# THE VOLUME OF A CERTAIN POLYTOPE, MULTIVARIATE EXTREMES AND UNIFORM SPACINGS

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Dedicated to

**Reuven Rubinstein** on his seventy's birthday

Joint work with **Shmuel Onn**

**Main Result:**

$$\Omega_k = \{\, \mathbf{v} \in \mathbf{R}^k \,:\, v_i \geq 0,\ \Sigma v_i = 1 \,\}$$

$$A_0 \,:\, \Omega_k \,\mapsto\, [1/k, 1]$$

$$A_0(\mathbf{v}) \,=\, \max_{1 \leq i \leq k} v_i\,, \quad \mathbf{v} \in \Omega_k$$

**Theorem**

$$\text{vol}(A_0) \,=\, \int_{\Omega_k} A_0(\mathbf{v}) d\mathbf{v}$$

$$= \frac{\sqrt{k}}{k!}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right).$$

So, what ?

## 1. Multivariate Extremes

Let $G(\mathbf{x})$ $(\mathbf{x} \in \mathbf{R}^k)$ be a d.f. with marginal d.f.'s $G_i$ $(i = 1, 2, \ldots, k)$.

$$B(\mathbf{x}) = \frac{-\log G(\mathbf{x})}{\sum_i -\log G_i(x_i)}$$

Pickands (1981): For multivariate extreme value distributions,

$$B(x_1, x_2, \ldots, x_k) = A(v_1, v_2, \ldots, v_k),$$

$$v_i = \frac{-\log G_i(x_i)}{\sum_j -\log G_j(x_j)} \Rightarrow$$

$$\mathbf{v} = (v_1, v_2, \ldots, v_k) \in \Omega_k$$

and

(i) $A$ is convex

(ii) $A$ satisfies

$$A_0(\mathbf{v}) = \max v_i \leq A(\mathbf{v}) \leq 1\,, \quad \mathbf{v} \in \Omega_k$$

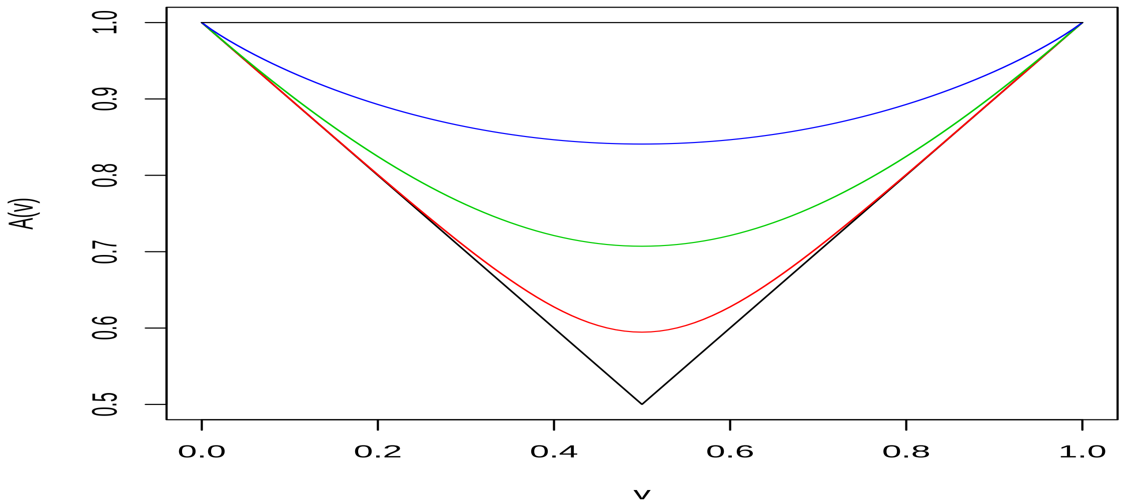
$$A \equiv 1 \ \Leftrightarrow \ \text{total independence}$$

$$A \equiv A_0 \ \Leftrightarrow \ \text{complete dependence}$$

A famous example is the *Logistic Model*:

$$A(\mathbf{v}) = \left( \sum_{i=1}^{k} v_i^{1/\alpha} \right)^{\alpha}, \quad 0 \leq \alpha \leq 1$$

$\alpha = 1 \;\; \Rightarrow$ total independence
$\alpha = 0 \;\; \Rightarrow$ complete dependence.
Here is Pickands dependence function for $k = 2$ and $\alpha = 0,\ .25,\ .50,\ .75,\ 1.$




Coefficient of Dependence:

$$0 \leq \tau = \frac{\int_{\Omega_k} (1 - A(\mathbf{v})) d\mathbf{v}}{\int_{\Omega_k} (1 - A_0(\mathbf{v})) d\mathbf{v}} \leq 1$$

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Numerator is case specific.

If we know $\text{vol}(A_0) = \int A_0$, half the job is done.

## 2. Uniform Spacings

$$Y_1 \leq Y_2 \leq \cdots \leq Y_{k-1}$$

are order statistics from i.i.d. $U[0,1]$.

$$V_1 = Y_1,\ V_2 = Y_2 - Y_1, \cdots, V_k = 1 - Y_{k-1}$$

are the spacings.

$$(V_1, V_2, \cdots, V_k) \sim U[\Omega_k].$$

Let $V_{(k)} = \max\{V_i\}$, then

$$E\,V_{(k)} = \frac{\int_{\Omega_k} \max v_i d\mathbf{v}}{\int_{\Omega_k} d\mathbf{v}} = \frac{\mathrm{vol}(A_0)}{\mathrm{vol}(\Omega_k)}$$

$$= \frac{\mathrm{vol}(A_0)}{\sqrt{k}/(k-1)!}$$

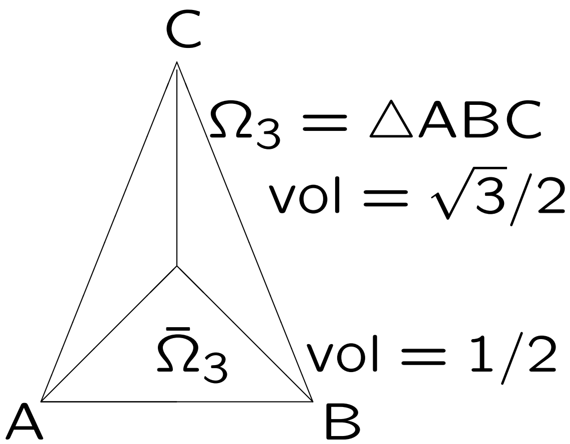
**NOTE:**

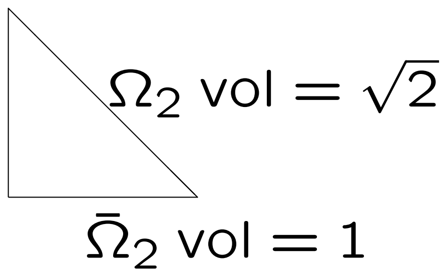
$$\Omega_k = \{\mathbf{v} \in \mathsf{R}^k : v_i \geq 0,\ \ \Sigma_{i=1}^{k} v_i = 1\,\}$$

$$\bar{\Omega}_k = \{\mathbf{v} \in \mathsf{R}^{k-1} : v_i \geq 0,\ \ \Sigma_{i=1}^{k-1} v_i \leq 1\,\}$$

$\Rightarrow$

$$\text{vol}(\Omega_k) = \frac{\sqrt{k}}{(k-1)!}, \quad \text{vol}(\bar{\Omega}_k) = \frac{1}{(k-1)!}$$



## Geometric Proof of Main Result

$$\text{vol}(A_0) = \int_{\Omega_k} A_0(\mathbf{v}) d\mathbf{v}$$

$$= \frac{\sqrt{k}}{k!}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right)$$

**Fact 1.**

$$\Omega_k = \bigcup_{\sigma} \Omega_\sigma$$

The union over all $k!$ permutations $\sigma$ of $\{1, 2, \ldots, k\}$, where

$$\Omega_\sigma = \{\mathbf{v} \in \Omega_k : v_{\sigma(k)} \leq \cdots \leq v_{\sigma(2)} \leq v_{\sigma(1)} \} .$$

$$\Rightarrow \text{vol}(A_0) = k! \int_{\Omega_e} A_0(\mathbf{v}) d\mathbf{v} = k! \int_{\Omega_e} v_1 d\mathbf{v},$$

where $\sigma = e$ is the identity permutation:

$$\Omega_e = \{\mathbf{v} \in \Omega_k : 0 \leq v_k \leq \cdots \leq v_2 \leq v_1 \}.$$

**Fact 2**

$$\int_{\Omega_e} v_1 d\mathbf{v} = \sqrt{k} \int_{\bar{\Omega}_e} v_1 d\mathbf{v},$$

where

$$\bar{\Omega}_e = \{(v_1, \ldots, v_{k-1}) :$$

$$0 \leq 1 - \Sigma_{i=1}^{k-1} v_i \leq v_{k-1} \leq \cdots \leq v_2 \leq v_1\}.$$

$$\Rightarrow \int_{\bar{\Omega}_e} v_1 d\mathbf{v} = \mathrm{vol}(P_k),$$

where $P_k$ is a convex polytope,

$$P_k = \{(v_0, v_1, \ldots, v_{k-1}) \ :$$

$$0 \leq 1 - \Sigma_{i=1}^{k-1} v_i \leq v_{k-1} \leq \cdots \leq v_2 \leq v_1 \, , \ 0 \leq v_0 \leq v_1\} \ .$$

Lemma 1. *The polytope $P_k$ has $2k$ vertices $\mathbf{a}^i, \mathbf{b}^i,\ i = 1, \ldots, k$, as follows:*

$$a^i_j \ = \ \begin{cases} 0 & j = 0 \\ \frac{1}{i} & 1 \le j \le i \\ 0 & i < j < k \end{cases} \quad b^i_j \ = \ \begin{cases} \frac{1}{i} & j = 0 \\ \frac{1}{i} & 1 \le j \le i \\ 0 & i < j < k \end{cases} \quad (j < k)\ .$$

Lemma 2. *For $s = 1, \ldots, k$, the polytope*

$$\Delta_s \quad = \quad \text{conv}\{\,\mathbf{a}^1, \ldots, \mathbf{a}^s, \mathbf{b}^s, \ldots, \mathbf{b}^k\,\} \quad ,$$

*is a $k$-dimensional simplex whose volume is*

$$vol(\Delta_s) \quad = \quad \frac{1}{s(k!)^2} \quad .$$

Lemma 3. *The $k$ simplices $\Delta_1, \ldots, \Delta_k$ form a triangulation of $P_k$.*

Combining the Facts and the Lemmas,

$$\int_{\bar{\Omega}_k} A_0(\mathbf{v})d\mathbf{v} \;=\; k!\int_{\bar{\Omega}_e} v_1 d\mathbf{v} = k!\mathrm{vol}(P_k)$$

$$=\; k!\sum_{s=1}^{k}\mathrm{vol}(\Delta_s) = k!\sum_{s=1}^{k}\frac{1}{s(k!)^2}$$

$$= \frac{1}{k!}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}\right).$$

This completes the geometric proof.

**Probabilistic Proof**

$$X_1, X_2, \cdots, X_k \text{ are i.i.d. } \exp(1)$$

$$T = \sum X_i$$

Then,

$$\left(\frac{X_1}{T}, \frac{X_2}{T}, \ldots, \frac{X_k}{T}\right) =_D (V_1, V_2, \ldots, V_k)$$

and the

$$\frac{X_i}{T} \text{ are independent of } T.$$

Now,

$$E\left(\frac{U}{W}\right) = \frac{EU}{EW} \Leftrightarrow \text{COV}\left(W, \frac{U}{W}\right) = 0.$$

Hence,

$$E\,V_{(k)} = E\frac{\max X_j}{T} = \frac{E\max X_j}{ET}$$

$$= \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}}{k} \left(= \frac{\text{vol}(A_0)}{\sqrt{k}/(k-1)!}\right).$$

# THANK YOU FOR YOUR ATTENTION