Optimisation under probability constraints: an approach via quantiles

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- Service industry to control measures of client satisfaction

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Distribution Formulation

$$\min_u J(u)$$
$$\text{subject to: } B(u) \le 0$$

Example

- We borrow one unit $$ at interest rate I, pay at end of period
- Decision: fraction u_1 to invest at fixed rate $b < I$
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$$\max_{u_1, u_2} \mathbb{E}\Big(U(1 - u_1 - u_2) + (1 + b)u_1 + (1 + \xi)u_2\Big)$$

$$\text{s.t.} \quad u \geq 0\,, \quad u_1 + u_2 \leq 1\,,$$

$$\mathbb{P}\Big((1 + b)u_1 + (1 + \xi)u_2 \geq 1 + I\Big) \geq p\,.$$

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Challenges with the problem include:

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- Gradient-based algorithms (stochastic) require
  - convexity of $B(u) = p - \mathbb{P}(g(\xi, u) \leq \alpha)$!!!!
  - estimation of gradient of a probability (discontinuities, lack of model for distribution)

Example: Optimal Cost

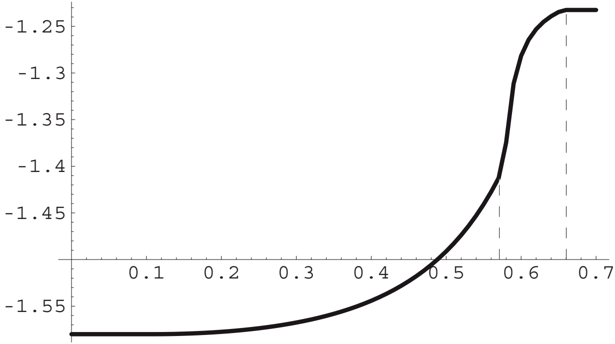

Figure: Optimal cost as a function of "confidence" level p.

Non-convexity and non saturated but active constraints.

Constrained Optimisation

Theorem

For a convex problem (strictly convex $J(u)$ and $B(u)$) the optimal (u^*, λ^*) is a saddle point and solves:

$$\min_{u \in \mathbb{R}^d} \max_{\lambda \geq 0} L(u, \lambda) = \max_{\lambda \geq 0} \min_{u \in \mathbb{R}^d} L(u, \lambda)$$

The Arrow Hurwicz Algorithm is:

$$u_{n+1} = u_n - \epsilon_n \left(\nabla_u J(u_n) + \lambda_n^T \nabla_u B(u_n) \right)$$

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The Arrow-Hurwicz Algorithm: Convergence

Theorem

Let $x_{n+1} = x_n + \epsilon V(x_n)$ and let $x_\epsilon(t) = x_n$, $t \in [n\epsilon, (n+1)\epsilon)$. If V is a Lipschitz continuous and bounded function, then as $\epsilon \to 0$, $x_\epsilon(\cdot)$ converges (in the sup norm) to the solution of the ODE:

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- Convexity is not required for this property to hold.
- Local convergence around stable points: study only the behaviour of active constraints: $\lambda > 0$ (continuity).
- Allows to characterise behaviour around stationary points.

The A-H Algorithm: Convergence and Optimality.

- The vector field of A-H has stable points that are saddlepoints of the Lagrangian (convex problems).
- Linearise around a stable point x^* using Taylor expansion

$$V(x) = V(x^*) + \mathbb{A}(x - x^*) + (O)(\|x - x^*\|^2),$$
$$x(t) - x^* \approx e^{\mathbb{A}t}, \quad \mathbb{A} = \nabla V(x^*)^T$$

- $\mathbb{A}$ Hurwitz: $\Re(eigenv(\mathbb{A})) < 0$ then x^* attractor (limit).

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Result

Convex problem, $x^* = (u^*, \lambda^*)$ optimal, constraint qualification $\nabla B(x^*)$ l.i. vector. Then $\mathbb{A}$ is Hurwitz, implying that the optimal solution and multiplier are attractors of the ODE.

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Non-convex problems: $V(x)$ for insight into algorithm behaviour.

Vector Fields: examples

Example

$$\min \tfrac{1}{2}u^2, \quad s.t. \quad \mathbb{P}(\xi - u \le \alpha) \ge p \qquad (u^0 = 0)$$

$$B(u) = p - F(u + \alpha).$$

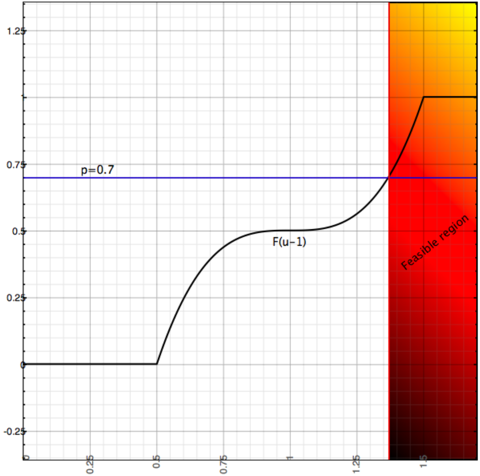
Case 1: uniform distribution

$$F(\xi) = 0.5 + 0.5(\xi - 0.5)\mathbf{1}_{\{-0.5<\xi\le 0.5\}}$$

Case 2: “beta”-like distribution

$$F(\xi) = 0.5 + 0.5(2\xi)^3\mathbf{1}_{\{-0.5<\xi\le 0.5\}}$$

Vector Fields: examples



- Depicted: Case 2.

picture aside

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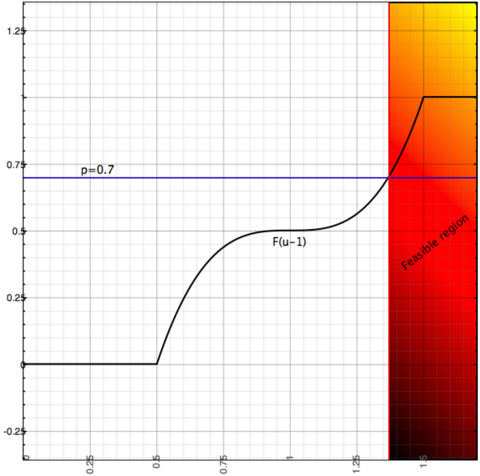

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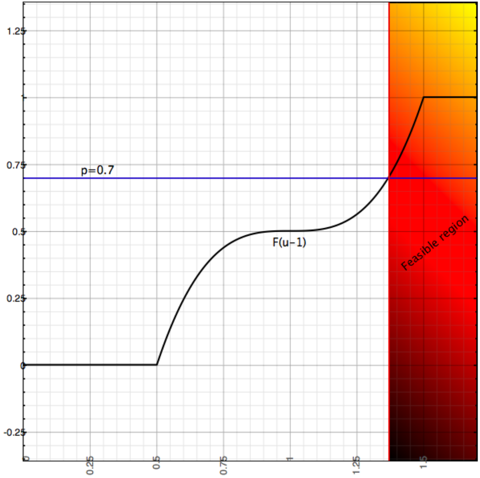

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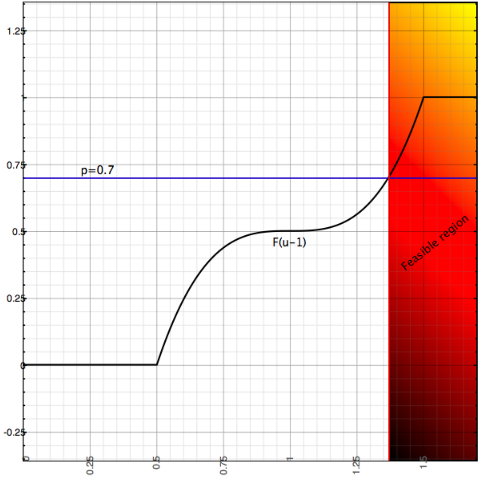

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- Solution is $u^* \approx 1.36 \neq u^0$.

picture aside

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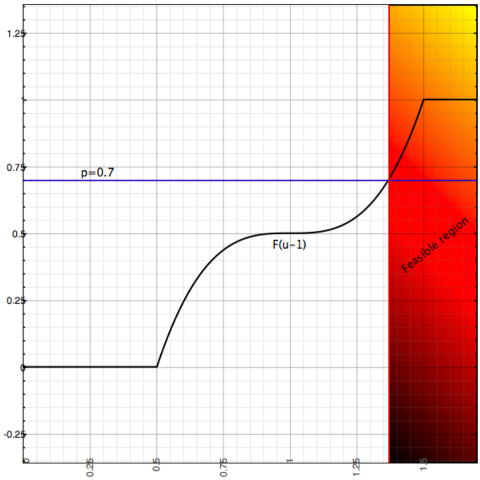

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- Unconstrained optimum at $u^0 = 0$.
- Feasible region is $F(u+1) \geq 0.7$
- Solution is $u^* \approx 1.36 \neq u^0$.
- Probability constraint is active at optimum.

picture aside

Vector Fields: examples

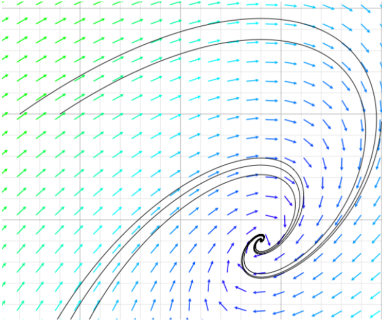

Figure: Left: Case 1: convex. Right: Case 2: non convex.

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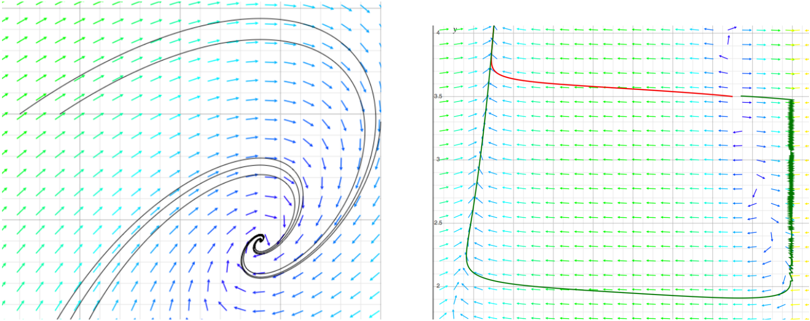

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Contribution

Conjecture: lack of convexity is the problem.

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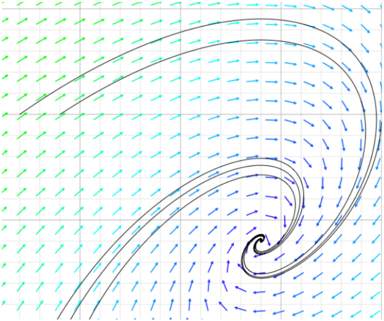

Figure: Case 1: convex.

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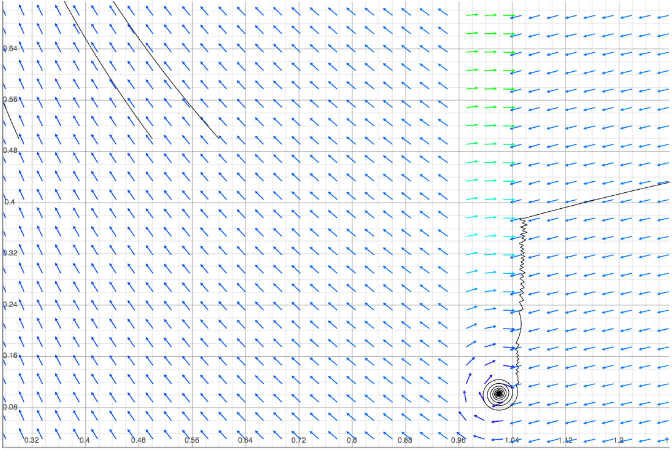

Figure: Convex distribution: zoom out

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- Identification of problem: distributions with bounded support (potential numerical problem for any distribution)
- Re statement of problem using a Quantile Formulation.

Pathology from bounded support

- For each u, $g(\cdot, u)$ is monotone increasing, $h(u, \xi) = g_u^{-1}(x)$.
- $g(\cdot, u)$ is continuously differentiable in u.
- Bounded support $F(\xi) = 0$, for all $\xi \leq \underline{\xi}$ and assume that $\mathcal{U} = \{u \colon h(u, a) \leq \underline{\xi}\} \neq \emptyset$

Theorem

Assume a unique optimal solution (u^*, λ^*) to the constrained problem

$$\min J(u) \quad s.t. \; B(u) = p - \mathbb{P}(g(\xi, u) \leq \alpha) \leq 0.$$

and that the unconstrained minimum $u^0 = \arg\min_u J(u) \in \mathcal{U}$. Then the A-H algorithm divergesbwhen initialising "close" to u^0; specifically, $u_n \to u^0 \neq u^*$ and $\lambda_n \to +\infty$.

Pathology from bounded support

Proof.

- A-H algorithm has a vector field:

$$u' = -J'(u) + \lambda f(g^{-1}(\alpha, u))(g^{-1}(\alpha, u))'$$
$$\lambda' = (p - F(g^{-1}(\alpha, u))\mathbf{1}_{\{\lambda \geq 0\}}$$

- When initialising inside $\mathcal{U}$, $F(u) = f(u) = 0$ so the algorithm behaves:

$$u' = -J'(u) \quad \Rightarrow \quad u \to u^0$$
$$\lambda' = p \quad \Rightarrow \quad \lambda \to +\infty$$

□

Quantile Formulation

Remark

Common methods to deal with no convexity can be used (penalties, augmented Lagrangian, A-H “beta” method for convexification, etc), but they will also suffer from the pathology of bounded support.

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Common methods to deal with no convexity can be used (penalties, augmented Lagrangian, A-H "beta" method for convexification, etc), but they will also suffer from the pathology of bounded support.

- Fact: if distribution function is convex (concave) then its inverse the quantile function is concave (convex)
- Conjecture: use one or another to deal with regions of non convexity
- But our results show that Quantile formulation always works! (under convexity of $g(\xi, \cdot)$).

Quantile Formulation

Lemma

Suppose that for every u, $g(\cdot, u)$ is monotone increasing. Then

$$\mathbb{P}(g(\xi, u) \leq \alpha) \geq p \Leftrightarrow g(Q(p), u) \leq \alpha.$$

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$$L(u, \lambda) = J(u) + \lambda(g(Q(p), u) - \alpha).$$

Theorem

If $J(\cdot)$ and $g(x, \cdot)$ are convex for every x, where x is a continuous random variable, then independently of the distribution function, AH has a unique attractor at the optimum.

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$$\dot{u}_t = -\nabla J(u_t) - \nabla_u g(Q(p), u_t)$$

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If decreasing then use $g(Q(1-p), u)$

Quantile Formulation

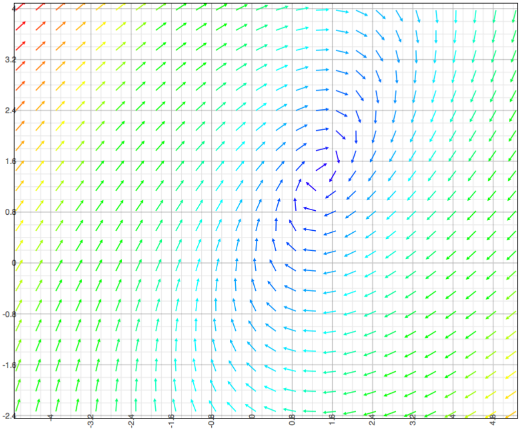

Figure: Quantile formulation

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$$\max_{u_1,u_2} \mathbb{E}\Big(U(1 - u_1 - u_2) + (1+b)u_1 + (1+\xi)u_2\Big)$$

s.t.

$$u \geq 0\,, \quad u_1 + u_2 \leq 1\,,$$

$$\mathbb{P}\Big((1+b)u_1 + (1+\xi)u_2 \geq 1 + I\Big) \geq p\,.$$

Example: Quantile Formulation

Here the constraint function is decreasing:
$g(u_1, u-2, \xi) = -(1+b)u_1 - (1+\xi)u_2, a = l+1$, so we use:
$B(u_1, u_2) = (l+1) - (1+b)u_1 - (1+Q(1-p))u_2$.

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$$\begin{aligned}
\dot{u_1} &= -U'(1-u_1-u_2) - \lambda_1 b - \lambda_2 \\
\dot{u_2} &= -U'(1-u_1-u_2) - \lambda_1 Q(1-p) - \lambda_2 \\
\dot{\lambda}_1 &= (l+1) - (1+b)u_1 - (1+\xi)u_2 \\
\dot{\lambda}_2 &= u_1 + u_2 - 1,
\end{aligned}$$

Fast convergence to optimal point, no problem for the algorithm. Note that now the multiplier gives sensitivity w.r.t. the level of constraint $a(-l+1)$ rather than to p.

On-going work

- Quantile formulation promises better algorithmic behaviour.
- Can the formulation be extended to piecewise monotonic functions?
- How to use the approach for simulation: open question.
- How to generalise to several variables: open question.
- Current research with France: aerospace control, needs a dynamical system and $g(\cdot, u)$ depends on whole trajectory.

The End

- Thank you for your attention
- Questions?