Gaussian Semimartingales and Moving Averages

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Stochastics in Turbulence and Finance

The set-up

We are interested in the semimartingale property of processes $(X_t)_{t\geq 0}$ on the form

$$X_t = \int_{-\infty}^{t} K_t(s)\, dW_s, \qquad t \geq 0, \tag{1}$$

where $(W_t)_{t\in\mathbb{R}}$ is a (two-sided) Brownian motion and $K = K_t(s)$ is a deterministic kernel such that the integral exists.

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Two observations:

- If $K_t(s)$ does not depend on t, then $(X_t)_{t\geq 0}$ is a martingale.
- If $K_t(s) = 1_{[0,1]}(t-s)$, then $X_t = W_t - W_{t-1}$, which is not a semimartingale.

Moving average processes

In the case where $K_t(s) = \varphi(t-s) - \psi(-s)$, that is

$$X_t = \int_{-\infty}^{t} \varphi(t-s) - \psi(-s)\, dW_s, \qquad t \in \mathbb{R}, \tag{2}$$

$(X_t)_{t\in\mathbb{R}}$ is called a moving average process.

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Some examples:

- The OU process, in this case $\psi = 0$ and $\varphi(t) = e^{-\beta t} 1_{[0,\infty)}(t)$ (this is a semimartingale).
- The fBm with Hurst parameter $H \in (0,1)$, in this case $\psi(t) = \varphi(t) = (t \vee 0)^{H-1/2}$ (this is not a semimartingale for $H \neq 1/2$).
- The model for the turbulent velocity field by Barndorff-Nielsen and Schmiegel in the special case of constant intermittency $(\sigma_t)_{t\in\mathbb{R}}$ reduces to a moving average process.

Definitions and notation

We will use the following notation: For each process $(Y_t)_{t\in\mathbb{R}}$, we let $(\mathcal{F}_t^Y)_{t\geq 0}$ denote the filtration given by $\mathcal{F}_t^Y = \sigma(Y_r : r \in [0,t])$ and let $(\mathcal{F}_t^{Y,\infty})_{t\geq 0}$ denote the filtration given by $\mathcal{F}_t^{Y,\infty} = \sigma(Y_r : r \in (-\infty, t])$.

Let $(\mathcal{F}_t)_{t\geq 0}$ denote a filtration. Then $(Y_t)_{t\geq 0}$ is said to be an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale if it can be written as

$$Y_t = Y_0 + M_t + A_t, \qquad t \geq 0,$$

where $(M_t)_{t\geq 0}$ is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$ local martingale, $(A_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process of bounded variation and X_0 is $\mathcal{F}_0$-measurable.

As seen from the definition, the semimartingale property is *very* filtration dependent. We have the following relation: Let $(\mathcal{G}_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$ denote two filtrations satisfying $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t \geq 0$. Moreover, let $(Y_t)_{t\geq 0}$ denote an $(\mathcal{F}_t)_{t\geq 0}$-semimartingale which is $(\mathcal{G}_t)_{t\geq 0}$-adapted then $(Y_t)_{t\geq 0}$ is also a $(\mathcal{G}_t)_{t\geq 0}$-semimartingale.

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Overview over results

Let $(X_t)_{t\geq 0}$ be given by (1). In this talk we consider the semimartingale property of $(X_t)_{t\geq 0}$ in the following three filtrations:

$$(\mathcal{F}_t^X)_{t\geq 0}, \quad (\mathcal{F}_t^{X,\infty})_{t\geq 0} \quad \text{and} \quad (\mathcal{F}_t^{W,\infty})_{t\geq 0}.$$

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- In *Basse(a)* we let $(X_t)_{t\geq 0}$ given by (1). In the filtrations $(\mathcal{F}_t^X)_{t\geq 0}$ and $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$ we derive necessary and sufficient conditions on the kernel K for $(X_t)_{t\geq 0}$ to be a semimartingale.
- In *Basse(b)* we let $(X_t)_{t\in\mathbb{R}}$ be a moving average process given by (2). We obtain necessary and sufficient conditions on φ and ψ for $(X_t)_{t\geq 0}$ to be an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. We also characterize the spectral measure of a general Gaussian process $(X_t)_{t\in\mathbb{R}}$ with stationary increments which is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale.
- In *Basse(c)* we study general Gaussian semimartingale. We derive a representation result for them and use it to obtain necessary and sufficient conditions on the covariance function for a Gaussian process to be an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale.

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A generalisation of F. Knight's result

The following result is due to F. Knight:
Let $(X_t)_{t\geq 0}$ be a moving average process given by (2). Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if

$$\varphi(t) = \alpha + \int_0^t h(r)\,dr, \qquad t \geq 0,$$

where $\alpha \in \mathbb{R}$ and $h \in L^2(\lambda)$.

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where $\alpha \in \mathbb{R}$ and $h \in L^2(\lambda)$. Let us rewrite this result:
Let $(X_t)_{t\geq 0}$ be given by (1) and assume $K_t(s) = \varphi(t-s) - \varphi(-s)$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if

$$K_t(s) = \alpha 1_{[0,\infty)}(s) + \int_0^t h(r+s)\,dr, \qquad s \leq t,$$

where $\alpha \in \mathbb{R}$ and $h \in L^2(\lambda)$ is 0 on $(-\infty, 0)$.

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Theorem: Let $(X_t)_{t\geq 0}$ be given by (1). Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{W,\infty})_{t\geq 0}$-semimartingale if and only if

$$K_t(s) = g(s) + \int_0^t \Psi_r(s)\,\mu(dr), \qquad s \leq t,$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is square integrable on $(-\infty, t]$ for all $t \geq 0$, μ is a Radon measure on $\mathbb{R}_+$ and $(t,s) \mapsto \Psi_r(s)$ is a measurable mapping such that $\|\Psi_r\|_{L^2(\mu)} = 1$ for all $r \geq 0$ and $\Psi_t(s) = 0$ if $t \geq s$.

Semimartingales w.r.t. $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$

Let $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ and for each measurable function $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$, let $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ be given by

$$\tilde{f}(t) = \int_{-\infty}^{\infty} \frac{e^{its} - 1_{[-1,1]}(s)}{is} f(s)\, ds, \qquad t \in \mathbb{R}.$$

Theorem: Let $(X_t)_{t\in\mathbb{R}}$ denote a moving average process given by (2) with $\varphi = \psi$. Then $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if φ is on the form

$$\varphi(t) = \beta + \alpha\tilde{f}(t) + \int_0^t \widehat{f\hat{h}}(s)\, ds, \qquad t \in \mathbb{R},$$

where $\alpha, \beta \in \mathbb{R}$, $h \in L^2(\lambda)$ and $f\colon \mathbb{R} \to S^1$ is measurable and satisfies $\overline{f} = f(-\cdot)$. If $\alpha \neq 0$, h is 0 on $(0, \infty)$.
Moreover, $(X_t)_{t\geq 0}$ is of bounded variation if and only if $\alpha = 0$ and $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-martingale if and only if $h = 0$.

Some applications

Let $(X_t)_{t\in\mathbb{R}}$ be a moving average process given by

$$X_t = \int \varphi(t-s) - \varphi(-s)\, dW_s, \qquad t \in \mathbb{R}.$$

Then $(X_t)_{t\in\mathbb{R}}$ is a (two-sided) Brownian motion if and only if

$$\varphi(t) = \beta + \alpha\tilde{f}(t)$$

for some $f\colon \mathbb{R} \to S^1$ satisfying $\overline{f} = f(-\cdot)$.

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Setting $f(t) = (t+i)(t-i)^{-1}$ we obtain $\tilde{f}$ equals $\varphi : t \mapsto (e^{-t} - 1/2)1_{\mathbb{R}_+}(t)$. Thus

$$X_t = \int_{-\infty}^{t} \varphi(t-s) - \varphi(-s)\, dW_s, \qquad t \geq 0,$$

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$$X_t = \int_{-\infty}^{t} \varphi(t-s) - \varphi(-s)\, dW_s, \qquad t \geq 0,$$

is a Brownian motion. Another way of putting this is:
Let $(X_t)_{t\geq 0}$ be the stationary OU-process given by

$$X_t = X_0 - \int_0^t X_s\, ds + W_t, \qquad t \geq 0,$$

with $X_0 \overset{\mathcal{D}}{=} N(0, 1/2)$ independent of the Brownian motion $(W_t)_{t\geq 0}$. Then $(Y_t)_{t\geq 0}$, given by

$$Y_t = W_t - 2\int_0^t X_s\, ds, \qquad t \geq 0,$$

is a Brownian motion.

$(\mathcal{F}_t^X)_{t\geq 0}$-semimartingales vs. $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingales

For each Gaussian process $(A_t)_{t\geq 0}$ which is right-continuous and bounded variation we let μ_A denote the Lebesgue-Stieltjes measure satisfying $\mu_A((0,t]) = E[\mathrm{V}_{[0,t]}(A)]$ for all $t \geq 0$.

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Theorem: Let $(X_t)_{t\in\mathbb{R}}$ be a Gaussian process which either is stationary or has stationary increments and $X_0 = 0$. Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale with canonical decomposition given by $X_t = X_0 + M_t + A_t$. Then $(M_t)_{t\geq 0}$ is a Brownian motion and μ_A is absolutely continuous with increasing density. Moreover, $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if μ_A has a bounded density.

Representation of Gaussian semimartingales

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Theorem: A process $(X_t)_{t\geq 0}$ is a Gaussian $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale if and only if it admits the following representation

$$X_t = X_0 + M_t + \Big(\int_0^t \Big(\int \Psi_r(s)\, dM_s \Big) \mu(dr) + \int_0^t Y_r\, \mu(dr) \Big),$$

where μ is a Radon measure, $(M_t)_{t\geq 0}$ is a Gaussian martingale starting at 0, $(Y_t)_{t\geq 0}$ is a measurable process which is bounded in $L^2(P)$ and satisfies $\{Y_t, X_0 : t \geq 0\}$ is Gaussian and independent of $(M_t)_{t\geq 0}$, $(s,r) \mapsto \Psi_r(s)$ is measurable and satisfies $(\Psi_r)_{r\geq 0}$ is bounded in $L^2(\mu_M)$ and $\Psi_t(s) = 0$ for $\mu_M \otimes \mu$-a.a. (s,t) with $s \geq t$.

The covariance function of Gaussian semimartingales

A measurable mapping $\mathbb{R}_+^2 \ni (t,s) \mapsto \Psi_t(s) \in \mathbb{R}$ is said to be a Volterra type kernel if $\Psi_t(s) = 0$ for all $s > t$. By $\mathbb{1}$ we denote the Volterra type kernel given by $\mathbb{1}_t(s) = 1_{[0,t]}(s)$. Based on the previous decomposition we derive the following new characterisation of the covariance function of a Gaussian semimartingale.

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Theorem: A centered Gaussian process $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^X)_{t\geq 0}$-semimartingale if and only if

$$\Gamma_X(t,u) = G(t,u) + \int \Phi_t(s)\Phi_u(s)\,\mu(ds), \qquad u,t \geq 0,$$

for a Radon measure μ on $\mathbb{R}_+$, a Volterra type kernel Φ such that $\mathbb{R}_+ \ni t \mapsto \Phi_t - \mathbb{1}_t \in L^2(\mu)$ is right-continuous and of bounded variation and finally a covariance function G satisfying

$$\sqrt{G(t,t) + G(s,s) - 2G(s,t)} \leq g(t) - g(s), \qquad 0 \leq s < t,$$

for some right-continuous and increasing function g.

A corollary

Corollary: Let $(X_t)_{t\geq 0}$ denote a Gaussian semimartingale with stationary increments. Then

- $(X_t)_{t\geq 0}$ is of bounded variation if and only if $(s,t)\mapsto \Gamma_X(s,t)$ is absolutely continuous.
- $(X_t)_{t\geq 0}$ is a martingale if and only if $(s,t)\mapsto \Gamma_X(s,t)$ is singular.

Let $(X_t)_{t\geq 0}$ denote a fBm with Hurst parameter $H\in(0,1)\setminus\{1/2\}$. We will show that $(X_t)_{t\geq 0}$ is not a semimartingale. Assume it is. Since $(s,t)\mapsto \Gamma_X(s,t)$ is absolutely continuous it follows by the above result that $(X_t)_{t\geq 0}$ is of bounded variation which is clearly not true.

Gaussian processes with stationary increments

Let $(X_t)_{t\in\mathbb{R}}$ be a centered Gaussian process with stationary increments such that $X_0 = 0$. Moreover, let μ denote the spectral measure of $(X_t)_{t\in\mathbb{R}}$, that is μ is a symmetric meausure which integrates $t \mapsto (1+t^2)^{-1}$ and satisfies

$$E[X_t X_u] = \int \frac{(e^{its}-1)(e^{-ius}-1)}{s^2}\,\mu(ds), \qquad t,u\in\mathbb{R}.$$

Decompose μ as $\mu = \mu_s + f\,d\lambda$.

Theorem: $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale if and only if μ_s is finite and $f = |\alpha + \hat{h}|^2$, where $\alpha\in\mathbb{R}$ and $h\in L^2(\lambda)$ is 0 on $(-\infty,0)$ if $\alpha\neq 0$.

Let us apply this result on the fBm: Let $(X_t)_{t\in\mathbb{R}}$ denote a fBm with Hurst parameter H. Then $\mu(ds) = c_H|s|^{1-2H}$. Assume $(X_t)_{t\geq 0}$ is an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale. Then $c_H|s|^{1-2H} = |\alpha + \hat{h}(s)|^2$, however this can only be satisfied for $H = 1/2$. Thus we have reproved that $(X_t)_{t\geq 0}$ is not an $(\mathcal{F}_t^{X,\infty})_{t\geq 0}$-semimartingale for $H \neq 1/2$.

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