# Sensitivity Analysis of Risk Functionals

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Happy birthday, Reuven!

# Sensitivities

Let $Y_x(t)$ be a stochastic process on $(\Omega, \mathcal{F}, \mu_x)$, where $x$ is a real parameter.

For any functional $\mathcal{A}_{\mu_x}[Y_x(\cdot)]$, we are interested in the sensitivity w.r.t. the parameter $x$, i.e. the gradient (subgradient, supergradient)

$$\frac{\partial}{\partial x}\mathcal{A}_{\mu_x}[Y_x(\cdot)].$$

Compare: The score function method (Rubinstein & Shapiro), perturbation analysis, etc.

# Examples

- Stochastic Optimization, especially for decision dependent probabilities:
  Queueing, Service, Manufacturing, Renewal Systems
  $\mathcal{A}[Y_x(\cdot)] = \mathbb{E}[f(Y_x(T))]$, the performance of the system at time $T$, where $f$ is a cost function, or
  $\mathcal{A}[Y_x(\cdot)] = \int_t^T \mathbb{E}[f(Y_x(t))]\, dt$, the integrated transient behavior
  or
  $\mathcal{A}[Y_x(\cdot)] = \mathbb{E}[f(Y_x(\infty))]$, the stationary behavior.
- Sensitivity Analysis, especially in Finance:
  - Calculation of the "Greeks"
  - Sensitivity of Risk functionals

# Weak derivatives

Let $(R, d)$ be a metric space. To the family of Borel probabilities on $(R, d)$, we associate a "dual space" $\mathbb{F}$ such as

- the space of all bounded, continuous functions
- the space of all continuous functions $f$, such that $|f(u)| \leq K_1 + K_2 d^p(u, u_0)$
- the space of all bounded measurable function
- the space of all measurable functions $f$, such that $|f(u)| \leq K_1 + K_2 d^p(u, u_0)$

**Definition.** The family of probability measures $(\mu_x)_{x \in X \subseteq \mathbb{R}}$ on $R$ is *weakly differentiable w.r.t. the dual space* $\mathbb{F}$, if there is a finite signed measure $\mu'_x$ such that for all $f \in \mathbb{F}$

$$\frac{1}{s}\left[\int f(w)\, d\mu_{x+s}(w) - \int f(w)\, d\mu_x(w)\right] \to \int f(w)\, d\mu'_x(w)$$

as $s \to 0$. (Heidergott, Vasquez-Abad, Leahu, G.P.)

# Decomposing the weak derivative

Any finite signed measure may be decomposed into its positive and negative part (Jordan decomposition). Since $\int 1 d\mu_x = 1$, we have that $\int 1 d\mu'_x = 0$, i.e. the positive and the negative part have the same mass. Thus we may decompose the derivative object $\mu'_x$

$$\mu'_x = c_x(\dot{\mu}_x - \ddot{\mu}_x)$$

where $\dot{\mu}_x$ and $\ddot{\mu}_x$ are probability measures. The representation as a multiple of the difference of two probability measures $\mu'_x = c(\mu_1 - \mu_2)$ is not unique, however the constant $c$ is minimal if the two parts $\mu_1$ and $\mu_2$ are orthogonal, i.e. if the decomposition is the Jordan decomposition.

Any triplet $(c_x, \dot{\mu}_x, \ddot{\mu}_x)$, such that for $f \in \mathbb{F}$

$$\begin{aligned}\frac{1}{s}\left[\int f(w)\, d\mu_{x+s}(w)\right. &- \left.\int f(w)\, d\mu_x(w)\right] \\ &\to c_x\left[\int f(w)\, d\dot{\mu}_x(w) - \int f(w)\, d\ddot{\mu}_x(w)\right],\end{aligned}$$

for $s \to 0$, is called a *weak derivative triplet*.

# Examples for weak derivatives

| Distribution $\mu_x$ ($x$ varies) | Constant $c_x$ | Positive part of the derivative: $\dot{\mu}_x$ | Negative part of the derivative: $\ddot{\mu}_x$ |
|---|---|---|---|
| Poisson$(x)$ | 1 | Poisson$(x) + 1$ | Poisson$(x)$ |
| Normal$(x, \sigma^2)$ | $1/\sigma\sqrt{2\pi}$ | $x$ + Weibull$(2, \frac{1}{2\sigma^2})$ | $x$ - Weibull$(2, \frac{1}{2\sigma^2})$ |
| Normal$(m, x^2)$ | $1/x$ | ds-Maxwell$(m, x^2)$ | Normal$(m, x^2)$ |
| Exponential$(x)$ | $1/x$ | Exponential$(x)$ | $x^{-1}$ Erlang(2) |
| Gamma$(\alpha, x)$ | $\alpha/x$ | Gamma$(\alpha, x)$ | Gamma$(\alpha + 1, x)$ |
| Weibull$(\alpha, x)$ | $1/x$ | Weibull$(\alpha, x)$ | $[\text{Gamma}(2, x)]^{1/\alpha}$ |

# Use of weak derivatives in sensitivity estimation

If $\dot{Y}_x$ resp. $\ddot{Y}_x$ are distributed according to $\dot{\mu}_x$ resp. $\ddot{Y}_x$, then for $f \in \mathbb{F}$

$$c_x[f(\dot{Y}_x) - f(\ddot{Y}_x)]$$

is a consistent estimate of $\frac{\partial}{\partial x}\mathbb{E}[f(Y_x)]$.
(Academic) Example. Let $F(x) = \mathbb{E}[\cos(Y_x)]$, where $Y_x$ is a Normal(0,$x^2$) variable. Then

$$1/x[\cos(\dot{Y}_x) - \cos(\ddot{Y}_x)]$$

where

$$\dot{Y}_x \sim doublesidedMaxwell(0, x^2) \text{ and } \ddot{Y}_x \sim Normal(0, x^2)$$

is a consistent estimate for $\frac{\partial}{\partial x}F(x)$.
Notice that no limits of infinitesimal quantities appear and that we do not have to know the derivative of cos.

# Coupling

A probability measure $\bar{\mu}$ on $R \times R$ has marginals $\mu_1$ and $\mu_2$, if

$$
\begin{aligned}
[\bar{\mu} J_1](A) &:= \int \gamma(A \times dy) = \mu_1(A) \\
[\bar{\mu} J_2](B) &:= \int \gamma(dx \times B) = \mu_2(B).
\end{aligned}
$$

$J_1, J_2$ are the *projection operators*. A *coupling* of two probability measures $\mu_1$ and $\mu_2$ w.r.t. $h$ is a probability measure $\bar{\mu}$ on $R \times R$ with given marginals $\mu_i$, which minimizes the expectation of the criterion function $h(u,v)$, i.e. is the solution of the following optimization problem:

$$
\left\|
\begin{array}{l}
\text{Minimize } \int h(u,v)\, d(\bar{\mu}(u,v)) \\
\text{subject to} \\
\gamma J_1 = \mu_1, \\
\gamma J_2 = \mu_2, \\
\gamma \text{ is a probability on } R^2
\end{array}
\right.
$$

The solution may not be unique. We denote the solution (or the set of solutions) by

$$\mu_1 \underset{h}{©} \mu_2$$

and call it "$\mu_1$ and $\mu_2$ coupled over $h$".

# Example

| Distribution $\mu_x$ ($x$ varies) | Constant $c_x$ | Positive part of the derivative: $\dot{\mu}_x$ | Negative part of the derivative: $\ddot{\mu}_x$ |
|---|---|---|---|
| Poisson($x$) | 1 | Poisson($x$) + 1 | Poisson($x$) |

Coupling over the Euclidean distance leads to taking $\dot{Y} = Y_x + 1$, $\ddot{Y} = Y_x$, i.e.:

For any integrable cost function $f$ and any Poisson variable $Y_x \sim \text{Poisson}(x)$ we have

$$\frac{\partial}{\partial x}\mathbb{E}[f(Y_x)] = \mathbb{E}[f(Y_x + 1)] - \mathbb{E}[f(Y_x)]$$

with very low variance.

Another example from my 1996 book:
Estimation of sensitivity of $x \mapsto \mathbb{E}[\sqrt{Y_x}]$, where
$Y_x \sim \mathit{Exponential}(x)$.
Then the variances are

| | |
|---|---|
| Numerical differences | 1069.9 |
| Score function method | 2.81 |
| Weak derivatives with coupling | 0.022 |

# Scenario processes and information processes

In multi-period decision problems, one is confronted with a stochastic process $Y_x(t)$, which we call the *scenario process* and the information process, which describes the available information. Information is measured in terms of filtrations (increasing sequences of $\sigma$-algebras).

Artzner's Example:

A fair coin is tossed three times. The payoff process $\xi^{(A)}$ is

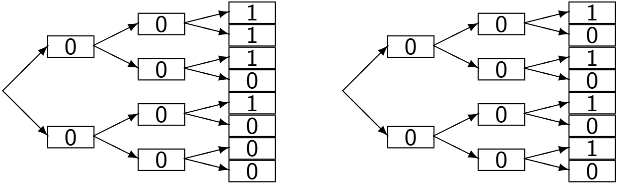
$$\xi_1^{(A)} = 0; \quad \xi_2^{(A)} = 0;$$

$$\xi_3^{(A)} = \begin{cases} 1 & \text{if heads is shown at least two times} \\ 0 & \text{otherwise} \end{cases}$$

We compare this process to another payoff process

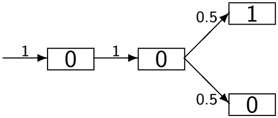
$$\xi_1^{(B)} = 0; \quad \xi_2^{(B)} = 0;$$

$$\xi_3^{(B)} = \begin{cases} 1 & \text{if heads is shown at least the last throw} \\ 0 & \text{otherwise} \end{cases}$$




Left: the process-and-information pair A, Right: the process-and-information pair B

In contrast, if we consider only the filtration generated by the payoff process $\xi^{(A)}$ or $\xi^{(B)}$, we get the following tree:




Notice that for finite probability spaces, filtrations and trees are equivalent. To formalize the concept of filtrations (and processes adapted to them) in a version-independent way, we introduce the notion of nested distributions.

# Nested distributions

Let $(\Xi, d)$ be a Polish space, i.e. complete separable metric space and let $\mathcal{P}_1(\Xi, d)$ be the family of all Borel probability measures $P$ on $(\Xi, d)$ such that

$$\int d(u, u_0)\, dP(u) < \infty$$

for some $u_0 \in \Xi$.
For two Borel probabilities, $P$ and $Q$ in $\mathcal{P}_1(\Xi, d)$, let $d(P, Q)$ denote the Kantorovich distance

$$d(P, Q) = \sup\{ \int f(u)\, dP(u) - \int f(u)\, dQ(u) : |f(u) - f(v)| \leq d(u, v)\}$$

$d$ metrizises the weak topology on $\mathcal{P}_1$.
$\mathcal{P}_1$ is a complete separable metric space (Polish space) under $d$.
Iterate the argument: $\mathcal{P}_1(\mathcal{P}_1(\Xi, d), d)$ is a polish space, a space of distributions over distributions (i.e. what Bayesians would call a hyperdistribution).

If $(\Xi_1, d_1)$ and $(\Xi_2, d_2)$ are Polish spaces then so is the Cartesian product $(\Xi_1 \times \Xi_2)$ with metric

$$d^2((u_1, u_2), (v_1, v_2)) = d_1(u_1, v_1) + d_2(u_2, v_2).$$

Consider some metric $d$ on $\mathbb{R}^m$, which makes it Polish (it needs not to be the Euclidean one). Then we define the following spaces

$$\begin{aligned} \Xi_1 &= (\mathbb{R}^m, d) \\ \Xi_2 &= (\mathbb{R}^m \times \mathcal{P}_1(\Xi_1, d), d^2) = (\mathbb{R}^m \times \mathcal{P}_1(\mathbb{R}^m, d), d^2) \\ \Xi_3 &= (\mathbb{R}^m \times \mathcal{P}_1(\Xi_2, d), d^2) = (\mathbb{R}^m \times \mathcal{P}_1(\mathbb{R}^m \times \mathcal{P}_1(\mathbb{R}^m, d), d^2), d^2) \\ &\vdots \\ \Xi_T &= (\mathbb{R}^m \times \mathcal{P}_1(\Xi_{T-1}, d), d^2) \end{aligned}$$

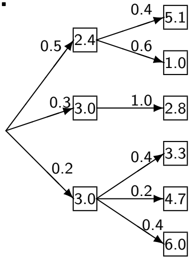
All spaces $\Xi_1, \ldots, \Xi_T$ are Polish spaces and they may carry probability distributions.

**Definition.** A Borel probability distribution $\mathbb{P}$ with finite first moment on $\Xi_T$ is called a *nested distribution of depth* $T$.
For any nested distribution $\mathbb{P}$, there is an embedded multivariate distribution $P$. We illustrate this for depth 2: Let $\mathbb{P}$ be a nested distribution on $\Xi_2$, which has components $\eta$ (a real random variable) and $\mu$ (a random distribution). Then

$$P(A_1 \times A_2) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{\eta \in A_1\}} \mu(A_2)].$$

The projection from the nested distribution to the embedded distribution is not injective:

**Example.**



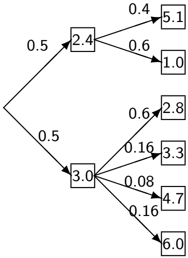

$$\left[\begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ \hline 3.0 & 3.0 & 2.4 \\ \left[\begin{array}{ccc} 0.4 & 0.2 & 0.4 \\ \hline 6.0 & 4.7 & 3.3 \end{array}\right] & \left[\begin{array}{c} 1.0 \\ \hline 2.8 \end{array}\right] & \left[\begin{array}{cc} 0.6 & 0.4 \\ \hline 1.0 & 5.1 \end{array}\right] \end{array}\right]$$

The embedded multivariate, but non-nested distribution of the stochastic process can be gotten from it:

$$\begin{bmatrix} 0.08 & 0.04 & 0.08 & 0.3 & 0.3 & 0.2 \\ \hline 3.0 & 3.0 & 3.0 & 3.0 & 2.4 & 2.4 \\ 6.0 & 4.7 & 3.3 & 2.8 & 1.0 & 5.1 \end{bmatrix}$$

Evidently, this multivariate distribution has lost the information about the nested structure. If one considers the filtration generated by the scenario process itself and forms the pertaining nested distribution, one gets

$$\begin{bmatrix} 0.5 & 0.5 \\ \hline 3.0 & 2.4 \\ \begin{bmatrix} 0.16 & 0.08 & 0.16 & 0.6 \\ \hline 6.0 & 4.7 & 3.3 & 2.8 \end{bmatrix} & \begin{bmatrix} 0.6 & 0.4 \\ \hline 1.0 & 5.1 \end{bmatrix} \end{bmatrix}$$



# Markov Chains as nested distributions

Markov Chains are special cases of nested distributions.
Let, for instance, $\gamma = (\gamma(1), \gamma(2), \gamma(3))$ be a starting distribution and let $P$ be a transition matrix on the state space $(z_1, z_2, z_3)$

$$P = \begin{pmatrix} p(1,1) & p(1,2) & p(1,3) \\ p(2,1) & p(2,2) & p(2,3) \\ p(3,1) & p(3,2) & p(3,3) \end{pmatrix}$$

The pertaining nested distribution is

$$\left[ \begin{array}{ccc} \gamma(1) & \gamma(2) & \gamma(3) \\ \hline z_1 & z_2 & z_3 \\ \left[ \begin{array}{ccc} p(1,1) & p(1,2) & p(1,3) \\ \hline z_1 & z_2 & z_3 \\ [\ldots] & [\ldots] & [\ldots] \end{array} \right] & \left[ \begin{array}{ccc} p(2,1) & p(2,2) & p(2,3) \\ \hline z_1 & z_2 & z_3 \\ [\ldots] & [\ldots] & [\ldots] \end{array} \right] & \left[ \begin{array}{ccc} p(3,1) & p(3,2) & p(3,3) \\ \hline z_1 & z_2 & z_3 \\ [\ldots] & [\ldots] & [\ldots] \end{array} \right] \end{array} \right]$$

# Derivatives of nested distributions

Let $\mathbb{P}_x$ be a family of nested distributions indexed by $x \in \mathbb{R}$. We consider the derivative object $\mathbb{P}'_x$ in the sense of weak derivatives for Borel measures in Polish spaces.
To begin with, let us consider the derivative of discrete probability measures

$$\text{distribution family: } \left[\begin{array}{cccc} p_x(1) & p_x(2) & \dots & p_x(m) \\ \hline z_1 & z_2 & \dots & z_m \end{array}\right]$$

$$\text{derivative object: } \left[\begin{array}{cccc} p'_x(1) & p'_x(2) & \dots & p'_x(m) \\ \hline z_1 & z_2 & \dots & z_m \end{array}\right]$$

triplet representation:

$$c_X \left[\begin{array}{cccc} \dot{p}_x(1) & \dot{p}_x(2) & \dots & \dot{p}_x(m) \\ \hline z_1 & z_2 & \dots & z_m \end{array}\right], \left[\begin{array}{cccc} \ddot{p}_x(1) & \ddot{p}_x(2) & \dots & \ddot{p}_x(m) \\ \hline z_1 & z_2 & \dots & z_m \end{array}\right]$$

We write the nested distribution $\mathbb{P}_x$ as

$$(\mathbb{P}_x^{(1)}, \mathbb{P}_x^{(2)}, \ldots, \mathbb{P}_x^{(T)})$$

i.e. as the composition of the conditional distributions, given the filtration. Notice that the filtration does not change with $x$.

# The Leibniz law for nested distributions

Let

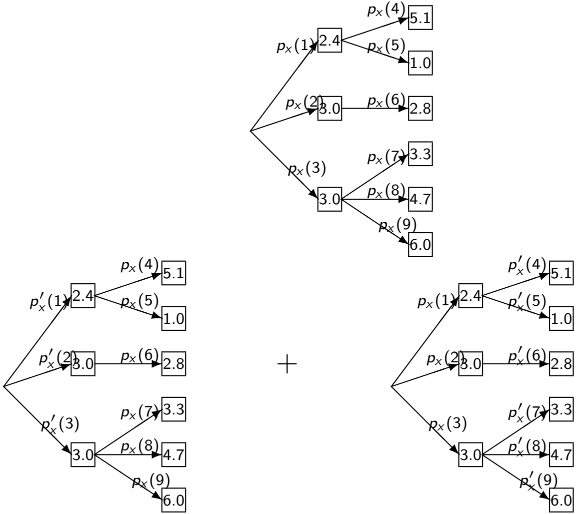
$$\mathbb{P}_x = (\mathbb{P}_x^{(1)}, \mathbb{P}_x^{(2)}, \ldots, \mathbb{P}_x^{(T)})$$

be a nested distribution. Then the derivative object is

$$\mathbb{P}'_x = \sum_{t=1}^{T} (\mathbb{P}_x^{(1)}, \mathbb{P}_x^{(2)}, \ldots \mathbb{P}_x'^{(t)}, \ldots \mathbb{P}_x^{(T)})$$

which may be decomposed into

$$\begin{aligned} \mathbb{P}'_x &= \sum_{t=1}^{T} c_x^{(t)} [(\mathbb{P}_x^{(1)}, \ldots, \mathbb{P}_x^{(t-1)}, \dot{\mathbb{P}}_x^{(t)}, \mathbb{P}_x^{(t+1)}, \ldots, ) \\ &- (\mathbb{P}_x^{(1)}, \ldots, \mathbb{P}_x^{(t-1)}, \ddot{\mathbb{P}}_x(t), \ldots \mathbb{P}_x^{(T)})]. \end{aligned}$$



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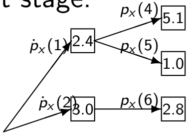
# Triplet representation

First stage:

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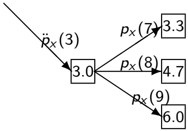
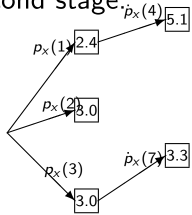



Second stage:

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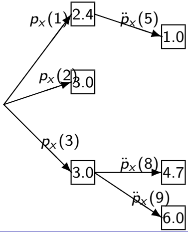
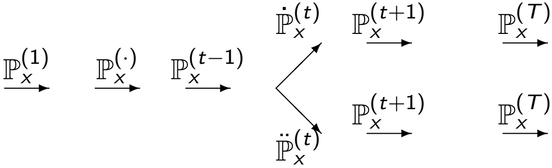


# Using weak derivatives and coupling

Do for every $t$




or choose $t$ randomly with uniform distribution in $\{1, \ldots, T\}$.

# Single-period functionals

Let $\mathcal{A}(Y)$ be a version-independent probability functional, i.e. if $Y_1$ and $Y_2$ have the same distributions, then $\mathcal{A}(Y_1) = \mathcal{A}(Y_2)$, irrespective on which probability space the random variables are defined.

We assume that $Y \mapsto \mathcal{A}(Y)$ is concave, i.e. for all $0 \le \lambda \le 1$

$$\mathcal{A}(\lambda Y_1 + (1-\lambda)Y_2) \ge \lambda \mathcal{A}(Y_1) + (1-\lambda)\mathcal{A}(Y_2).$$

Concave u.s.c. functionals have the (dual) representation

$$\mathcal{A}(Y) = \inf\{\int \mathbb{E}[Y \cdot Z] - \mathcal{A}^+(Z) : Z \in \mathcal{Z}\}$$

where $\mathcal{A}^+$ is the (concave) dual functional.

Examples are

- $\mathcal{A}(Y) = \mathbb{E}(Y)$
- $\mathcal{A}(Y) = \mathbb{E}(Y) - \frac{1}{2}\mathbb{M}\mathrm{ad}(Y)$
- $\mathcal{A}(Y) = \mathbb{E}(Y) - \mathbb{S}\mathrm{td}(Y)$
- $\mathcal{A}(Y) = \mathbb{AV}@\mathrm{R}_\alpha(Y) = \frac{1}{\alpha}\int_0^1 G_Y^{-1}(p)\, dp$

# Sensitivity of concave functionals

Suppose that

- the functional $\mathcal{A}(Y)$ is version-independent, concave and has the representation

$$\mathcal{A}\{\mu_x\} = \inf\{\mathbb{E}_{\mu_x}[Y \cdot Z] - \mathcal{A}^+_{\mu_x}(Z) : Z \in \mathcal{Z}\}, \qquad (1)$$

- $x \mapsto \mu_x$ is (appropriately) weakly differentiable,
- $x \mapsto \mathcal{A}^+_{\mu_x}(Z)$ is differentiable for all $Z \in \mathcal{Z}$.

Then the mapping $x \mapsto \mathcal{A}\{\mu_x\}$ is (super-)differentiable with supergradient set

$$\partial_x \mathcal{A}\{\mu_x\} = \left\{\mathbb{E}_{\mu'_x}[Y \cdot Z] - \frac{\partial}{\partial x}\mathcal{A}^+_{\mu_x}(Z) : Z \in \mathrm{argmin}\{(1)\}\right\}.$$

($Y$ is identity here)

## Example: Sensitivity of the average value-at-risk

Suppose that we choose the dual space $\mathbb{F}$ as the continuous functions with at most linear growth. The $\mathbb{AV@R}$: $\mathbb{AV@R}_\alpha\{\mu_x\} = \frac{1}{\alpha}\int_0^\alpha G_x^{-1}(u)\,du$, where $G_x = \int_0^u d\mu_x$ has the following dual representation

$$\mathbb{AV@R}_\alpha\{\mu_x\} = \inf\{\mathbb{E}_{\mu_x}[Y \cdot Z] : \mathbb{E}_{\mu_x}[Z] = 1, 0 \le Z \le 1/\alpha\}.$$

The minimizer is (cum grano salis)

$$Z = \frac{1}{\alpha}\mathbb{1}_{\{Y \le G_x^{-1}(\alpha)\}}.$$

The full supergradient set is

$$\partial_x \mathbb{AV@R}_\alpha\{\mu_x\} = conv\{\int \frac{y}{\alpha}\mathbb{1}_{\{y<u\}} + \frac{y}{\alpha}\left(\frac{\alpha - \alpha_{G_x}^-}{\alpha_{G_x}^+ - \alpha_{G_x}^-}\right)\mathbb{1}_{\{y=u\}}\, d\mu_x'(y)$$
$$u \in [G_x^{-1}(\alpha), G_x^{-1}(\alpha+)]\}$$

where $\alpha_{G_x}^+ = G_x(G_x^{-1}(\alpha)), \alpha_{G_x}^- = \lim_{h\downarrow 0} G_x(G_x^{-1}(\alpha) - h)$.

# Sensitivity of the multiperiod average value-at-risk

Let $\mathbb{P}_x$ be a nested distribution of depth $T$.
The multiperiod average value-at-risk is defined as

$$\mathbb{AV@R}(\mathbb{P}_x) = \sum_{t=1}^{T} \mathbb{E}[\mathbb{AV@R}_\alpha(Y_x(t)|\mathcal{F}_{t-1})]$$

The supergradient of this functional is

$$\begin{aligned} & \sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}_x'^{(t-1)}}[\mathbb{AV@R}_\alpha(Y_x(t)|\mathcal{F}_{t-1})] \\ + & \sum_{t=1}^{T} \mathbb{E}_{\mathbb{P}_x^{(t-1)}}[\mathbb{E}_{\mathbb{P}_x'^{(t)}}[Y_x(t)Z_x(t)]|\mathcal{F}_{t-1})] \end{aligned}$$

where

$$Z_x(t) = \frac{1}{\alpha}\mathbb{1}_{\{Y<G_x^{-1}(t)\}} + \frac{1}{\alpha}\left(\frac{\alpha - \alpha^-_{G_x(t)}}{\alpha^-_{G_x(t)}\alpha^-_{G_x(t)}}\right)\mathbb{1}_{\{Y=G_x^{-1}(t)(\alpha)\}}$$

# Conclusions

- The triplet representation of derivative objects allows to implement low variance estimates through coupling.
- There is something beyond chains of conditional distributions: nested distributions. They capture also the information aspect through filtrations.
- Not only the sensitivity of expectations can be calculated: By virtue of the supergradient representation of functionals, we may represent their sensitivities also as integrals w.r.t. the derivative objects. This allows to find estimation formulas using the triplet representation and coupling.