Randomized methods based on new Monte Carlo schemes for control and optimization

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Efficient Monte Carlo: From Variance Reduction to Combinatorial Optimization — A Conference on the Occasion of R.Y.Rubinstein's 70th Birthday (Soenderborg)

Outline

- "Ideal" Monte Carlo and convergence estimates
- Implementable random algorithms: boundary oracle
  - Hit-and-Run
  - Shake-and-Bake
- Numerical simulation
- Conclusions

Historical background

- First random search methods
  Rastrigin (1960-ies)
- Pessimism on effectiveness of randomized algorithms
  Nemirovski and Yudin (1983)
- Revival of randomized approaches for optimization
  Bertsimas and Vempala (2004), Polyak and Shcherbakov (2006) and Dabbene (2008), Campi (2008)

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Problem statement

$$\min \; c^T x$$
$$\text{s.t. } x \in X$$

X is a convex bounded closed set in $\mathbb{R}^n$ with nonempty interior

Remark: Any convex optimization problem

$$\min \; f(x)$$
$$\text{s.t. } x \in Q,$$
$$Q \text{ and } f \text{ are convex}$$

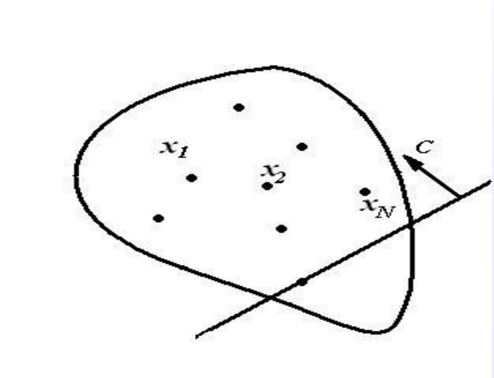
can be converted to this format.

"Ideal" Monte Carlo

$x_1, x_2, \ldots x_N$ independent uniformly distributed points in X

Cutting plane method

1. Set $X_1 = X$
2. Generate uniform $x_1, x_2, \ldots x_N \in X_k$
3. Find $f_k = \min c^T x_i$
4. Set $X_{k+1} = X_k \bigcap \{x : c^T x \leq f_k\}$
   go to Step 2.



Convergence estimates

$$f^* = \max_{x \in X} \; c^T x, \quad f_* = \min_{x \in X} \; c^T x, h = f^* - f_*$$

Theorem

After k iterations of the algorithm

$$E\left[f_k\right] - f_* \leq q^k, \quad q = \frac{h}{n} B\left(N+1, \frac{1}{n}\right),$$

where $B(a,b)$ is Euler beta-function.

Case of a special interest: $N = 1, \; k = 1$

Theorem

Let x_1 be a random point uniformly distributed in X. Then

$$E\left[c^T x_1\right] - f_* \leq h\left(1 - \frac{1}{n+1}\right).$$

Radon theorem and center of gravity method

Remark:

$$E\left[x_1\right] = g \text{ (center of gravity of } X\text{)}, \quad \Rightarrow \quad c^T g - f_* \leq h\left(1 - \frac{1}{n+1}\right)$$

[Radon theorem (1916)]

Deterministic version:

center of gravity method

$$x^k = g^k, \quad X_{k+1} = X_k \bigcap \{x : c^T x \leq c^T g^k\}$$

Implementable random algorithms: boundary oracle

uniform random samples $\Rightarrow$ *asymptotically* uniformly distributed via Markov-chain Monte Carlo schemes

Given $x_0 \in X$, d — vector specifying the direction in $\mathbb{R}^n$

Boundary oracle

$L = \{t \in \mathbb{R} : x^0 + td \in X\}$

For convex sets $L = (\underline{t}, \bar{t})$,
where $\underline{t} = \inf\{t : x^0 + td \in X\}$, $\bar{t} = \sup\{t : x^0 + td \in X\}$

Complete boundary oracle

$L = \{t \in \mathbb{R} : x^0 + td \in X\}$ + inner normals to X at the boundary points

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Boundary oracle is available for numerous sets

- LMI set

$$X = \left\{ x \in \mathbb{R}^n : A_0 + \sum_{i=1}^{n} x_i A_i \leq 0 \right\}$$

- LMI constrained set of symmetric matrices P

$$X = \left\{ P : AP + PA^T + C \leq 0,\ P \geq 0 \right\}$$

- Quadratic matrix inequalities set

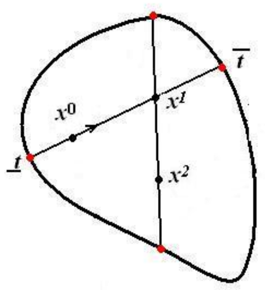
$$X = \left\{ P : AP + PA^T + PBB^T P + C \leq 0,\ P \geq 0 \right\}$$

- Linear algebraic inequalities set

$$X = \left\{ x \in \mathbb{R}^n : c_i^T x \leq a_i,\ i = 1, \ldots, m \right\}$$

Hit-and-Run

1. $i = 0$, $x^0 \in X$
2. Choose random direction d uniformly distributed on the unit sphere
3. $x^{i+1} = x^i + t_1 d$,
   t_1 is uniformly distributed on $L = (\underline{t}, \overline{t})$
4. L is updated with respect to x^{i+1}, go to Step 2.



Theorem (Smith (1984))

Let X be bounded open or coincides with the closure of interior points of X. Then for any measurable set $A \subset X$ probability $P_i(A) = P(x^i \in A | x^0)$ can be estimated as $|P_i(A) - P(A)| \leq q^i$, where $P(A) = \dfrac{\mathrm{Vol(A)}}{\mathrm{Vol(X)}}$ and $q < 1$ does not depend on x^0.

Shake-and-Bake: an alternative way for generating points

Points are *asymptotically* uniformly distributed in the boundary of X.
Complete boundary oracle is exploited.

- LMI set

$$X = \left\{ x \in \mathbb{R}^n : A_0 + \sum_{i=1}^{n} x_i A_i \leq 0 \right\}$$

$n_i = -(A_i e, e),$ where e is the eigenvector corresponding to zero eigenvalue of the matrix $A_0 + \sum_{i=1}^{n} x_i^0 A_i.$

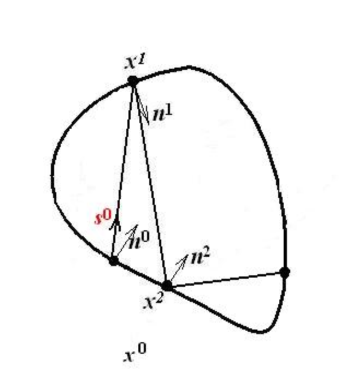
- LMI constrained set of symmetric matrices P

$$X = \left\{ P : AP + PA^T + C \leq 0 \right\}$$

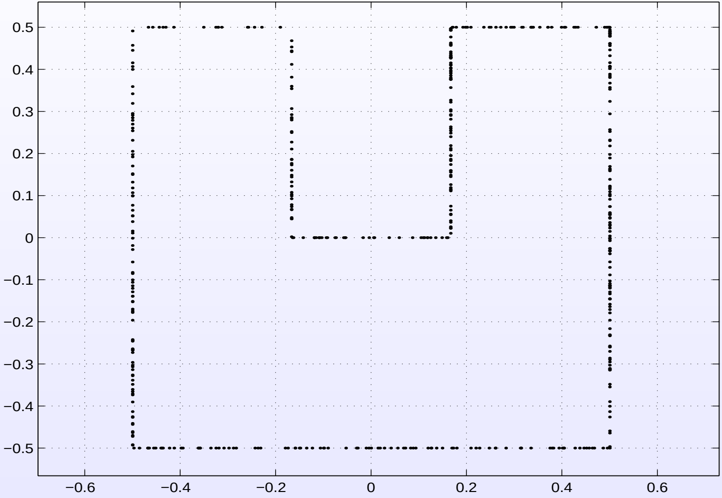
$N = -(ee^T A + A^T ee^T),$ where e is the eigenvector corresponding to zero eigenvalue of the matrix $AP_0 + P_0 A^T + C.$

Shake-and-Bake: the algorithm

1. $i = 0$, $x^0 \in \partial X$, n^0 is the normal.
2. Choose random direction s^i,
   $s^i = \sqrt{1 - \xi^{\frac{2}{n-1}}} n^0 + r$,
   ξ uniform random in $(0, 1)$,
   r is random unit uniform direction $(n^0, r) = 0$.
3. $x^{i+1} = x^i + \bar{t}s$,
   $\bar{t}$ is given by the boundary oracle for the direction s.
4. L is updated with respect to x^{i+1}, go to Step 2.



Shake-and-Bake for nonconvex sets



Numerical simulation

Standard SDP of the form

$$\min \ c^T x$$

$$\text{s.t. } A_0 + \sum_{i=1}^{n} x_i A_i \leq 0$$

A_i, $i = 0, 1, \dots n$ — symmetric real matrices $m \times m$; $c = [0, \dots, 0, 1]$
We applied modified HR where $\min \ x_i$ was replaced with averaged X_i $+$ various heuristic acceleration methods (scaling, projecting, accelerating step)
Open problem: number of HR points in every step.

Numerical simulation

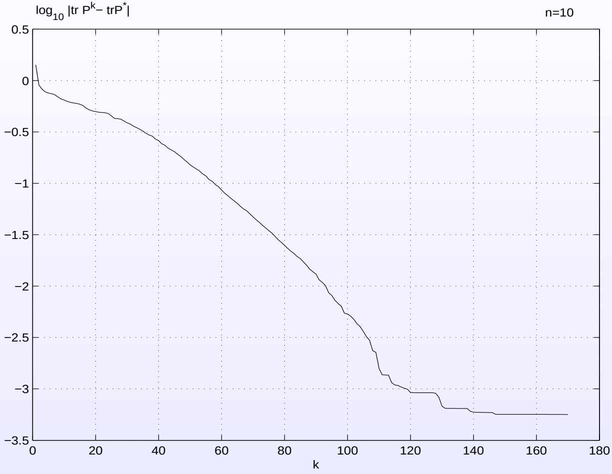
Standard SDP of the form

$$\min \; trP$$
$$\text{s.t. } AP + PA^T \leq 0$$
$$P \geq I$$

A is stable matrix $n \times n$

We applied SB for various dimensions
$+$ extension for the nonsmooth boundary

Simulation results:small size problem



Applications to control

Sets with available boundary oracle

- Stability set for polynomials

$$\mathcal{K} = \{k \in \mathbb{R}^n : p(s,k) = p_0(s) + \sum_{i=1}^{n} k_i p_i(s) \text{ is stable}\}$$

- Stability set for matrices

$$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{l\times n}$$

$$\mathcal{K} = \{K \in \mathbb{R}^{m\times l} : A + BKC \text{ is stable}\}$$

- Robust stability set for polynomials

$$\mathcal{K} = \{k : P_0(s,q) + \sum_{i=1}^{n} k_i P_i(s,q) \text{ is stable } \forall q \in Q\}, \quad Q \subset \mathbb{R}^m$$

- Quadratic stability set

$$\dot{x} = Ax$$

$$\mathcal{K} = \{P > 0 : AP + PA^T \leq 0\}$$

Stability set for polynomials

$$\mathcal{K} = \{k \in \mathbb{R}^n : p(s,k) = p_0(s) + \sum_{i=1}^{n} k_i p_i(s) \text{ is stable}\}$$

$k^0 \in \mathcal{K}$ i.e. $p(s,k^0)$ is stable,

$d = s/||s||, s =$ `randn(n,1)` — random direction

Boundary oracle: $L = \{t \in \mathbb{R} : k^0 + td \in \mathcal{K}\}$,

i.e. $\{t \in \mathbb{R} : p(s,k^0) + t \sum d_i p_i(s) \text{ is stable}\}$.

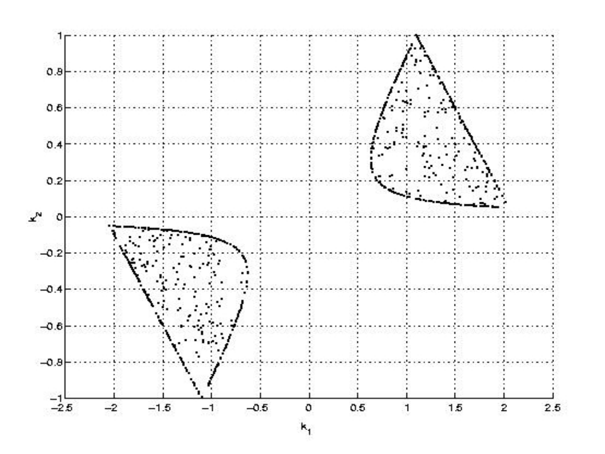
D**-decomposition problem for real scalar parameter** t**!**

Gryazina E. N., Polyak B. T. Stability regions in the parameter space: D-decomposition revisited //Automatica. 2006. Vol. 42, No. 1, P. 13–26.

Example: Generating points in the disconnected set

$$\mathcal{K} = \{k \in \mathbb{R}^n : p(s,k) = p_0(s) + \sum_{i=1}^{n} k_i p_i(s) \text{ is stable}\},$$

$$p(s,k) = 2.2s^3 + 1.9s^2 + 1.9s + 2.2 + k_1(s^3 + s^2 - s - 1) + k_2(s^3 - 3s^2 + 3s - 1)$$



Stability set for matrices

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad u = Ky$$

$$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{l\times n}; \quad \mathcal{K} = \{K \in \mathbb{R}^{m\times l} : A + BKC \text{ is stable}\}$$

$K^0 \in \mathcal{K}$, i.e. $A + BK^0C$ is stable
$D = Y/||Y||, Y =$ `randn`(m, l) — random direction in the matrix space K

$A + B(K^0 + tD)C = F + tG$, where $F = A + BK^0C$, $G = BDC$

Boundary oracle: $L = \{t \in \mathbb{R} : F + tG \text{ is stable}\}$
Total description of L is hard:
$f(t) = \max Re \,\text{eig}(F + tG)$
numerical solution of the equation $f(t) = 0$, $t > 0$ (MatLab command `fsolve`)

Quadratic stability

$$\dot{x} = Ax + Bu, \quad u = Kx$$

$$\mathcal{K} = \{K : \exists P > 0, A_c^T P + P A_c \le 0\}, \quad A_c = A + BK$$

$\mathcal{K}$ is convex and bounded.

$$Q = P^{-1} > 0, \quad QA^T + AQ + BY + Y^T B^T < 0, \quad Y = KQ.$$

$k^0 \in \mathcal{K}$, $Q_0 = P_0^{-1}$, $Y_0 = K_0 Q_0$ — starting points

$Q = Q_0 + tJ$, $Y = Y_0 + tG$, where J and G are random directions in the matrix space.

initial inequality $\iff F + tR < 0$

Boundary oracle: $L = (-\underline{t}, \overline{t})$,

where $\overline{t} = \min \lambda_i$, $\underline{t} = \min \mu_i$;

λ_i — real positive eigenvalues for the pair of matrices

$F = Q_0 A^T + AQ_0 + BY_0 + Y_0^T B^T$ and $-R = JA^T + AJ + BG + G^T B^T$;

μ_i correspondingly for matrices F, R.

Polyak B.T., Shcherbakov P.S. The D-decomposition technique for linear matrix inequalities // Automation and Remote Control. 2006. No. 11. P. 1847–1861

Conclusions

- Randomized approaches for optimization are promising.
- Proposed methods are simple in implementation and give an opportunity to solve large-dimensional problems.