v \cdot u \le 0 \text{ for all } u \in Tan(X, x) \}.$$

The unit normal bundle of X is the subset of $\mathbb{R}^d \times S^{d-1}$

nor
$$X = \{(x, n) \in \mathbb{R}^d \times S^{d-1} : x \in X, n \in \operatorname{Nor}(X, x)\}$$

Proposition 3.3. If reach X > 0 then Tan(X, x) is a convex cone for any x and the unit normal bundle nor X is closed.

Define the mapping $\xi_X : z \mapsto (\Pi_X(z), \frac{z - \Pi_X(z)}{|z - \Pi_X(z)|}), z \in \text{Unp } X \setminus X$. The image of ξ_X is nor X. Modifying a bit the local Steiner formula, we can define the curvature measures (called also support measures) $C_k(X; \cdot)$ on $\mathbb{R}^d \times S^{d-1}, k = 0, \ldots, d-1$:

$$\lambda^{d}((X_{r} \setminus X) \cap \xi_{X}^{-1}(A)) = \sum_{k=0}^{d-1} \omega_{d-k} r^{d-k} \bar{C}_{k}(X; A), \quad 0 < r < \operatorname{reach} X.$$

It is clear that spt $C_k(X; \cdot) \subseteq$ nor X and $\overline{C}_k(X; \cdot)$ is the first coordinate projection of $C_k(X; \cdot)$, $k = 1, \ldots, d-1$.

If 0 < r < reach X then ∂X_r is a $C^{1,1}$ smooth (d-1)-dimensional submanifold of \mathbb{R}^d (i.e., a C^1 submanifold with Lipschitz normal field n(z)). Thus, n(z) is differentiable \mathcal{H}^{d-1} -almost everywhere on ∂X_r , and the principal curvatures $\kappa_i^{(r)}(z) \in \mathbb{R}$ and principal directions $a_i^{(r)}(z) \in S^{d-1}$ are defined as eigenvalues and eigenvectors of its differential, $i = 1, \ldots, d-1$ (note that the eigenvalues are real since the differential is selfadjoint). The mapping

$$(x,n) \mapsto x + rn, \quad (x,n) \in \operatorname{nor} X$$

defines a bi-Lipschitz correspondence between nor X and ∂X_r such that n(x+rn) = n is the unit outer normal of ∂X_r at $f_r(x, n)$. The principal directions $a_i^{(r)}(z)$ can be chosen so that $a_i(x, n) := a_i^{(r)}(x+rn)$ does not depend on r and the limits $\kappa_i(x, n) := \lim_{r \to 0} \kappa_i^{(r)}(x+rn)$ exist, $i = 1, \ldots, d-1$. The vectors $a_i(x, n)$ are called (generalized) principal directions and $\kappa_i(x, n)$ (generalized) principal curvatures of A at $(x, n), i = 1, \ldots, d-1$. They are defined \mathcal{H}^{d-1} -almost everywhere on nor X and $\kappa_i(x, n) \in (-\infty, \infty]$.

Theorem 3.4 (Zähle [9]). If reach X > 0 and A is a Borel subset of $\mathbb{R}^d \times S^{d-1}$ then

$$C_k(X;A) = \int_{A \cap \operatorname{nor} X} \frac{\binom{d-1}{k}}{(d-k)\omega_{d-k}} H_{d-1-k}(X;x,n) \mathcal{H}^{d-1}(d(x,n)),$$

where

$$H_j(X;x,n) = \frac{1}{\binom{d-1}{j}} \sum_{1 \le i_1 < \dots < i_j \le d-1} \frac{\kappa_{i_1}(x,n) \cdots \kappa_{i_j}(x,n)}{\sqrt{1 + \kappa_1(x,n)^2} \cdots \sqrt{1 + \kappa_{d-1}(x,n)^2}}.$$

The functions $H_j(X; \cdot)$ are called generalized symmetric functions of principal curvatures of X. If some of the principal curvatures is infinite then its contribution is given by the convention $\frac{1}{\sqrt{1+\infty^2}} = 0$, $\frac{\infty}{\sqrt{1+\infty^2}} = 1$.

3.3. Current representation of curvature measures. If reach X > 0 then the unit normal bundle nor X is locally (d-1)-rectifiable. Thus, the approximate tangent cone $\operatorname{Tan}^{d-1}(\operatorname{nor} X, (x, n))$ is a (d-1)-dimensional subspace of \mathbb{R}^{2d} for \mathcal{H}^{d-1} -almost all $(x, n) \in \operatorname{nor} X$. If the principal curvatures and directions exist at (x, n) then

$$\left(\frac{1}{\sqrt{1+\kappa_i(x,n)^2}}a_i(x,n),\frac{\kappa_i(x,n)}{\sqrt{1+\kappa_i(x,n)^2}}a_i(x,n)\right),\quad i=1,\ldots,n$$

are approximate tangent vectors. Assume that the principal directions are ordered in such a way that $a_1(x, n), \ldots, a_{d-1}(x, n), n$ form a positively oriented orthonormal basis of \mathbb{R}^d , and define

$$a_X(x,n) = \bigwedge_{i=1}^{d-1} \left(\frac{1}{\sqrt{1 + \kappa_i(x,n)^2}} a_i(x,n), \frac{\kappa_i(x,n)}{\sqrt{1 + \kappa_i(x,n)^2}} a_i(x,n) \right).$$

Then a_X is a unit simple (d-1)-vectorfield orienting nor X. Thus,

$$N_X = (\mathcal{H}^{d-1} \llcorner \operatorname{nor} X) \land a_X$$

is a (d-1) dimensional rectifiable current in \mathbb{R}^{2d} . Since $\partial N_X = 0$, we call it *normal cycle* of X.

The kth Lipschitz-Killing curvature form φ_k is the d-1-form in \mathbb{R}^{2d} given by

$$\left\langle \bigwedge_{i=1}^{d-1} (u_0^i, u_1^i), \varphi_k(x, n) \right\rangle = \frac{1}{(d-k)\omega_{d-k}} \sum_{\substack{\varepsilon_1, \dots, \varepsilon_{d-1}=0, 1\\\varepsilon_1+\dots+\varepsilon_{d-1}=d-1-k}} \left\langle \bigwedge_{i=1}^{d-1} u_{\varepsilon_i}^i \wedge n, \Omega_d \right\rangle,$$

where $\Omega_d = dx_1 \wedge \cdots \wedge dx_d$. Note that $\varphi_k(x, n) = \varphi_k(n)$ depends only on the second coordinate vector n.

Theorem 3.5 (Zähle [9]). If reach X > 0 and A is a Borel subset of $\mathbb{R}^d \times S^{d-1}$ then

$$C_k(X;A) = (N_X \sqcup \mathbf{1}_A)(\varphi_k), \quad k = 0, \dots, d-1.$$

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3.4. Intersection formulae. We consider for simplicity flat sections. Let reach X > 0 and let F be a *j*-dimensional affine subspace od \mathbb{R}^d . Assuming that X and F do not "osculate", i.e., nor $X \cap$ nor $F = \emptyset$, the section $X \cap F$ has positive reach and its unit normal bundle (relatively to F) is

nor
$$(X \cap F) = \{(x, \pi_{F_0}n), (x, n) \in \text{nor } X\}$$

(here F_0 is the linear subspace parallel to F, $\pi_{F_0}(u) = \frac{p_{F_0}u}{|p_{F_0}u|}$ and p_{F_0} is the orthogonal projection to F_0). Applying a coarea theorem for currents, one can derive the following translative Crofton formula for curvature measures. Analogous formulae exist for intersections of two (or finitely many) sets with positive reach.

Theorem 3.6 (Translative Crofton fomula). Let L be a j-subspace of \mathbb{R}^d , $1 \leq j \leq d$, and reach X > 0. Assume that

$$\lambda^{d-j} \{ z \in L^{\perp} : \operatorname{nor} X \cap \operatorname{nor} (L+z) \neq \emptyset \} = 0.$$

Then for any $0 \leq k \leq j-1$ and any bounded Borel measurable function g on $\mathbb{R}^d \times (L \cap S^{d-1})$ with compact support we have

$$\int_{L^{\perp}} g(x,v) C_k(X \cap (L+z); d(x,v)) \lambda^{d-j}(dz) = \frac{\binom{d-1}{d+k-j}}{(j-k)\omega_{j-k}} \int_{\operatorname{nor} X} g(x, \pi_L(n)) H_{j-k-1}(X; L; x, n) \mathcal{H}^{d-1}(d(x, n)),$$

where

$$H_l(X;L;\cdot) = \frac{1}{\binom{d-1}{l}} \sum_{1 \le i_1 < \dots < i_l \le d-1} \frac{\kappa_{i_1} \cdots \kappa_{i_l} J(L, \text{Lin}\{a_{i_1}, \dots, a_{i_l}\})}{\sqrt{1 + \kappa_1^2} \cdots \sqrt{1 + \kappa_{d-1}^2}}$$

(for the definition of $J(\cdot, \cdot)$ see §1.6).

Integrating the formula above with respect to all linear subspaces, we obtain the Crofton formula.

3.5. Additive extensions. Curvature measures are additive (in the sense that $C_k(X \cup Y, \cdot) + C_k(X \cap Y; \cdot) = C_k(X; \cdot) + C_k(Y; \cdot)$ provided that all the sets $X, Y, X \cap Y, X \cup Y$ have positive reach. This property makes it possible to extend curvature measures additively to finite unions of sets with positive reach. Let \mathcal{U}_{PR} denote the system of all sets representable as finite union of sets with positive reach such that any nonempty intersection of the components has positive reach as well. (In particular, any polyconvex set belongs to \mathcal{U}_{PR} .) If $X \in \mathcal{U}_{PR}$, we define the *index function*

$$i_X(x,n) = \mathbf{1}_X(x) \Big(1 - \lim_{r \to 0_+} \lim_{s \to 0_+} \chi \big(X \cap B(x + ((r+s)n, r)) \big) \Big),$$

 $x \in \mathbb{R}^d$, $n \in S^{d-1}$. The carrier nor $X := \{(x, n) : i_X(x, n) \neq 0\}$ is locally (d-1)-rectifiable and the index function is \mathcal{H}^{d-1} -integrable, hence,

$$N_X = (\mathcal{H}^{d-1} \llcorner \operatorname{nor} X) \land a_X i_X$$

is a rectifiable current. It is a cycle again and it extends additively the normal cycle for sets with positive reach.

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3.6. Geometric sets of Fu. Fu [4] introduced the index function of a set $X \subseteq \mathbb{R}^d$

$$\iota_X(x,n) = \lim_{r \to 0_+} \lim_{s \to 0_+} \left(\chi(X \cap B(x,r) \cap H_{n,s}(x)) - \chi(X \cap B(x,r) \cap H_{n,-s}(x)) \right),$$

where $H_{n,s}(x) = \{z : (z-x) \cdot n \leq s\}$. (If $X \in \mathcal{U}_{PR}$ then $i_X(x,n)$ and $\iota_X(x,n)$ equal up to sign.) We call a current N_X normal cycle of X if

- (a) N_X is a rectifiable current in \mathbb{R}^{2d} with support in $X \times S^{d-1}$;
- (b) $\partial N_X = 0$ (N_X is a cycle);
- (c) $N_X \sqcup \alpha = o$, where $\alpha(x, n) = n_1 dx_1 + \dots + n_d dx_d$ (N_X is Legendrian); (d) $N_X(g\varphi_0) = \frac{1}{d\omega_d} \int_{S^{d-1}} \sum_{x \in \mathbb{R}^d} g(x, n) \iota_X(x, n) \mathcal{H}^{d-1}(d(x, n))$ for any C^1 smooth function g on \mathbb{R}^{2d} with compact support.

Fu called a compact set $X \subseteq \mathbb{R}^d$ geometric if it admits a normal cycle. He also showed that if X is geometric then its normal cycle is uniquely determined [5]. The assumption of compactness of X can be weakend by assuming ∂X to be compact.

The property (d) can be considered as a local Gauss-Bonet formula for sections of X with hyperplanes.

(1) \mathcal{U}_{PR} -sets. **Example 3.1** (Examples of geometric sets).

(2) Closures of complements of full-dimensional sets with positive reach. Let X be a bounded set with positive reach such that for all $x \in X$, Nor(X, x)contains no line. (As example, consider a full-dimensinal convex body.) Then $\widetilde{X} := \operatorname{cl}(\mathbb{R}^d \setminus X)$ is geometric and

$$N_{\tilde{X}} = -\rho_{\#}N_X,$$

where $\rho: (x, n) \mapsto (x, -n)$. Consequently, the curvature measures satisfy

$$C_k(\widetilde{X}; A) = (-1)^{d-1-k} C_k(X; \rho(A)), \quad A \subseteq \mathbb{R}^d \times S^{d-1}$$
 Borel.

- (3) Lipschitz submanifolds of \mathbb{R}^d with "bounded curvatures" [8].
- (4) Subanalytic sets [5].

3.7. Approximation with parallel sets. If X is a compact set with positive reach then the closed r-neighbourhoods X_r converge to X as $r \to 0$ in the Hausdorff metric and the corresponding normal cycles N_{X_r} converge to N_X weakly.

If X is polyconvex then X_r (the closure of the complement of X_r) has positive reach for r small enough and $N_{\widetilde{X}_r}$ converge to $\rho_{\#}N_X$ weakly [7]. This makes it possible to approximate curvature measures of polyconvex sets by those of dilated sets, without the index function.

The property that X_r has positive reach for r small enough is preserved by many very general sets (i.e., in \mathbb{R}^2 or \mathbb{R}^3 , it holds for any set and λ^1 -almost all r, which has been shown by Fu [3]). Suppose that for such a set X, the following two condition hold.

- (1) $\chi(X_r \cap H) \to \chi(X \cap H), r \to 0$, for almost all hyperplanes H;
- (2) $\liminf_{r\to 0} M(N_{X_r}) < \infty$.

Then, by the compactness theorem for currents, there exists a sequence $r_i \rightarrow 0$ such that $N_{X_{r_i}}$ converges weakly to some rectifiable current T. One easily verifies that T keeps the properties (a)-(c) of normal cycles. To verify (d), one needs the convergence of the Euler characteristic of hyperplane sections (1). Thus, T is the normal cycle of X (which is unique) and X is geometric. This procedure works for certain Lipschitz manifolds.

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Exercises.

- (1) Evaluate $V_j(C)$, j = 0, 1, 2, 3, for the unit cube C in \mathbb{R}^3 , using the Steiner formula.
- (2) Let $X \subseteq \mathbb{R}^2$ be the graph of the function $y = \sin(x), 0 \le x \le \pi$. Determine the curvature function of X. Compute $C_0(X; X \times S^1_+)$, where S^1_+ is the halfsphere $S^1 \cap \{y > 0\}$.
- (3) Let $X \subseteq \mathbb{R}^2$ be the graph of the function $y = \frac{1}{n^2} \sin nx$ $(n \in \mathbb{N})$. Compute reach X.
- (4) Evaluate the index function $i_X(o, n)$ for the coordinate cross $X = \{x = 0\} \cup \{y = 0\}$ in \mathbb{R}^2 , $n \in S^1$. Find $\overline{C}_0(X; \{o\})$.
- (5) Evaluate the index function $i_X(x,n)$ for the set $X = \{x \leq 0\} \cup \{y \leq 0\} \cup \{z \leq 0\} \subseteq \mathbb{R}^3, x \in \partial X, n \in S^2$.
- (6) Show that if reach X > 0 then for all $0 \le k \le d-1$ and \mathcal{H}^{d-1} -almost all $(x, n) \in \operatorname{nor} X$,

$$\langle a_X(x,n), \varphi_k(n) \rangle = \operatorname{const} H_{d-1-k}(X;x,n).$$

(7) (*) Show that $\partial N_X = 0$ whenever reach X > 0. (Hint: If ∂X is C^1 smooth, apply the Stokes theorem. In the general case, approximate X by X_r , $r \to 0$.)

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