Comparing Two Systems Using Gaussian Copulae[1]

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- Standard sampling: $\mathbf{U}_X$ is independent of $\mathbf{U}_Y$
- CRN sampling: $\mathbf{U}_X = \mathbf{U}_Y$
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What is essential?

Let $\mathbf{U} = (\mathbf{U}_X, \mathbf{U}_Y)$

$$\mathbf{U} = (\underbrace{U_1, U_2, \ldots, U_d}_{IID}, \underbrace{U_{d+1}, U_{d+2}, \ldots, U_{2d}}_{IID})$$

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What is new ...

... is that we use a new class of copulas and have a computational method for searching over it

Gaussian Copulae

Let $\mathbf{Z} = (\mathbf{Z}_X, \mathbf{Z}_Y)$ be jointly Gaussian, standard normal marginals, covariance matrix

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$$X = f_U(\mathbf{U}_X) = f(\mathbf{Z}_X)$$
$$Y = g_U(\mathbf{U}_Y) = g(\mathbf{Z}_Y)$$

Example 1

Take $d = 2$

$$X = f(\mathbf{Z}_X) = \frac{Z_X[1] + Z_X[2]}{\sqrt{2}}$$
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An optimal Gaussian copula:

$$\mathbf{Z}_Y[1] = \frac{Z_X[1] + Z_X[2]}{\sqrt{2}}$$
$$\mathbf{Z}_Y[2] = \frac{Z_X[1] - Z_X[2]}{\sqrt{2}}$$

so that $X = Y$ or, equivalently,

$$\boldsymbol{\Sigma}_{XY} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Example 2

Take $d = 1$

$$X = f_U(\mathbf{U}_X) = I(\mathbf{U}_X \in [0.5, 0.6])$$
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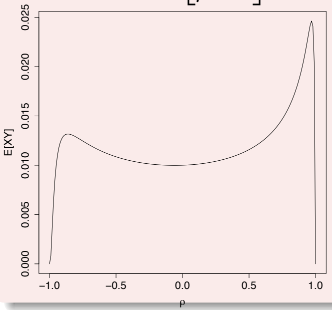
Optimal Gaussian copula:

$\text{var}(X - Y) = 0.15$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$



Optimizing the Copula

- $\min \operatorname{var}(X - Y) \Leftrightarrow \max \operatorname{cov}(X, Y) \Leftrightarrow \max Ef(\mathbf{Z}_X)g(\mathbf{Z}_Y)$

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- max $Ef\left(\left(\boldsymbol{\Sigma}^{1/2}\mathbf{N}\right)[1, \ldots, d]\right) g\left(\left(\boldsymbol{\Sigma}^{1/2}\mathbf{N}\right)[d+1, \ldots, 2d]\right)$
  subject to
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  subject to
  $$\mathbf{\Sigma} = \begin{bmatrix} I_d & \mathbf{\Sigma}_{XY} \\ \mathbf{\Sigma}_{XY}^T & I_d \end{bmatrix} \succeq 0$$
- Nonlinear semidefinite program
- Can be tackled, but gradients of objective are tricky
- Reformulate using Cholesky factors directly by observing that $\mathbf{\Sigma}_{XY}$ is "sub-orthogonal" ...

An Alternative Formulation

Proposition

$$\mathbf{\Sigma} = \begin{bmatrix} I_d & \mathbf{\Sigma}_{XY} \\ \mathbf{\Sigma}_{XY}^T & I_d \end{bmatrix} \succeq 0 \Leftrightarrow \exists \mathbf{M}_2 : \mathbf{M}^T\mathbf{M} = I \textit{ where } \mathbf{M} := \begin{bmatrix} \mathbf{\Sigma}_{XY} \\ \mathbf{M}_2 \end{bmatrix}$$

Furthermore, $\mathbf{\Sigma}$ *is covariance matrix of*

$$\begin{bmatrix} \mathbf{Z}_X \\ \mathbf{Z}_Y \end{bmatrix} := \begin{bmatrix} \mathbf{N}[1,\ldots,d] \\ \mathbf{M}^T\mathbf{N} \end{bmatrix}$$

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Notice that

$$\begin{bmatrix} I_d & 0_d \\ \boldsymbol{\Sigma}_{XY}^T & M_2^T \end{bmatrix} \begin{bmatrix} I_d & \boldsymbol{\Sigma}_{XY} \\ 0_d & M_2 \end{bmatrix} = \boldsymbol{\Sigma}$$

Solving the Alternative Formulation

- max $Ef(\mathbf{Z}_X)g(\mathbf{M}^T\mathbf{N})$ subject to $\mathbf{M}^T\mathbf{M} = I$
- Nonlinear optimization over a Stiefel manifold
- Gradients w.r.t. $\mathbf{M}$ easily obtained

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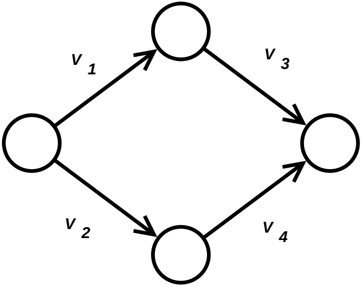
- Use, e.g., sample-average approximation
- Sample and fix $\mathbf{N}_1, \ldots, \mathbf{N}_m$, and

$$\begin{aligned} \text{max} \quad & \frac{1}{m}\sum_{i=1}^{m} f(\mathbf{N}_i[1,\ldots,d])g(\mathbf{M}^T\mathbf{N}_i) \\ \text{subject to} \quad & \mathbf{M}^T\mathbf{M} = I \end{aligned}$$

- Freely available `sgmin` in MATLAB
- Use solution to above in subsequent conditionally independent simulation

Example 3



$$X : V_1, V_2, V_3, V_4 \sim \text{ IID } \exp(1)$$
$$Y : V_1, V_2 \sim \text{ IID } \exp(1)$$
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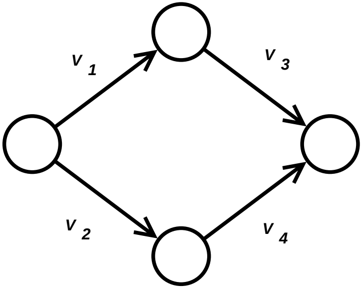

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IID	5.3
CRN	0.57
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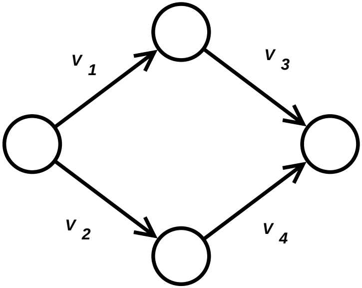

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$$\Sigma^*_{XY} = \begin{bmatrix} .958 & -.038 & .160 & .237 \\ -.037 & .960 & .239 & .141 \\ -.158 & -.238 & .957 & -.048 \\ -.239 & -.143 & -.026 & .960 \end{bmatrix}$$

An Observation for Linear Functions

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Proposition

If f and g are linear, then an optimal Σ_{XY} is orthogonal and corresponds to a Householder transformation that "aligns" the two linear functions.

Conclusions and Future Research

- One might use more general joint distributions than CRN for comparisons
- Gaussian copula is particularly convenient
- Could use other copulas, e.g., chessboards, but computation needs to be feasible
- Simple examples demonstrate that large gains are possible

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- One might use more general joint distributions than CRN for comparisons
- Gaussian copula is particularly convenient
- Could use other copulas, e.g., chessboards, but computation needs to be feasible
- Simple examples demonstrate that large gains are possible
- When is optimization problem unimodal?
- Clarify connection to existing optimality results for CRN?
- More complicated (interesting?) examples?