Rare-event Simulation of Brownian Motion Avoiding Hard Obstacles

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Columbia IEOR Department

Rubinstein's Celebration

Agenda

- **Introduction: Rare-event Simulation**
- Brownian Motion Avoiding Obstacles
- Explaining the Strategy
- Conclusions

- **Assumptions:**

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- **Question:**
- Design an efficient simulation algorithm to estimate $P(\mathcal{T} > t)$ for large t

Motivation

- Undetected objects for long period of time
- Motivation as a problem in random media (study of polymers in random environments)
- Materials properties (obstacles represent impurities)
- Introduced by Smoluchowsky (1918) in Chemistry and Physics / now proposed as model of molecules in motion in cells (http://jb.asm.org/cgi/content/full/187/1/23)
- *It provides an interesting example of importance sampling that involves infinite dimensional simulation (control) problem... .*

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- Simulate iid replications of Y to estimate $P(Z \in A) = E^Q Y$.
- Want to reduce the variance of Y

An Obvious Observation and a Powerful Principle

- Select $Q(\cdot)$ as conditional distribution given $Z \in A$

$$Q(d\omega) = \frac{I(Z(\omega) \in A) P(d\omega)}{P(Z \in A)},$$

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- *Then* $Y = P(Z \in A)$ *is unbiased with zero variance...*

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- Obviously useless to implement BUT *yields a powerful principle: WE SHALL CALL IT GISP ("Good Importance Sampling Principle").*
- **GISP**: *"To design a good importance sampling try to mimic the conditional distribution of the process given the rare event" (Asmussen and Rubinstein '85)*

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- **Remark:** Need to also consider the computer time to generate Z_n that is typically polynomial in $|\log \alpha_n|$ so doesn't contribute significantly to complexity.

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Volume of the Wiener Sausage

- Recall T = 1st time $B(\cdot)$ hits a Poissonian obstacle

$$\begin{aligned} P(T > t) &= E\left(P\left(T > t \mid B(s) : 0 \leq s \leq t\right)\right) \\ &= E\left(P\left(\text{No obstacle in trajectory} \mid B(\cdot)\right)\right) \\ &= E\exp\left(-V(t,a)\right), \end{aligned}$$

where

$$V(t,a) = Vol\left(\bigcup_{0\leq s\leq t} Square\left(center = B(s), vol = a\right)\right)$$

Brownian Scaling

- By the invariance principle, if $\tau = t^{d/(d+2)}$ then we have

$$E\exp\left(-V\left(t,a\right)\right) = E\exp\left(-\tau V\left(\tau, a\tau^{-1/d}\right)\right)$$

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- How to obtain a "GISP" here? What does large deviations tell us?

Large Deviation Asymptotics

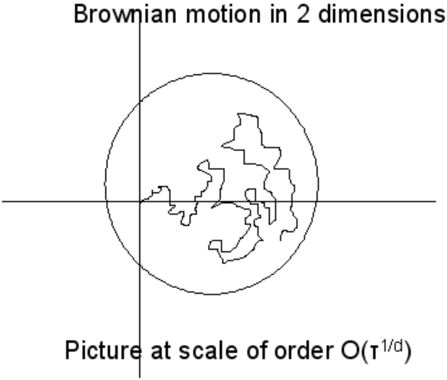
- Donsker and Varadhan '75 proved that when $\delta \downarrow 0 \geq a\tau^{-1/d}$ as $\tau \searrow 0$ (also Bolthausen '90, Sznitman '89) then

$$\begin{aligned} & \frac{1}{\tau} \log \alpha\left(\tau, \delta\right) \\ \longrightarrow \quad & -\inf_{f: \int f = 1} \left(vol(supp(f)) + \frac{1}{8} \int \frac{\|\nabla f\|^2}{f} \right) \\ = \quad & -\inf_{G \text{ open}} \left(vol(G) + \lambda_G \right), \end{aligned}$$

where λ_G = principal e-value of $\triangle/2$ on G −> **discuss optimal path**

Asymptotic Conditional Distribution

- **Conditional description (Schmock $d = 1$, Sznitman $d = 2$, Povel $d > 2$)**: *B. Motion travels $O\left(\tau^{1/d}\right)$ distance to find an optimal center (random even at $\tau^{1/d}$ scales!) and it confines itself inside a ball with optimal radius at spatial scales of $O\left(\tau^{1/d}\right)$...*



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- **Question:**
- *How to describe a change-of-measure that mimics the conditional distribution close enough to obtain an asymptotically optimal estimator – GISP?*
- *Such change-of-measure must find an optimal ball with the right distribution and do it step-by-step from the Brownian path...*

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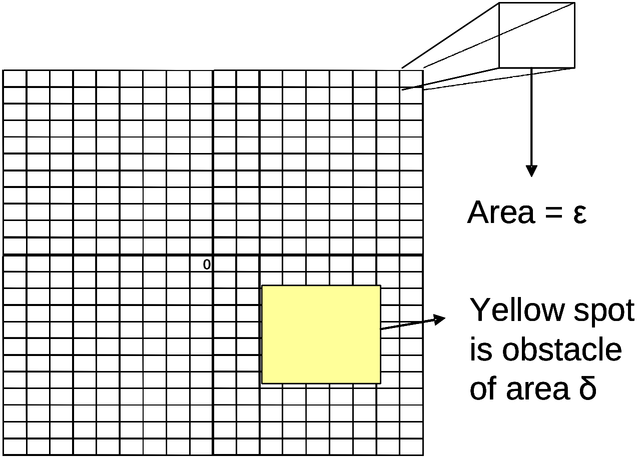
The Strategy

- $\alpha(\tau,\delta) = E_0 \exp(-\tau V(\tau,\delta))$
- Divide the space in cubes of volume

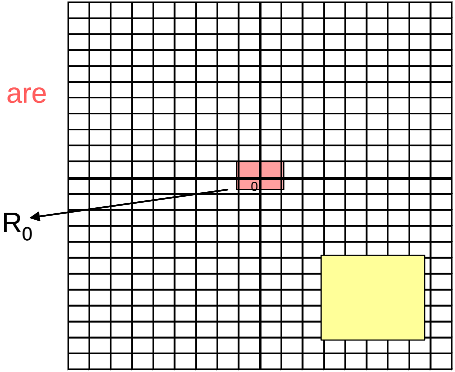


The Strategy

- We generate a *suitable process* that keeps exploring regions as follows:

Initial "explored" region.

Explored regions are painted pink



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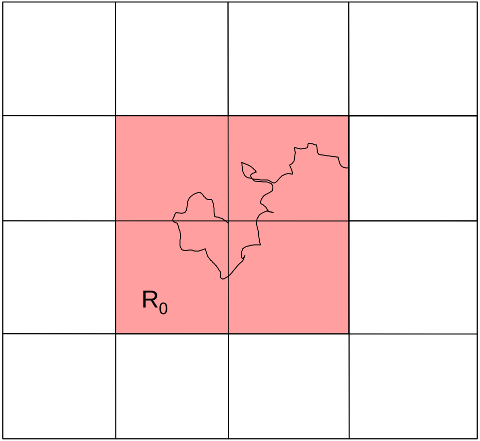
- The distribution of the process adapts according to explored regions (we'll see how!)



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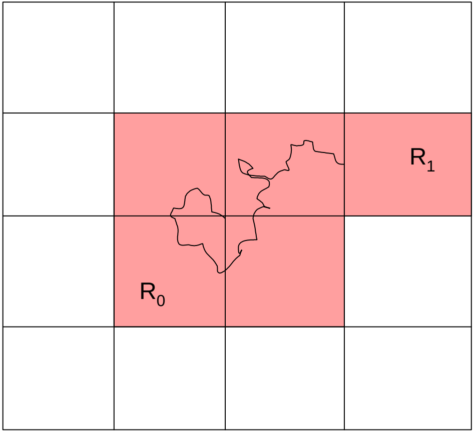
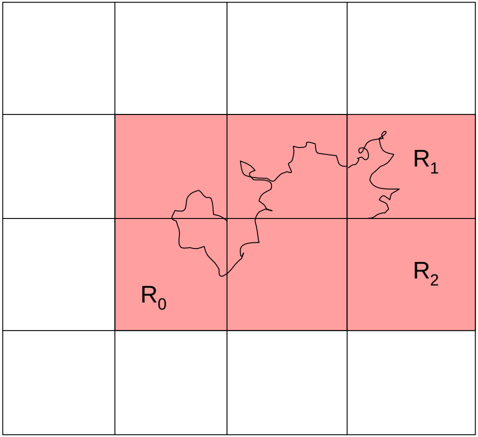


The Strategy

- And one goes on sequentially —> now we'll explain the evolution



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- Want to spend as much as possible in the explored region (which is free of obstacles!)
- Let $T_M = \inf\{t \geq 0 : B(t) \notin \mathcal{R}_M\}$...
- Select θ_M such that

$$E\left(\exp(\theta_M T_M) \middle| \text{Visited region } \mathcal{R}_M\right) = \exp(\gamma \varepsilon \tau),$$

AND γ which will be chosen...

The Strategy

- Implement the strategy sequentially: Given region $\mathcal{R}_M$ sample according to the SDE

$$dX(t) = \nabla \log v_{\mathcal{R}_M}(X(t), \theta_M)\, dt + dB(t),$$

where $v_{\mathcal{R}_M}(x) = E_x(\exp(\theta_M T_M)|\,\text{Visited region } \mathcal{R}_M)$ for $x \in \mathcal{R}_M$.

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- The likelihood ratio

$$L_\tau = \frac{1}{v_{\mathcal{R}_{M(\tau)}}\left(B_\tau, \theta_{M(\tau)}\right)} \exp\left(\gamma \varepsilon \tau M_\tau - \int_0^\tau \theta_{M(s)}\, ds\right),$$

The Strategy

- So, the I.S. estimator is

$$L_{\tau} \exp\left(-\tau V\left(\tau,\delta\right)\right) = \frac{\exp\left(-\tau\left(V\left(\tau,\delta\right)-\gamma\varepsilon M_{\tau}\right)-\int_{0}^{\tau}\theta_{M(s)}ds\right)}{v_{\mathcal{R}_{M(\tau)}}\left(B_{\tau},\theta_{M(\tau)}\right)},$$

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2. $\mathbf{P}\left(\left.T_M > x\right|\mathcal{R}_M\right)=\exp\left(-\lambda_{\mathcal{R}_M}x+o\left(x\right)\right)$
3. **By the choice of θ_M, we have that** $\theta_{M(s)}=\lambda_{\mathcal{R}_{M(s)}}+o\left(1/\tau\right)$

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4. $\lambda_{\mathcal{R}_{M(s)}} \geq \lambda_{\mathcal{R}_{M(\tau)}} \geq 0$ **for** $r \geq s$

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4. $\lambda_{\mathcal{R}_{M(s)}}\geq\lambda_{\mathcal{R}_{M(\tau)}}\geq 0$ for $r\geq s$
5. $\varepsilon \mathbf{M}_\tau = \mathbf{Vol}\left(\mathcal{R}_{M(\tau)}\right)\leq \mathbf{V}\left(\tau,\delta\right)$ **(for** $\varepsilon\leq\delta/2$)

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- Take $\varepsilon = \delta$ and this takes us to...

$$
\begin{aligned}
& \text{IS EST} \\
\leq \; & \exp\left(-\tau \min_{G:\text{ open}} \left(Vol\left(G\right)\left(1-\gamma\right)+\lambda_G\right)\right) \\
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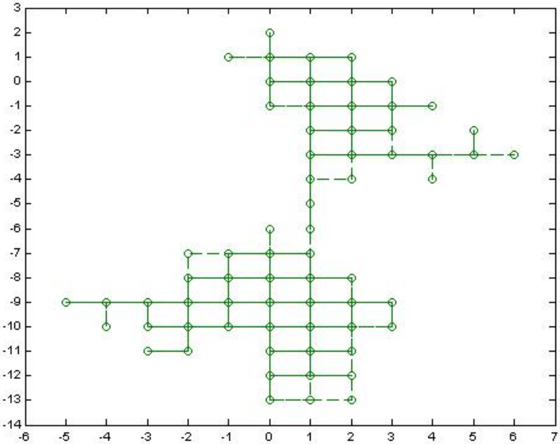
- Therefore

$$
E\left(Estimator^2\right) = \alpha\left(\tau,\delta\right)^2 \exp\left(\tau O\left(\gamma\right)+o\left(\tau\right)\right)
$$

and *weak efficiency follows* (for $\delta, \gamma \longrightarrow 0$ sufficiently slow as $\tau \longrightarrow \infty$)

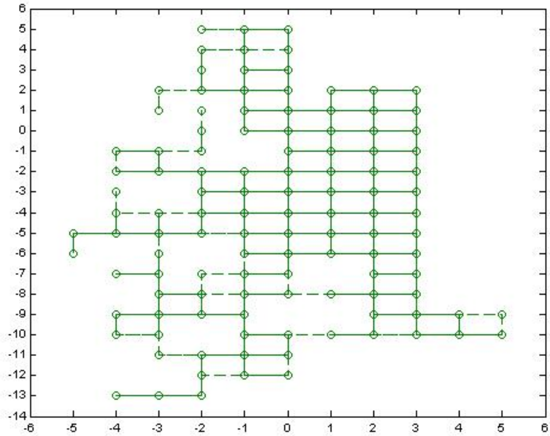
Pictures

A path for $\tau = 1000$, using $\gamma = .1$



Pictures

- A path for $\tau = 5000$, using $\gamma = .1$



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Conclusions

- Brownian motion avoiding obstacles gives an example where history dependent importance sampling should be performed to achieve efficiency
- Strategy induces confinement −> particle tries to stay inside explored region, which is obstacle free
- Eventually, explored region is basically a ball with optimal radius and specific distribution center