State-dependent Large-deviations Based Importance Sampling for a Slow-down Tandem Queue

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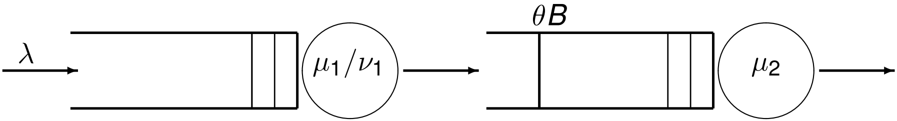
Outline

Outline

A Slow-down Tandem Queue



Arrival process:	*Poisson*(λ)
Service rates:	*exp*(μ_1), exp(ν_1), exp(μ_2)
Slow-down mechanism:	$\nu_1 < \mu_1$

Underlying processes

- Joint queue-length process

$$Q(t) = (Q_1(t), Q_2(t))$$

- Scaled queue-length process

$$X(t) = Q(Bt)/B$$

- Embedded (discrete-time) queue-length process

$$X_k = (X_{1,k}, X_{2,k})$$

Goal

Find probability of collecting B jobs in the second buffer before emptying the system, starting from any state

Formally

$$p_B^x := \mathbb{P}(X_2(T_B) = 1 | X(0) = x)$$

where

$$T_B := \min\{t > 0 | X_2(t) = 1 \text{ or } X(t) = 0\}$$

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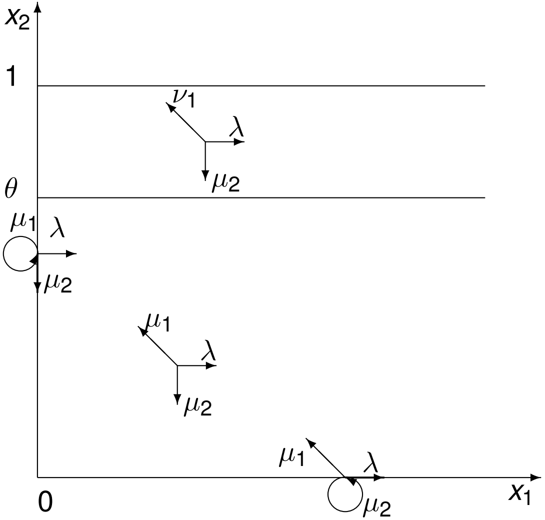
State Space



x_2
1
ν_1
λ
μ_2
θ
μ_1
λ
μ_2
μ_1
λ
μ_2
μ_1
λ
0
μ_2
x_1

Method

- State-dependent Importance Sampling
  - simulating the system under the new measure $\mathbb{Q}$, such that $\mathbb{Q}(X_2(T_B) = 1) \gg \mathbb{P}(X_2(T_B) = 1)$
  - the new measure is changing dynamically

- Asymptotic efficiency

$$\lim_{B \to \infty} \frac{\log \mathbb{E}(L^2 \mathbb{I})}{\log \mathbb{E}(L \mathbb{I})} \geq 2$$

Here, L is the likelihood ratio of the particular sample path; $\mathbb{I}(X_2(T_B) = 1) = 1$ and $\mathbb{I}(X(T_B) = 0) = 0$

Outline

Cost-functions

Following holds for any Poisson process $N(t)$ with intensity λ

$$\mathbb{P}(N(t) \approx \tilde{\lambda} t) \approx e^{I(\tilde{\lambda}|\lambda)t}$$

where $I(\tilde{\lambda}|\lambda)$ is the cost-function, given by

$$I(\tilde{\lambda}|\lambda) := \sup_{\theta} \left(\theta\tilde{\lambda} - \log \mathbb{E} e^{\theta N(1)} \right)$$

after some optimization

$$I(\tilde{\lambda}|\lambda) = \lambda - \tilde{\lambda} + \tilde{\lambda} \log \frac{\tilde{\lambda}}{\lambda}$$

Cost of the path

The cost of moving one unit up in the interior under the new measure is

$$\frac{I(\tilde{\lambda}|\lambda) + I(\tilde{\mu}_1|\mu_1) + I(\tilde{\mu}_2|\mu_2)}{\tilde{\mu}_1 - \tilde{\mu}_2} \qquad \text{if } x_2 < \theta \text{, and}$$

$$\frac{I(\tilde{\lambda}|\lambda) + I(\tilde{\nu}_1|\nu_1) + I(\tilde{\mu}_2|\mu_2)}{\tilde{\nu}_1 - \tilde{\mu}_2} \qquad \text{if } x_2 \geq \theta$$

$\gamma(x)$ is the minimal cost of the path $x \to (\cdot, 1)$

Large deviations result

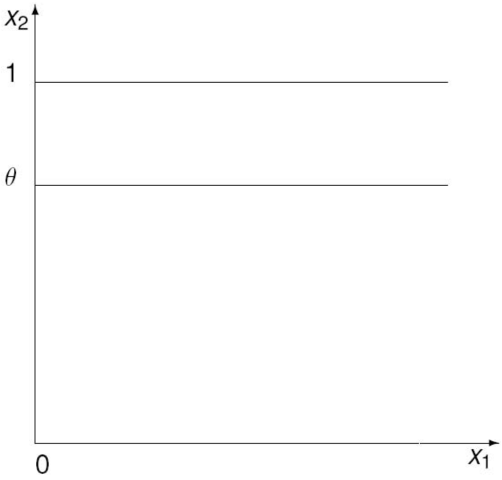
Theorem

$$\lim_{B \to \infty} \frac{1}{B} \log p_B^x = -\gamma(x)$$

Remark

$\gamma(x)$ is the exponential decay rate of the interest probability.

Five subsets with constant jump rates



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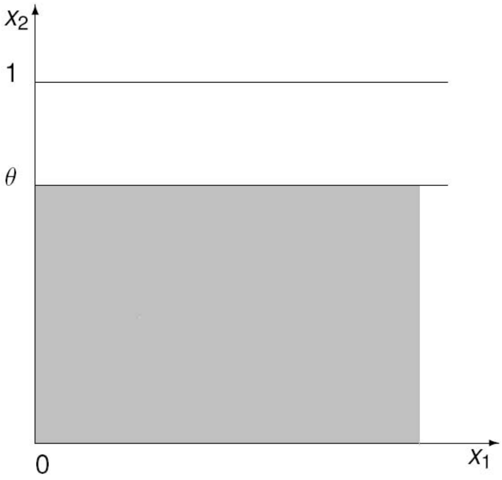

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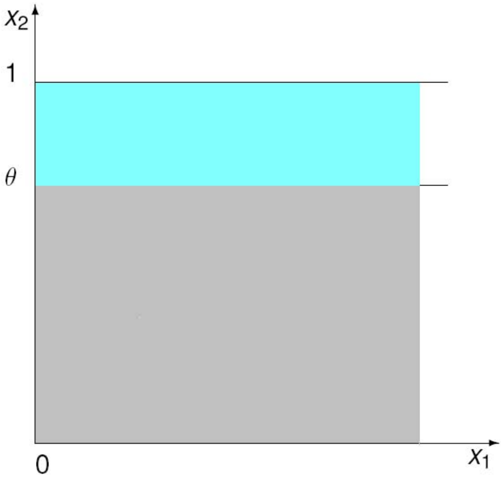

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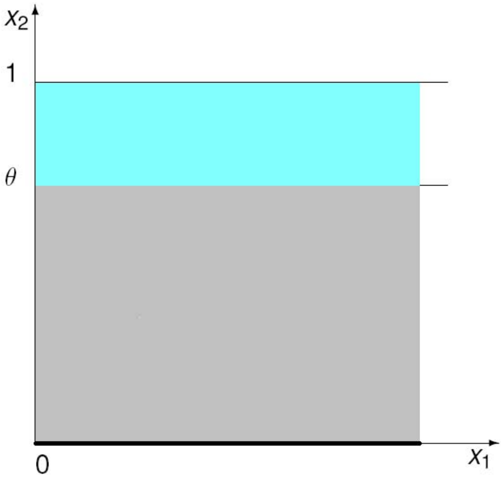

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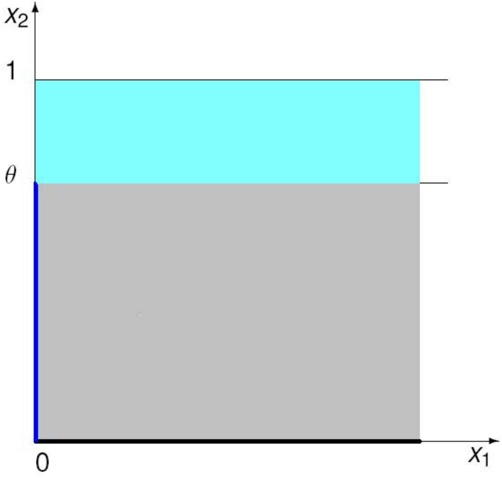

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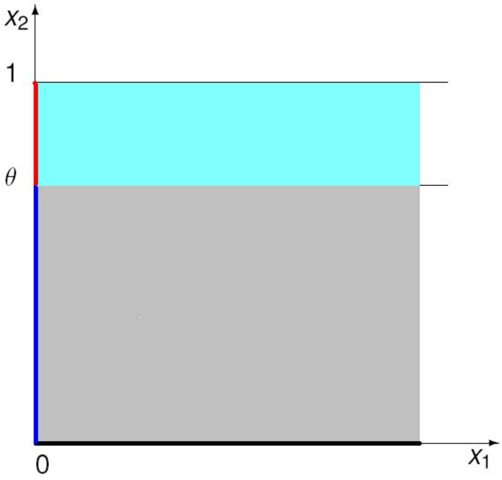

The path structure

Lemma

The optimal path between any two states in the same subset is a straight line.

Lemma

The optimal path does not have more than

- *one subpath in each subset if* $\mu_2 < \mu_1$*, and*
- *two subpaths in each subset if* $\mu_2 \geq \mu_1$

The optimal path illumination

- Number of paths is limited (by Lemmas)
- Cost of each candidate is known
- So, we can find the shape of the optimal path to overflow in the second queue

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Outline

Three possible cases

- $\mu_2 < \nu_1 < \mu_1$ i.e., the second server is the bottleneck
- $\nu_1 \leq \mu_2 < \mu_1$ i.e., the shifting bottleneck
- $\nu_1 < \mu_1 \leq \mu_2$ i.e., the first server is the bottleneck

We compute new jump rates

$$(\tilde{\lambda}(x), \tilde{\mu}_1(x), \tilde{\mu}_2(x)) \text{ and } (\tilde{\lambda}^+(x), \tilde{\nu}_1^+(x), \tilde{\mu}_2^+(x))$$

below and above the slow-down threshold

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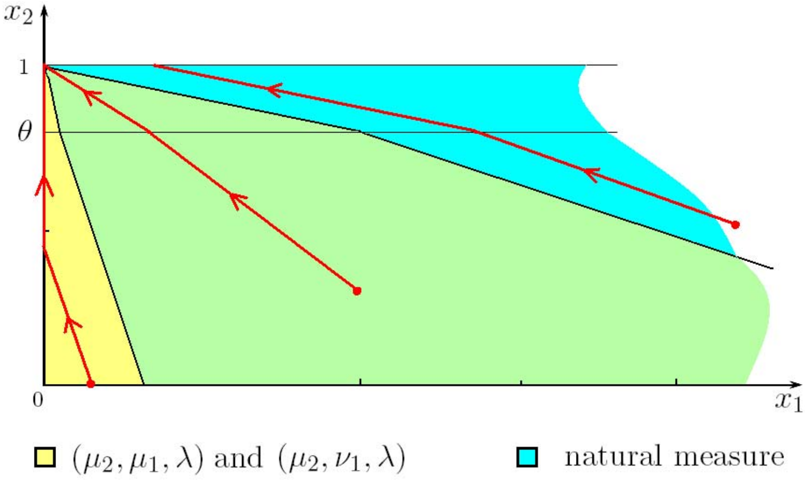
below and above the slow-down threshold

The second server is the permanent bottleneck

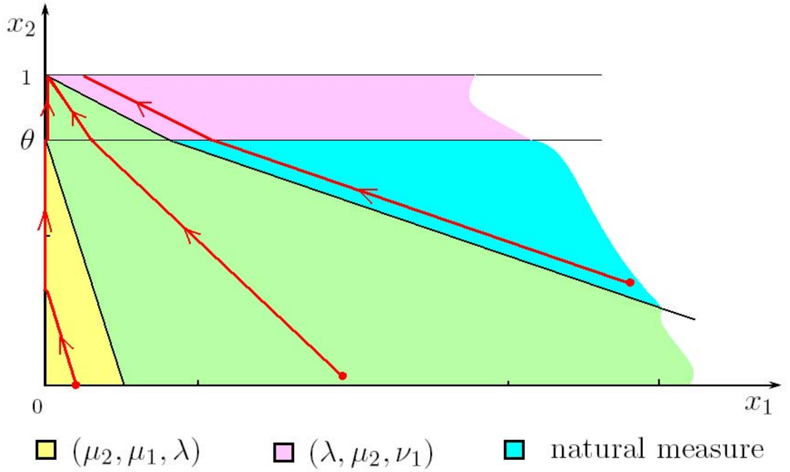


The green area

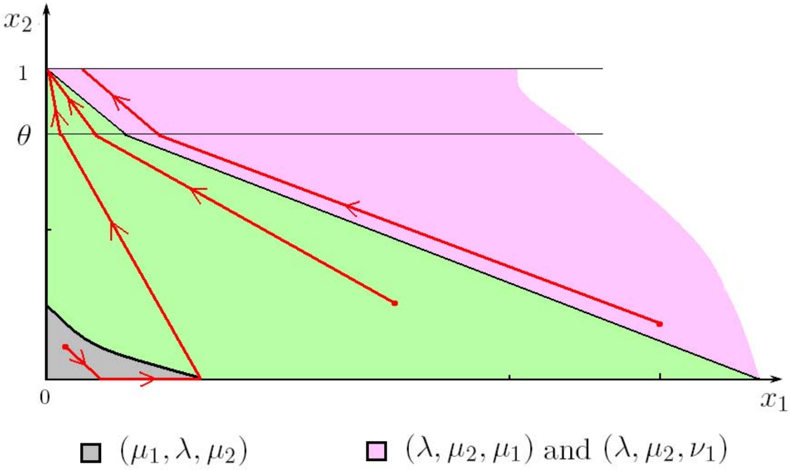
Solve

$$\begin{cases} \tilde\lambda = \tilde\mu_1 + \frac{\kappa^*(x)-x_1}{\theta-x_2}(\tilde\mu_1-\tilde\mu_2) \\ \tilde\lambda+\tilde\mu_1+\tilde\mu_2 = \lambda+\mu_1+\mu_2 \\ \tilde\lambda\tilde\mu_1\tilde\mu_2 = \lambda\mu_1\mu_2 \\ \\ \tilde\lambda^+ = \tilde\nu_1^+ - \frac{\kappa^*(x)}{1-\theta}(\tilde\nu_1^+-\tilde\mu_2^+) \\ \tilde\lambda^+ + \tilde\nu_1^+ + \tilde\mu_2^+ = \lambda+\nu_1+\mu_2 \\ \tilde\lambda^+\tilde\nu_1^+\tilde\mu_2^+ = \lambda\nu_1\mu_2 \\ \\ \tilde\lambda \le \tilde\mu_1, \quad \tilde\mu_1 > \tilde\mu_2, \quad \tilde\lambda^+ \le \tilde\nu_1^+ \text{ and } \tilde\nu_1^+ > \tilde\mu_2^+ \\ \tilde\lambda, \tilde\mu_1, \tilde\mu_2, \tilde\lambda^+, \tilde\nu_1^+, \tilde\mu_2^+ > 0 \\ \\ \kappa^*(x) := x_1 - \frac{\tilde\mu_1-\tilde\lambda}{\tilde\mu_1-\tilde\mu_2}(\theta-x_2) = y_1 + \frac{\tilde\nu_1^+-\tilde\lambda^+}{\tilde\nu_1^+-\tilde\mu_2^+}(y_2-\theta), \end{cases}$$

The shifting bottleneck

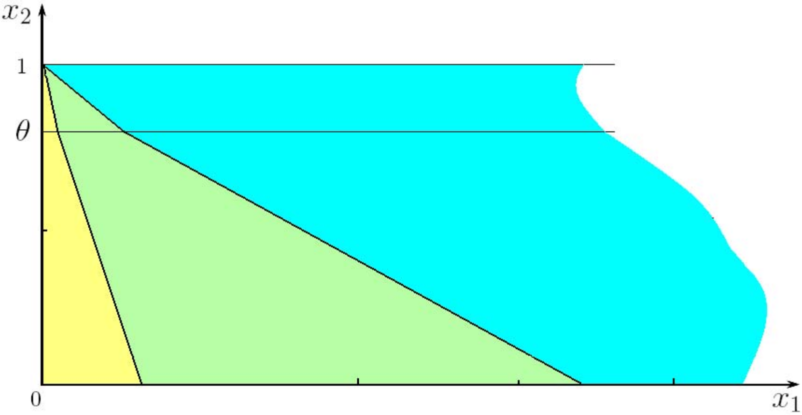


The first server is the permanent bottleneck



Outline

The protection line



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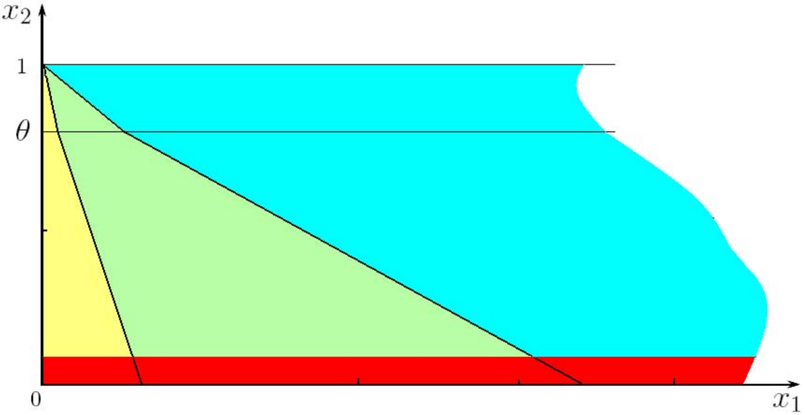

$W-$function

We define

$$
\begin{aligned}
W_1(x) &= 2\gamma_1(x_1, x_2) + 2\gamma_2(\kappa^*(x), \theta) - \delta && \text{if } x_2 < \theta, \\
W_1(x) &= 2\gamma_2(x_1, x_2) - \delta, && \text{if } x_2 \geq \theta \\
W_2(x) &= 2\gamma(x_1, \delta/2\gamma) - \delta, \\
W_3(x) &= 2\gamma - 3\delta,
\end{aligned}
$$

where

$$\gamma_1(x_1, x_2) := -(x_1 - \kappa^*(x)) \log \frac{\tilde{\lambda}(x_1, x_2)}{\lambda} - (\theta - x_2) \log \frac{\tilde{\mu}_2(x_1, x_2)}{\mu_2}$$

$$\gamma_2(\kappa^*(x), \theta) := -\kappa^*(x) \log \frac{\tilde{\lambda}^+(\kappa^*(x), \theta)}{\lambda} - (1 - \theta) \log \frac{\tilde{\mu}_2^+(\kappa^*(x), \theta)}{\mu_2},$$

$W-$function, mollification

$$W(x) = -\epsilon \log \sum_{i=1}^{3} e^{-W_i(x)/\epsilon}$$

Assumption

For positive parameters $\delta \equiv \delta_B$ and $\epsilon \equiv \epsilon_B$ the following holds:

$$\lim_{B\to\infty} \epsilon_B = 0, \qquad \lim_{B\to\infty} \delta_B = 0,$$

$$\lim_{B\to\infty} B\epsilon_B = \infty, \qquad \lim_{B\to\infty} \frac{\epsilon_B}{\delta_B} = 0$$

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The new measure

$$
\begin{aligned}
\bar{\lambda}(x) &= \lambda e^{-\langle DW(x), v_0\rangle/2} e^{\mathbb{H}(DW(x))/2}, && \text{if } x_2 < \theta \\
\bar{\mu}_i(x) &= \mu_i e^{-\langle DW(x), v_i\rangle/2} e^{\mathbb{H}(DW(x))/2}, && \text{if } x_2 < \theta, i = 1, 2, \\
\bar{\lambda}^+(x) &= \frac{\lambda}{\lambda+\nu_1+\mu_2} e^{-\langle DW(x), v_0\rangle/2} e^{\mathbb{H}^+(DW(x))/2}, && \text{if } x_2 \geq \theta \\
\bar{\nu}_1^+(x) &= \frac{\nu_1}{\lambda+\nu_1+\mu_2} e^{-\langle DW(x), v_1\rangle/2} e^{\mathbb{H}^+(DW(x))/2}, && \text{if } x_2 \geq \theta, \\
\bar{\mu}_2^+(x) &= \frac{\mu_2}{\lambda+\nu_1+\mu_2} e^{-\langle DW(x), v_2\rangle/2} e^{\mathbb{H}^+(DW(x))/2}, && \text{if } x_2 \geq \theta,
\end{aligned}
$$

Normalization

$$\mathbb{H}(DW(x)) =$$

$$2\log\left[\lambda e^{-\langle DW(x),v_0\rangle/2} + \mu_1 e^{-\langle DW(x),v_1\rangle/2} + \mu_2 e^{-\langle DW(x),v_2\rangle/2}\right]^{-1}$$

and

$$\mathbb{H}^{+}(DW(x)) =$$

$$2\log\left[\lambda e^{-\langle DW(x),v_0\rangle/2} + \nu_1 e^{-\langle DW(x),v_1\rangle/2} + \mu_2 e^{-\langle DW(x),v_2\rangle/2}\right]^{-1}$$

Outline

Likelihood ratio analysis

$$\begin{aligned}\log L(A) &= \frac{B}{2}\sum_{j=0}^{\sigma-1}\langle DW(X_j), X_{j+1}-X_j\rangle \\ &+ \sum_{k=1}^{2}\frac{1}{2}\sum_{j=0}^{\sigma-1}\langle DW(X_j), v_k\rangle I\{X_j = X_{j+1}, X_{k,j}=0\} \\ &- \frac{1}{2}\sum_{j=0}^{\sigma-1}\left(\mathbb{H}(DW(X_j))I_{\{X_{2,j}<\theta\}} + \mathbb{H}^+(DW(X_j))I_{\{X_{2,j}\geq\theta\}}\right).\end{aligned}$$

Minor terms

Lemma

For some positive constants γ^* *and* δ^*

$$\sum_{k=1}^{2} \frac{1}{2} \sum_{j=0}^{\sigma-1} \langle DW(X_j), v_k \rangle I\{X_j = X_{j+1} \in \partial_k\} \leq \gamma^* e^{-\delta^*/\epsilon} \sigma$$

Lemma

$$-\frac{1}{2}\left(\mathbb{H}(DW(X_j)) I_{\{X_j \in D\}} + \mathbb{H}^+(DW(X_j)) I_{\{X_j \in D^+\}}\right)$$
$$\leq -\tfrac{1}{2} e^{-(\theta - \frac{\delta}{2\gamma})\gamma/\epsilon} \mathbb{H}^+(DW_2(0, \theta))$$

Major term

Lemma

For any path $A = (X_j, j = 0, ..., \sigma)$, positive constants C and C^+

$$\frac{B}{2}\left|\sum_{j=0}^{\sigma-1}\langle DW(X_j), X_{j+1} - X_j\rangle - (W(X_\sigma) - W(X_0))\right| \leq \frac{C}{B\epsilon}\sigma + C^+\sigma^+,$$

where σ^+ is the number of the slow-down threshold crossings .

Lemma

For any υ_B such that $\lim_{B\to\infty}\upsilon_B = 0$ and T_B:

$$\lim_{B\to\infty}\frac{1}{B}\log\mathbb{E}(e^{\upsilon_B T_B}|X_2(T_B) = 1) = 0.$$

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Asymptotic efficiency

Theorem

$$\lim_{B\to\infty} \frac{1}{B} \log \mathbb{E}\left[L(A^x)\mathbb{I}(X_2(T_B)=1)\right] \le -2\gamma(x) = 2 \lim_{B\to\infty} \frac{1}{B} \log p_B^x,$$

The new measure

$(\bar{\lambda}(x), \bar{\mu}_1(x), \bar{\mu}_2(x))$ and $(\bar{\lambda}^+(x), \bar{\nu}_1^+(x), \bar{\mu}_2^+(x))$

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- The method is flexible and open for generalizing
- Slight modification of this method provides easy-to-implement and asymptotically efficient IS schemes

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