Efficient Simulation of Tail Probabilities of Sums of Lognormal Random Variables with Gaussian Copula

Leonardo Rojas-Nandayapa

joint work with José Blanchet and Sandeep Juneja

A Conference on the Occasion of R.Y. Rubinstein's 70th Birthday,
July 15, 2008. Sandbjerg Gods, Denmark.

Outline

Outline

Outline

Estimation of rare event probabilities

Rare Event Simulation Algorithm

Let $\{A_x\}$ be an indexed set of events such that

$$\lim_{x \to x_0} \mathbb{P}(A_x) \to 0.$$

An algorithm for estimating $\mathbb{P}(A_x)$ is a set of r.v. variables $\{Z_x\}$

$$\mathbb{E}\, Z_x = \mathbb{P}(A_x)\ \forall\ x.$$

Efficient algorithms

$$\limsup_{x \to x_0} \frac{\mathbb{V}\text{ar}\, Z_x}{\mathbb{P}^{2-\epsilon}(A_x)} < \infty.$$

Either for $\epsilon = 0$ (Bounded Relative Error) or for all $\epsilon > 0$ (Logarithmic Efficient).

Estimation of rare event probabilities

Rare Event Simulation Algorithm

Let $\{A_x\}$ be an indexed set of events such that

$$\lim_{x \to x_0} \mathbb{P}(A_x) \to 0.$$

An algorithm for estimating $\mathbb{P}(A_x)$ is a set of r.v. variables $\{Z_x\}$

$$\mathbb{E}\, Z_x = \mathbb{P}(A_x) \ \forall\ x.$$

Efficient algorithms

$$\limsup_{x \to x_0} \frac{\mathbb{V}\mathrm{ar}\, Z_x}{\mathbb{P}^{2-\epsilon}(A_x)} < \infty.$$

Either for $\epsilon = 0$ (Bounded Relative Error) or for all $\epsilon > 0$ (Logarithmic Efficient).

Outline

Rare event simulation involving sums of r.v.'s.

Rare event probabilities of the type

$$\mathbb{P}(X_1 + \ldots + X_N > u) \qquad u \to \infty.$$

N possibly random.

Light tails

Most established tool is Importance Sampling.

Heavy Tails

Asmussen, Binswanger and Højgaard (1998) Severe difficulties.

Rare event simulation involving sums of r.v.'s.

Rare event probabilities of the type

$$\mathbb{P}(X_1 + \ldots + X_N > u) \qquad u \to \infty.$$

N possibly random.

Light tails

Most established tool is Importance Sampling.

Heavy Tails

Asmussen, Binswanger and Højgaard (1998) Severe difficulties.

Rare event simulation involving sums of r.v.'s.

Rare event probabilities of the type

$$\mathbb{P}(X_1 + \ldots + X_N > u) \qquad u \to \infty.$$

N possibly random.

Light tails

Most established tool is Importance Sampling.

Heavy Tails

Asmussen, Binswanger and Højgaard (1998) Severe difficulties.

Rare event simulation involving i.i.d. r.v.

State Independent

- Asmussen and Binswanger (1997) Logarithmic Efficiency
- Asmussen and Kroese (2006) Bounded Relative Error
- Juneja (2007) Zero Relative Error.

State-Dependent

- Dupuis et. al. (2007) IS for Regularly Varying.

Simulation with Heavy Tailed Random Variables

Subexponential Distributions

In the independent case

$$\mathbb{P}(X_1 + \ldots + X_n > u) \sim \sum_{i=1}^{n} \mathbb{P}(X_i > u)$$

Intuitive Idea

S_n becomes large as a consequence of single large jump.

Dependent Case

Ideas from the Independent Case might not work.

Simulation with Heavy Tailed Random Variables

Subexponential Distributions

In the independent case

$$\mathbb{P}(X_1 + \ldots + X_n > u) \sim \sum_{i=1}^{n} \mathbb{P}(X_i > u)$$

Intuitive Idea

S_n becomes large as a consequence of single large jump.

Dependent Case

Ideas from the Independent Case might not work.

Simulation with Heavy Tailed Random Variables

Subexponential Distributions

In the independent case

$$\mathbb{P}(X_1 + \ldots + X_n > u) \sim \sum_{i=1}^{n} \mathbb{P}(X_i > u)$$

Intuitive Idea

S_n becomes large as a consequence of single large jump.

Dependent Case

Ideas from the Independent Case might not work.

Outline

Lognormal marginals with Gaussian Copula

Definition

Let $(Y_1, \ldots, Y_n)$ be a multivariate Gaussian random vector. Take $X_k = \mathrm{e}^{Y_k}$. The vector $(X_1, \ldots, X_n)$ is a lognormal random vector with gaussian copula.

Lognormal Random Variables as Heavy Tailed

- Light among Subexponential Distributions.
- All moments exist, however it does not have mgf.

Lognormal marginals with Gaussian Copula

Definition

Let $(Y_1, \ldots, Y_n)$ be a multivariate Gaussian random vector. Take $X_k = \mathrm{e}^{Y_k}$. The vector $(X_1, \ldots, X_n)$ is a lognormal random vector with gaussian copula.

Lognormal Random Variables as Heavy Tailed

- Light among Subexponential Distributions.
- All moments exist, however it does not have mgf.

Lognormal marginals with Gaussian Copula

Definition

Let $(Y_1, \ldots, Y_n)$ be a multivariate Gaussian random vector. Take $X_k = \mathrm{e}^{Y_k}$. The vector $(X_1, \ldots, X_n)$ is a lognormal random vector with gaussian copula.

Lognormal Random Variables as Heavy Tailed

- Light among Subexponential Distributions.
- All moments exist, however it does not have mgf.

Lognormal marginals with Gaussian Copula

Definition

Let $(Y_1, \ldots, Y_n)$ be a multivariate Gaussian random vector. Take $X_k = \mathrm{e}^{Y_k}$. The vector $(X_1, \ldots, X_n)$ is a lognormal random vector with gaussian copula.

Lognormal Random Variables as Heavy Tailed

- Light among Subexponential Distributions.
- All moments exist, however it does not have mgf.

Objective

Construct algorithms to estimate

$$\mathbb{P}(X_1 + \ldots + X_n > u)$$

where $(X_1, \ldots, X_n) \sim \mathrm{LN}(\overline{\mu}, \Sigma)$.

Main Contributions

Algorithm A

- ✓ Importance Sampling.
- ✓ Logarithmic efficient.

Algorithm B

- ✓ Conditional Monte Carlo.
- ✓ Logarithmic efficient.

Algorithm C

- ✓ Algorithm A or B plus IS.
- ✓ Bounded Relative Error.

Main Contributions

Algorithm A

- ✓ Importance Sampling.
- ✓ Logarithmic efficient.

Algorithm B

- ✓ Conditional Monte Carlo.
- ✓ Logarithmic efficient.

Algorithm C

- ✓ Algorithm A or B plus IS.
- ✓ Bounded Relative Error.

Main Contributions

Algorithm A

- ✓ Importance Sampling.
- ✓ Logarithmic efficient.

Algorithm B

- ✓ Conditional Monte Carlo.
- ✓ Logarithmic efficient.

Algorithm C

- ✓ Algorithm A or B plus IS.
- ✓ Bounded Relative Error.

Outline

Algorithm A

IS for Estimating the Probability of an Event A

Increases the probability of the event A while resembling the original distribution.

How to Build a Proposal

Remember that if $\widetilde{X} \sim \mathrm{LN}(\mu, \theta^2\sigma^2)$ then

$$\mathbb{E}(X) = e^{\mu+\theta^2\sigma^2/2} \qquad \mathbb{V}\mathrm{ar}(X_i) = e^{2\mu+2\theta^2\sigma^2} - e^{2\mu+\theta^2\sigma^2}$$

Hence, it seems reasonable to propose as IS distribution

$$\mathrm{LN}_n(\overline{\mu}, \theta^2(u)\Sigma)$$

Algorithm A

IS for Estimating the Probability of an Event A

Increases the probability of the event A while resembling the original distribution.

How to Build a Proposal

Remember that if $\widetilde{X} \sim \text{LN}(\mu, \theta^2\sigma^2)$ then

$$\mathbb{E}(X) = e^{\mu+\theta^2\sigma^2/2} \qquad \mathbb{V}\text{ar}(X_i) = e^{2\mu+2\theta^2\sigma^2} - e^{2\mu+\theta^2\sigma^2}$$

Hence, it seems reasonable to propose as IS distribution

$$\text{LN}_n(\overline{\mu}, \theta^2(u)\Sigma)$$

Algorithm A

IS for Estimating the Probability of an Event A

Increases the probability of the event A while resembling the original distribution.

How to Build a Proposal

Remember that if $\widetilde{X} \sim \text{LN}(\mu, \theta^2\sigma^2)$ then

$$\mathbb{E}\left(X\right) = e^{\mu+\theta^2\sigma^2/2} \qquad \mathbb{V}\text{ar}\left(X_i\right) = e^{2\mu+2\theta^2\sigma^2} - e^{2\mu+\theta^2\sigma^2}$$

Hence, it seems reasonable to propose as IS distribution

$$\text{LN}_n(\overline{\mu}, \theta^2(u)\Sigma)$$

Algorithm A

IS for Estimating the Probability of an Event A

Increases the probability of the event A while resembling the original distribution.

How to Build a Proposal

Remember that if $\widetilde{X} \sim \text{LN}(\mu, \theta^2\sigma^2)$ then

$$\mathbb{E}(X) = e^{\mu+\theta^2\sigma^2/2} \qquad \mathbb{V}\text{ar}(X_i) = e^{2\mu+2\theta^2\sigma^2} - e^{2\mu+\theta^2\sigma^2}$$

Hence, it seems reasonable to propose as IS distribution

$$\text{LN}_n(\overline{\mu}, \theta^2(u)\Sigma)$$

Algorithm A

Intuitive Idea

Let $\theta(u)$ grow moderately to as $u \to \infty$.

Formal Statement

Algorithm A is logarithmic efficient if and only if

$$log\ \theta(u) = \mathrm{o}\big(\log^2 u\big)$$

How to choose it?

A convenient way to choose $\theta(u)$ is as the solution of

$$\mathrm{e}^{\mu_1+\theta^2\sigma^2/2} + \ldots + \mathrm{e}^{\mu_n+\theta^2\sigma^2/2} = u.$$

Algorithm A

Intuitive Idea

Let $\theta(u)$ grow moderately to as $u \to \infty$.

Formal Statement

Algorithm A is logarithmic efficient if and only if

$$log\ \theta(u) = o\big(\log^2 u\big)$$

How to choose it?

A convenient way to choose $\theta(u)$ is as the solution of

$$e^{\mu_1+\theta^2\sigma^2/2} + \ldots + e^{\mu_n+\theta^2\sigma^2/2} = u.$$

Algorithm A

Intuitive Idea

Let $\theta(u)$ grow moderately to as $u \to \infty$.

Formal Statement

Algorithm A is logarithmic efficient if and only if

$$log\ \theta(u) = \mathrm{o}\big(\log^2 u\big)$$

How to choose it?

A convenient way to choose $\theta(u)$ is as the solution of

$$\mathrm{e}^{\mu_1+\theta^2\sigma^2/2} + \ldots + \mathrm{e}^{\mu_n+\theta^2\sigma^2/2} = u.$$

Algorithm B

Conditional Monte Carlo

Use all known information. Simulate less.

Key Ideas

Let $Z_1, \ldots, Z_n$ i.i.d. N(0, 1) r.v.'s and define

$$\widetilde{Z}_i := \frac{Z_i}{\sqrt{Z_1^2 + \ldots + Z_n^2}}$$

We know $Z_1^2 + \ldots + Z_n^2 \sim \chi_n^2$ and how to simulate $\widehat{Z}_i$'s.

Algorithm B

Conditional Monte Carlo

Use all known information. Simulate less.

Key Ideas

Let $Z_1, \ldots, Z_n$ i.i.d. $\mathrm{N}(0,1)$ r.v.'s and define

$$\widetilde{Z}_i := \frac{Z_i}{\sqrt{Z_1^2 + \ldots + Z_n^2}}$$

We know $Z_1^2 + \ldots + Z_n^2 \sim \chi_n^2$ and how to simulate $\widehat{Z}_i$'s.

Algorithm B

Conditional Monte Carlo

Use all known information. Simulate less.

Key Ideas

Let $Z_1, \ldots, Z_n$ i.i.d. $N(0,1)$ r.v.'s and define

$$\widetilde{Z}_i := \frac{Z_i}{\sqrt{Z_1^2 + \ldots + Z_n^2}}$$

We know $Z_1^2 + \ldots + Z_n^2 \sim \chi_n^2$ and how to simulate $\widehat{Z}_i$'s.

Algorithm B

Key Ideas

Take $R^2 \sim \chi^2_n$ and a decomposition $\Sigma = CC^*$ such that C is square.

$$(Y_1, \ldots, Y_n) := R\, C\, (\widetilde{Z}_1, \ldots, \widetilde{Z}_n)^t + \overline{\mu} \ \sim \ N(\overline{\mu}, \Sigma)$$

CMC Algorithm

$$\begin{aligned}\mathbb{P}(S_n > u) &= \mathbb{P}(\mathrm{e}^{Y_1} + \ldots + \mathrm{e}^{Y_n} > u) \\ &= \mathbb{E}\Big[\mathbb{P}\Big(\mathrm{e}^{R(C_{11}\widetilde{Z}_1 + \ldots + C_{1n}\widetilde{Z}_n) + \mu_1} + \ldots \\ &\qquad + \mathrm{e}^{R(C_{n1}\widetilde{Z}_1 + \ldots + C_{nn}\widetilde{Z}_n) + \mu_n} > u \,|\, \widetilde{Z}_1, \ldots, \widetilde{Z}_n\Big)\Big]\end{aligned}$$

Algorithm B

Key Ideas

Take $R^2 \sim \chi^2_n$ and a decomposition $\Sigma = CC^*$ such that C is square.

$$(Y_1, \ldots, Y_n) := R\,C\,(\widetilde{Z}_1, \ldots, \widetilde{Z}_n)^t + \overline{\mu} \;\sim\; N(\overline{\mu}, \Sigma)$$

CMC Algorithm

$$\begin{aligned}
\mathbb{P}(S_n > u) &= \mathbb{P}(\mathrm{e}^{Y_1} + \ldots + \mathrm{e}^{Y_n} > u) \\
&= \mathbb{E}\Big[\mathbb{P}\Big(\mathrm{e}^{R(C_{11}\widetilde{Z}_1 + \ldots + C_{1n}\widetilde{Z}_n) + \mu_1} + \ldots \\
&\qquad + \mathrm{e}^{R(C_{n1}\widetilde{Z}_1 + \ldots + C_{nn}\widetilde{Z}_n) + \mu_n} > u \big| \widetilde{Z}_1, \ldots, \widetilde{Z}_n\Big)\Big]
\end{aligned}$$

Algorithm B

CMC Algorithm

Simulate $\widetilde{\mathbf{Z}} = (\widetilde{Z}_1, \ldots, \widetilde{Z}_n)$ and return

$$\mathbb{P}\big(R < \Psi_1(u, \widetilde{\mathbf{Z}})\big) + \mathbb{P}\big(R > \Psi_2(u, \widetilde{\mathbf{Z}})\big)$$

Efficiency

The algorithm B has logarithmic efficiency.

Algorithm B

CMC Algorithm

Simulate $\widetilde{\mathbf{Z}} = (\widetilde{Z}_1, \ldots, \widetilde{Z}_n)$ and return

$$\mathbb{P}\big(R < \Psi_1(u, \widetilde{\mathbf{Z}})\big) + \mathbb{P}\big(R > \Psi_2(u, \widetilde{\mathbf{Z}})\big)$$

Efficiency

The algorithm B has logarithmic efficiency.

Preliminaries of Algorithm C

Asymptotic Result

A consequence of Asmussen and Rojas-Nandayapa (2008)

$$\mathbb{P}(S_n > u) \sim \mathbb{P}(\max\{X_i : i = 1, \ldots, n\} > u) \sim \sum_{i=1}^{n} \mathbb{P}(X_i > u)$$

in the Dependent Case.

Intuitive Ideas

- Asymptotically $\mathbb{P}(M_n > u)$ accounts for most of the total probability $\mathbb{P}(S_n > u)$.
- In the event $(M_n > u)$ the random variables $X_1, \ldots, X_n$ behave as independent random variables.

Preliminaries of Algorithm C

Asymptotic Result

A consequence of Asmussen and Rojas-Nandayapa (2008)

$$\mathbb{P}(S_n > u) \sim \mathbb{P}(\max\{X_i : i = 1, \dots, n\} > u) \sim \sum_{i=1}^{n} \mathbb{P}(X_i > u)$$

in the Dependent Case.

Intuitive Ideas

- Asymptotically $\mathbb{P}(M_n > u)$ accounts for most of the total probability $\mathbb{P}(S_n > u)$.
- In the event $\{M_n > u\}$ the random variables $X_1, \dots, X_n$ behave as independent random variables.

Preliminaries of Algorithm C

Asymptotic Result

A consequence of Asmussen and Rojas-Nandayapa (2008)

$$\mathbb{P}(S_n > u) \sim \mathbb{P}(\max\{X_i : i = 1, \ldots, n\} > u) \sim \sum_{i=1}^{n} \mathbb{P}(X_i > u)$$

in the Dependent Case.

Intuitive Ideas

- Asymptotically $\mathbb{P}(M_n > u)$ accounts for most of the total probability $\mathbb{P}(S_n > u)$.
- In the event $\{M_n > u\}$ the random variables $X_1, \ldots, X_n$ behave as independent random variables.

Preliminaries of Algorithm C

Asymptotic Result

A consequence of Asmussen and Rojas-Nandayapa (2008)

$$\mathbb{P}(S_n > u) \sim \mathbb{P}(\max\{X_i : i = 1, \ldots, n\} > u) \sim \sum_{i=1}^{n} \mathbb{P}(X_i > u)$$

in the Dependent Case.

Intuitive Ideas

- Asymptotically $\mathbb{P}(M_n > u)$ accounts for most of the total probability $\mathbb{P}(S_n > u)$.
- In the event $\{M_n > u\}$ the random variables $X_1, \ldots, X_n$ behave as independent random variables.

Preliminaries of Algorithm C

Key Idea

First proposed in Juneja (2008)

$$\mathbb{P}(S_n > u) = \mathbb{P}(S_n > u, M_n < u) + \mathbb{P}(M_n > u)$$

Estimation of $\mathbb{P}(S_n > u, M_n < u)$

The same as in Algorithms A and B. Smaller variance.

Estimation of $\mathbb{P}(M_n > u)$

Design a new method for the Gaussian Copula (IS).

Preliminaries of Algorithm C

Key Idea

First proposed in Juneja (2008)

$$\mathbb{P}(S_n > u) = \mathbb{P}(S_n > u, M_n < u) + \mathbb{P}(M_n > u)$$

Estimation of $\mathbb{P}(S_n > u, M_n < u)$

The same as in Algorithms A and B. Smaller variance.

Estimation of $\mathbb{P}(M_n > u)$

Design a new method for the Gaussian Copula (IS).

Preliminaries of Algorithm C

Key Idea

First proposed in Juneja (2008)

$$\mathbb{P}(S_n > u) = \mathbb{P}(S_n > u, M_n < u) + \mathbb{P}(M_n > u)$$

Estimation of $\mathbb{P}(S_n > u, M_n < u)$

The same as in Algorithms A and B. Smaller variance.

Estimation of $\mathbb{P}(M_n > u)$

Design a new method for the Gaussian Copula (IS).

Estimation of $\mathbb{P}(M_n > u)$

Importance Sampling Distribution

Take K supported over $\{1, \ldots, n\}$. Consider the distribution of

$$(X_1, \ldots, X_n | X_K > u)$$

Main Features

- ✓ We know how to simulate it.
- ✓ We know its density.
- ✓ It is supported exactly over $\{M_n > u\}$.

Distribution of K

Our proposal

$$\mathbb{P}(K = k) = \frac{\mathbb{P}(X_k > u)}{\sum_{\ell=1}^{n} \mathbb{P}(X_\ell > u)}.$$

Estimation of $\mathbb{P}(M_n > u)$

Importance Sampling Distribution

Take K supported over $\{1, \dots, n\}$. Consider the distribution of

$$(X_1, \dots, X_n | X_K > u)$$

Main Features

- ✓ We know how to simulate it.
- ✓ We know its density.
- ✓ It is supported exactly over $\{M_n > u\}$.

Distribution of K

Our proposal

$$\mathbb{P}(K = k) = \frac{\mathbb{P}(X_k > u)}{\sum_{\ell=1}^{n} \mathbb{P}(X_\ell > u)}.$$

Estimation of $\mathbb{P}(M_n > u)$

Importance Sampling Distribution

Take K supported over $\{1, \ldots, n\}$. Consider the distribution of

$$(X_1, \ldots, X_n | X_K > u)$$

Main Features

- ✓ We know how to simulate it.
- ✓ We know its density.
- ✓ It is supported exactly over $\{M_n > u\}$.

Distribution of K

Our proposal

$$\mathbb{P}(K = k) = \frac{\mathbb{P}(X_k > u)}{\sum_{\ell=1}^{n} \mathbb{P}(X_\ell > u)}.$$

Algorithm C

Efficiency

The following algorithms has Bounded Relative Error

- The IS algorithm for $\mathbb{P}(M_n > u)$.
- Algorithm C for $\mathbb{P}(S_n > u)$ based on Algorithm A.
- Algorithm C for $\mathbb{P}(S_n > u)$ based on Algorithm B.

Algorithm C

Efficiency

The following algorithms has Bounded Relative Error

- The IS algorithm for $\mathbb{P}(M_n > u)$.
- Algorithm C for $\mathbb{P}(S_n > u)$ based on Algorithm A.
- Algorithm C for $\mathbb{P}(S_n > u)$ based on Algorithm B.

Algorithm C

Efficiency

The following algorithms has Bounded Relative Error

- The IS algorithm for $\mathbb{P}(M_n > u)$.
- Algorithm C for $\mathbb{P}(S_n > u)$ based on Algorithm A.
- Algorithm C for $\mathbb{P}(S_n > u)$ based on Algorithm B.

Algorithm C

Efficiency

The following algorithms has Bounded Relative Error

- The IS algorithm for $\mathbb{P}(M_n > u)$.
- Algorithm C for $\mathbb{P}(S_n > u)$ based on Algorithm A.
- Algorithm C for $\mathbb{P}(S_n > u)$ based on Algorithm B.

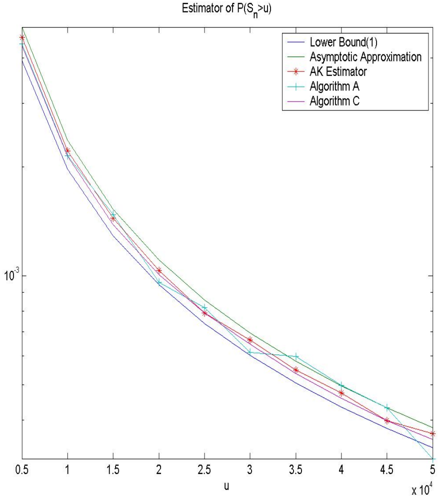
Outline

Example 1

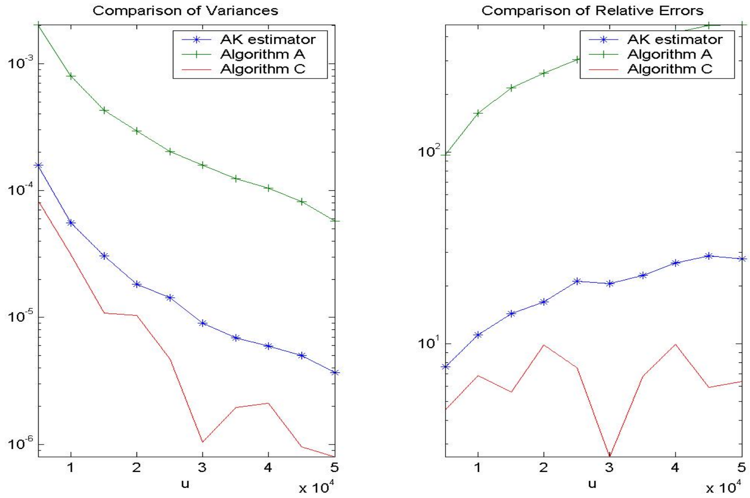
Example

10 lognormal r.v. with Gaussian Copula

- $\mu_i = i - 10$
- $\sigma_i^2 = i$
- $\sigma_{ij} = 0.4\sigma_i\sigma_j$
- $R = 10000$ *(Estimator)*
- $R = 1000000$



Example 1


Comparison of Variances
AK estimator
Algorithm A
Algorithm C
10^-3
10^-4
10^-5
10^-6
1
2
3
4
5
u
x 10^4
Comparison of Relative Errors
AK estimator
Algorithm A
Algorithm C
10^2
10^1
1
2
3
4
5
u
x 10^4