# Overview of State-Dependent Importance Sampling for Simulating Jackson Networks

Victor F. Nicola

{vfnicola@ieee.org}

# Outline

- Motivation
- Model and Notation
- MAP-Based Approach
  (Kroese and Nicola 1999)
- Adaptive IS Heuristics
  (Rubinstein 1999, de Boer and Nicola 2002)
- State-Dependent Heuristics
  (Nicola and Zaburnenko 2004)
- Game-Based Approach
  (Dupuis, Sezer and Wang 2007)
- Conclusions and Further Research

# Motivation

- Efficient simulation of rare events in queueing models (e.g., buffer overflow)

- Applications
  - Computer Systems:
  performance/dependability evaluation
  - Telecommunication Networks:
  QoS management and control
  - Manufacturing:
  scheduling and resource allocation

- Importance sampling (IS) technique
  - State-independent (static) IS
  (does not always work!)
  - state-dependent (dynamic) IS
  (generally more effective!)

- Brief overview of recent advances in dynamic IS

## Model and Notation

- Jackson queueing network with $n$ nodes

  $\lambda_i$ external arrival rate at Node $i$

  $\mu_i$ service rate at Node $i$

  $p_{ij}$ routing probability from $i$ to $j$

  $p_{ie}$ routing probability from $i$ to $exit$

  $\Lambda_i$ total arrival rate at Node $i$
  (from traffic equations):

$$\Lambda_i = \lambda_i + \sum_{\forall j} \Lambda_j \, p_{ji}, \; i = 1, \ldots, n$$

  $\Lambda_i < \mu_i, \; i = 1, \ldots, n$ (all queues are stable!)

# Model and Notation (contd.)

- Notation

  - $X_{i,t}$ $(i = 1, \ldots, n)$ number at Node $i$ at $t \geqslant 0$

  - $\mathbf{X}_t = (X_{1,t}, \ldots, X_{n,t})$, $t \geqslant 0$, Markov process

  - $S_t = \sum_{\forall i} \beta_i \; X_{i,t}$ , with $\beta_i = \{0, 1\}, i = 1, ..., n$

  When $\forall i \, \beta_i = 1$, $S_t \equiv$ network population

  When only $\beta_i = 1$, $S_t \equiv$ buffer $i$ content

- Starting from $S_0 = i \geqslant 1$

  $\tau_K$ first time $S_t = K$

  $\tau_0$ first time $S_t = 0$

  - rare event of interest:

  $S_t$ hits $K$ before return to 0

  - associated probability: $\gamma_i(K) = \mathbb{P}(\tau_K < \tau_0)$

# State-Independent (Static) IS

- heuristics based on large deviations, e.g.,
  (Parekh and Walrand 1989)
  (Frater et al. 1989, 1991 and 1994)
  (Juneja and Nicola 2005)

- heuristics based on effective bandwidth, e.g.,
  (Chang et al. 1994)
  (De Veciana et al. 1994)
  (L'Ecuyer and Champoux 2001)

- not always asymptotically efficient!
  (Glasserman et al. 1995 and 1997)
  (Randhawa and Juneja 2004)
  (de Boer 2004)

# State-Depdependent (Dynamic) IS

- Two basic ideas:
  - approximate the zero-variance C.O.M.

  - "push" the system close to the most likely path to the rare set (MLP C.O.M.)

- Zero-variance and MLP C.O.M.s:
  - not known (require the true probability!)
  - state-dependent (even for the $M/M/1$ queue!)
  - strong dependence close to boundaries (when one or more queues are empty!)
  - weak or no dependence in the interior

- Challenges:
  - determine C.O.M.s (at boundaries/interior)
  - appropriately combine these C.O.M.s
  - dependence range along boundaries is crucial for asymptotic efficiency!
  - boundaries increase with number of queues

# State-Depdependent (Dynamic) IS (contd.)

- Generally more effective than static IS!

- Formal MAP approach
  (Kroese and Nicola 2002)
  - for 2-node tandem networks
  - provably effective
  - difficult for networks with many nodes?

- Adaptive IS heuristics
  - cross-entropy (CE) (pioneerd by Rubinstein)
  - CE for QN models (de Boer and Nicola 2002)
  - stoch. approximation (Ahamed et al. 2005)
  - less effective for networks with many nodes:
  excessive computational effort;
  convergence problems?

# State-Depdependent (Dynamic) IS (contd.)

- State-dependent IS heuristics
  (Nicola and Zaburnenko 2007)
  - less computational effort than adaptive IS
  - effective for tandem/parallel topologies
  - not developed for general Jackson networks?

- Formal game-theoretic approach
  (Dupuis and Wang 2008)
  - rigorous optimal control framework
  - provably effective
  - implementation and performance problems
  for networks with many nodes?

## MAP-Based Approach

- First (formal) approach to state-dependent IS (Kroese and Nicola 1999, 2002)

- Stable two-node tandem network $(\lambda, \mu_1, \mu_2)$

- Rare event: starting from $(X_{1,0} = i,\ X_{2,0} = 1)$, second buffer $(X_{2,t})$ hits some high level $K$ before returning to 0; associated prob. $\gamma_i(K)$

- $(X_{1,t})$: birth-death process $(\lambda, \mu_1)$

  $(D_t)$: departure process from Node 1

  $(E_t)$: pure death process (rate $\mu_2$)

- Define $S_t \stackrel{\text{def}}{=} X_{2,0} + (D_t - E_t),\ t \geq 0,$

  then $S_t \equiv X_{2,t}\,,\ \ 0 \leqslant t \leqslant \min\{\tau_0, \tau_K\}$

# MAP-Based Approach (contd.)

- $(X_{1,t}, S_t)$: Markov additive process (MAP)

  $(X_{1,t})$: modulating (driving) process;
  continuous-time birth-death process $(\lambda, \mu_1)$

  $(S_t)$: modulated (additive) process,
  with increments $D_t$ and death rate $\mu_2$

- $M_t(\theta)$: matrix of moment gen. functions

  $(M_t(\theta))_{i,j} \stackrel{\text{def}}{=} \mathbf{E}_i \, \mathrm{e}^{\theta S_t} \, I_{\{X_t = j\}}, \; 0 \le i, j \le \infty$

- Then $M_t(\theta) = \mathrm{e}^{t\,G(\theta)}$, $t \ge 0$, with $G(\theta)$ given by

$$\begin{pmatrix} -\lambda - \mu_2 + \mu_2 \,\mathrm{e}^{-\theta} & \lambda & & \\ \mu_1 \,\mathrm{e}^{\theta} & -\lambda - \mu_1 - \mu_2 + \mu_2 \,\mathrm{e}^{-\theta} & \lambda & \\ & \ddots & \ddots & \ddots \\ & & \mu_1 \,\mathrm{e}^{\theta} & -\mu_1 - \mu_2 + \mu_2 \,\mathrm{e}^{-\theta} \end{pmatrix}$$

# MAP-Based Approach (contd.)

- $\kappa(\theta)$: largest positive eigen value of $G(\theta)$

- $\mathbf{w}(\theta) = \{w_i(\theta),\ i = 0, 1, \ldots\}$:
right-eigenvector corresponding to $\kappa(\theta)$

- Exponential C.O.M.: MAP $(\tilde{X}_{1,t}, \tilde{S}_t)$ given by

  $(\tilde{X}_{1,t})$: modulating process; a continuous-time birth-death process with conjugate rates

$$\tilde{\lambda}(i) = G_{i,i+1}(\theta)\,\frac{w_{i+1}(\theta)}{w_i(\theta)},\ \ i = 0, 1, \ldots$$

$$\tilde{\mu}_1(i) = G_{i,i-1}(\theta)\,\frac{w_{i-1}(\theta)}{w_i(\theta)},\ \ i = 1, 2, \ldots$$

  rates depend on $i$ (content of the first buffer)

- $(\tilde{S}_t)$: modulated additive process, with increments $\tilde{D}_t$ and death rate $\tilde{\mu}_2 = \mu_2\,\mathrm{e}^{-\theta}$

# MAP-Based Approach (contd.)

- To estimate $\gamma_i(K)$, perform IS simulation with the optimal tilting parameter $\theta^*$, as determined from $\kappa(\theta^*) = 0$ (Asmussen and Rubinstein 1995)

- IS estimator is provably asymptotically efficient!

- MAP-based approach application to larger networks seems difficult?

- Conclusion: more research on state-dep. IS!

# Adaptive IS Heuristics

- The method of Cross-Entropy (CE) for rare event simulation (Rubinstein 1999, de Boer and Nicola 2000, 2002)

- The CE method is an iterative procedure; each iteration involves two basic steps:

  1. generate samples according to a given prob. measure (specified by a set of parameters)

  2. update parameters of the prob. measure (based on the samples collected in Step 1) to produce "better" samples in the next iteration

- The goal is to converge to a prob. measure sufficiently close to the zero-variance C.O.M.

## Adaptive IS Heuristics (contd.)

- Consider the simulation of an $M/M/1$ queue

  - original prob. measure $f(t;\mathbf{v})$:
  $\mathbf{v} = (\lambda, \mu)$ (with $\lambda + \mu = 1$)

  - for overflow level $K$, we wish to estimate $\gamma(K)$ using importance sampling

  - change of measure $f(t;\widehat{\mathbf{v}})$:
  $\widehat{\mathbf{v}} = (\widehat{\lambda}, \widehat{\mu})$ (also, $\widehat{\lambda} + \widehat{\mu} = 1$)

- The CE method is used to determine an optimal parameter vector $\widehat{\mathbf{v}}^*$

# Adaptive IS Heuristics (contd.)

- The CE algorithm (to determine $\hat{\mathbf{v}}^*$):

1. Set $\hat{\mathbf{v}}_0 = \mathbf{v}$ and $j = 1$ (iteration counter)

2. Update $\hat{\mathbf{v}}_j$:

- generate cycles $w_1, \ldots, w_m$ using $f(t; \hat{\mathbf{v}}_{j-1})$

- select the "best" $1\%$ cycles (with highest levels). Set $K_j \leqslant K$ to the lowest level reached

- update $\hat{\mathbf{v}}_j = (\hat{\lambda}_j, \hat{\mu}_j)$ (CE minimization):

$$\hat{\lambda}_j = \frac{\sum_{i=1}^m I_i(K_j)\, L_i \times A_{ij}}{\sum_{i=1}^m I_i(K_j)\, L_i \times (A_{ij} + D_{ij})}, \qquad \hat{\mu}_j = 1 - \hat{\lambda}_j$$

$A_{ij}$: arrivals in cycle $i$ before reaching $K_j$

$D_{ij}$: departures in cycle $i$ before reaching $K_j$

3. if $K_j = K$, set $\hat{\mathbf{v}}^* = \hat{\mathbf{v}}_j$ and go to Step 4; otherwise, set $j = j + 1$ and go to Step 2

4. use $\hat{\mathbf{v}}^*$ to estimate $\gamma(K)$ via IS

# Adaptive IS Heuristics (contd.)

- The CE algorithm can be used to optimize a state-dependent C.O.M. $\underline{\widehat{v}} = (\underline{\widehat{\lambda}}, \underline{\widehat{\mu}})$ with

  $\underline{\widehat{\lambda}} = (\widehat{\lambda}(0), \ldots, \widehat{\lambda}(b))$

  $\underline{\widehat{\mu}} = (\widehat{\mu}(1), \ldots, \widehat{\mu}(b))$

  $b$ is the dependence range along boundary (generally critcical for asymptotic efficiency!)

- Adaptive IS:

  - robust and effective for small networks

  - less effective for large networks:
  excessive computational effort
  convergence problems?

- Related work based on stoch. approximation (Ahamed, Borkar and Juneja 2005)

## State-Dependent Heuristics

- Goal: "push" the system close to the most likely path (MLP) to the rare set (Nicola and Zaburnenko 2007)

- Methodology:

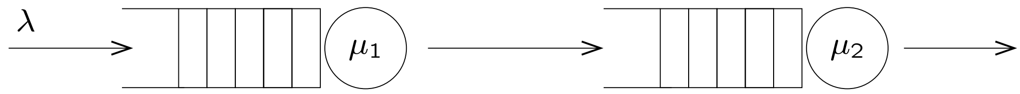
  - approximate the MLP C.O.M. along a few important boundaries and in the interior of the state-space (e.g., via time-reversal arguments)

  - appropriately combine these C.O.M.s (e.g., via simple linear interpolation)

  - set/adjust the dependence range (boundary thickness) to attain asymptotic efficiency

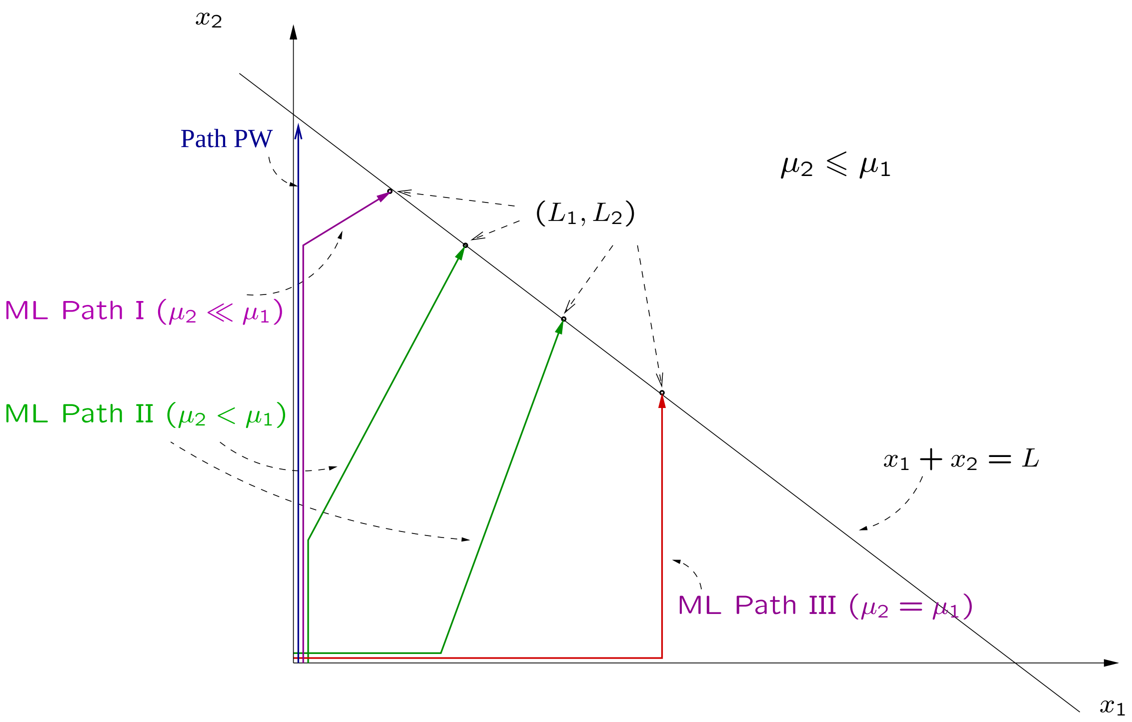
# State-Dependent Heuristics (contd.)

Time reversal of the 2-node tandem network



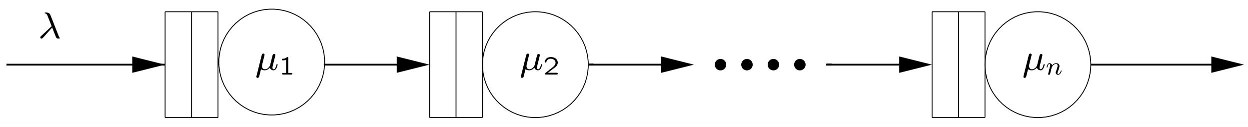

a) Forward time process


b) Reverse time process



# State-Dependent Heuristics (contd.)

SDH for $n$-node tandem networks




$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$

From an empty network:

- initially $(x_i = 0, \forall i)$, no change of measure;

- as $x_1 \geqslant 1$, start 'pushing' Node 1;

- as $x_2 \geqslant 1$, gradually go to 'pushing' Node 2;

- as $x_3 \geqslant 1$, gradually go to 'pushing' Node 3;

- etc.

# State-Dependent Heuristics (contd.)

- Advantages:

  - relatively simple to develop and implement

  - no computational effort to determine C.O.M.

  - empirical results show effectiveness for

  tandem/parallel networks
  with any number of nodes

  feed-forward/feedback networks
  of small and moderate size

- Challenges:

  - approximating MLP C.O.M. along boundaries

  - guidelines to set the dependence range

  - proof of asymptotic efficiency

# Game-based Approach

- A formal game-theoretic foundation for the development of provably efficient state-dependent IS schemes (Dupuis, Sezer and Wang 2007)

- Goal: "push" the system along the most likely path (MLP) to the rare set

- Methodology:

  - stochastic optimal control formulation to minimize the variance of the IS estimator (in the limit as $K \to \infty$, converges to a deterministic optimal control problem)

  - approximate solution to the associated DPE provides "key ingredients" to determine MLP C.O.M.s along distinct boundaries

  - a proper weighted sum of these C.O.M.s yields an asymptotically efficient IS scheme

# Game-based Approach (contd.)

- Advantage:

  - rigorous and systematic framework for the construction of asymptotically efficient IS schemes for Jackson networks

  - simultaneous estimation of probabilities for different overflow events

- Challenges:

  - despite asymptotic efficiency, performance is sensitive to boundary thickness; rel. error may grow quickly (more than linear) with $K$

  - number of boundaries $(2^n - 1)$ increases exponentially with the number of nodes $(n)$

  - implementation and performance issues for large networks

# Conclusions and Further Research

- Significant and promising recent advances

  - Adaptive IS heuristics

  - State-dependent IS heuristics

  - rigorous control-theoretic approach

- Further research

  - formal study of boundary thickness' impact on asymptotic efficiency and actual performance

  - effectiveness for large networks (applicability and performance issues)

  - reversibility-based approach (work in progress)

  - extensions to non-Jackson queueing networks