HOMOGENEOUS AND ISOTROPIC STATISTICAL SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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The aim of the talk is to expoud results on Isotropic Statistical Solution from

S.Dostoglou, A.V.Fursikov, J.D.Kahl: *Homogeneous and Isotropic Statistical Solutions of the Navier-Stokes Equations*.- Math. Physics Electronic Journal, http://www.ma.utexas.edu/mpej/ volume 12, paper No. 2, 2006

These results are founded on results on Homogeneous Statistical Solutions published in

Vishik, V.I. and A.V. Fursikov: *Mathematical problems of statistical hydromechanics.* Kluwer, 1988.

Initial definitions

Let H be a Hilbert space of vector fields $u(x): u = (u_1, u_2, u_3),\ x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

$T_h : H \to H,\ h \in \mathbb{R}^3$ is translation operator if:

$$T_h u(x) = u(x + h).$$

$O(3)$ denotes the group of all orthogonal 3×3-matrices ω (with $\det \omega = \pm 1$)

$R_\omega : H \to H,\ \omega \in O(3)$ is called rotation operator if

$$(R_\omega u)(x) = \omega u(\omega^{-1} x)$$

In fact R_ω can be also reflection operator for some ω

Homogeneous and isotropic measures

$\mathcal{B}(H)$ is the σ-algebra of Borel sets of H. Let H_1, H_2 be Hilbert spaces. For a measure $\mu(A), A \in \mathcal{B}(H_1)$, **push forward of μ under the map** $\Psi : H_1 \to H_2$ is the measure

$$\Psi^* \mu(B) = \mu(\Psi^{-1} B) \quad \forall\ B \in \mathcal{B}(H_2).$$

where

$$\Psi^{-1} B := \{m \in H_1 : \Psi(m) \in B\} \in \mathcal{B}(H_1).$$

A measure μ defined on $\mathcal{B}(H)$ is called **homogeneous** if it is translation invariant:

$$T_h^* \mu(A) = \mu(A), \quad \forall A \in \mathcal{B}(H), h \in \mathbb{R}^3 \quad (1)$$

A homogeneous measure μ defined on $\mathcal{B}(H)$ is called **isotropic** if

$$R_\omega^* \mu(A) = \mu(A), \quad \forall A \in \mathcal{B}(H), \omega \in O(3) \quad (2)$$

Why isotropic measures are usefull?

Correlation tensor for general (nonisotropic) measure is defined as follows:

$$K_{i,j}(x_1, x_2) = \int u(x_1) \otimes u(x_2) \mu(du),$$

where $i, j = 1, \ldots 3$, $x_1, x_2 \in \mathbb{R}^3$.

If μ is isotropic measure, its correlation tensor has the form:

$$K_{i,j}(x_1, x_2) = k(|x_1 - x_2|)\delta_{i,j}$$

where $k(\lambda)$, $\lambda \in \mathbb{R}_+$ is a scalar function, and δ_i, j is Kroneker symbol.

In Kolmogorov turbulence theory isotropic measures are essentially used.

Navier-Stokes Equations

$$\partial u(t,x) - \Delta u + (u, \nabla u) + \nabla p(t,x) = 0, \ \mathrm{div} u = 0,$$

$$u(t,x)|_{t=0} = u_0(x)$$

where $t \in (0,T), x \in \mathbb{R}^3$.

Energy inequality

$$\|u(t,\cdot)\|^2 + \int_0^T \|\nabla u(\tau,\dot{})\|^2 d\tau \le \|u_0\|^2, \ t \in (0,T),$$

$(\|\cdot\| = \|\cdot\|_{L_2(\mathbb{R}^3)})$ is the basic property to prove existence of generalized solution.

Energy inequality can be used to prove existence of statistical solution that is, roughly speaking, the measure P supported on the set of generalized solution and satisfying the following analog of energy estimate:

$$\int (\|u(t,\cdot)\|^2 + \int_0^T \|\nabla u(\tau,\dot{})\|^2 d\tau) P(du) \le \int \|u_0\|^2 \mu(du_0)$$

where μ is a given measure on the set of initial conditions.

Properties of homogeneous measures

1)Unique homogeneous measure supported on $L_2(\mathbb{R}^3)$ is δ-measure, i.e. $\delta(A) = 1$ for $A \in \mathcal{B}(L_2(\mathbb{R}^3))$ if $0 \in A$ and $\delta(A) = 0$ if $0 \notin A$

2)Each homogeneous measure μ is supported on bounded functions defined on $\mathbb{R}^3$. Moreover, if homogeneous measure $\mu \neq \delta$ (and does not contain component $\alpha\delta, \alpha \in (0,1)$), then $\mu(L_2(\mathbb{R}^3)) = 0$.

That is why the Hilbert space H on which there is reason to consider homogeneous (and isotropic) measures is as follows:

$$\mathcal{H}^0(r) = \{u(x), x \in \mathbb{R}^3 : \ \mathrm{div} u = 0,$$

$$\|u\|^2_{\mathcal{H}^0(r)} = \int (1 + |x|^2)^r |u(x)|^2 dx < \infty\}$$

where $r < -3/2$.

Since $\|u\|_{L_2(\mathbb{R}^3)} = \infty$ for typical $u \in \mathcal{H}^0(r)$, there is no hope to use energy estimate to prove existence of homogeneous statistical solution for Navier-Stokes equation.

Density of energy for homogeneous measures

For a homogeneous measure μ the pointwise averages

$$\int |u|^2(x)\ \mu(du), \quad \int |\nabla u|^2(x)\ \mu(du) \qquad (3)$$

can be defined by the equalities

$$\int\int |u(x)|^2\phi(x)\ dx\ \mu(du) = \int |u(x)|^2\ \mu(du) \int \phi(x)\ dx$$

$$\int\int |\nabla u(x)|^2\phi(x)\ dx\ \mu(du)$$

$$= \int |\nabla u(x)|^2\ \mu(du) \int \phi(x)\ dx \quad \forall\ \phi \in L_1(\mathbb{R}^3)$$

The first expression in (3) is **the energy density** and the second one is **the density of the energy dissipation**. In terms of these qualities we will get analog of energy estimate for statistical solution.

Some definitions

The set of all **generalized solutions** of the Navier-Stokes system:

$$\mathcal{G}_{NS} = \Big\{ u \in L^2(0,T;\mathcal{H}^0(r)) : L(u,\phi)$$

$$\equiv \int_0^T \left(\langle u, \frac{\partial \phi}{\partial t} \rangle_2 + \langle u, \Delta\phi \rangle_2 + \sum_{j=1}^{3} \langle u_j u, \frac{\partial \phi}{\partial x_j} \rangle_2 \right) \, dt = 0,$$

for all $\phi \in C_0^\infty \Big((0,T) \times \mathbb{R}^3 \Big) \cap C((0,T);\mathcal{H}^0(r)) \Big\}$,
where $\langle u, v \rangle_2 = \int_{\mathbb{R}^3} u(x) \cdot v(x) \; dx$.

A measure $P(A), \;\; A \in \mathcal{B}(L^2(0,T;\mathcal{H}^0(r)))$ is called **homogeneous in** x if $T_h^* P = P$
$\forall h \in \mathbb{R}^3$

The space $\mathcal{H}^1(r)$ is defined as follows:

$\mathcal{H}^1(r) = \{ u(x) \in \mathcal{H}^0(r) : \|u\|_{\mathcal{H}^1(r)} \equiv \|\nabla u\|^0_{\mathcal{H}}(r) \leq \infty \}$

Definition of homogeneous statistical solution

Given homogeneous probability measure μ on $\mathcal{B}(\mathcal{H}^0(r))$ possessing finite energy density, a **homogeneous statistical solution of the Navier-Stokes equations with initial condition** μ is a probability measure P on $\mathcal{B}(L^2(0,T;\mathcal{H}^0(r)))$ such that:

1. P is homogeneous in x.

2. $P(\widehat{W}) = 1$, where $\widehat{W} = L^2(0,T;\mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS},\ s > \frac{11}{2}$.

3. $P(\gamma_0^{-1}A) = \mu(A), \quad \forall A \in \mathcal{B}(\mathcal{H}^0(r))$ where $\gamma_0^{-1}A = \{u \in \widehat{W} : \gamma_0 u \in A\}$.

4. $\int \left(|u(t,x)|^2 + \int_0^t |\nabla u|^2(\tau,x)\ d\tau\right) P(du)$ $\leq C \int |u(x)|^2\ \mu(du)$, for each $t \in [0,T]$.

Definition of isotropic statistical solution

A homogeneous measure $P(A), A \in \mathcal{B}(L^2(0,T;\mathcal{H}^0(r))$ is called **isotropic in** x if $R_\omega^* P = P, \forall \omega \in O(3)$

Let μ be an isotropic measure on $\mathcal{B}(\mathcal{H}^0(r))$ possessing finite energy density.

An **isotropic statistical solution of the Navier-Stokes equations with initial condition** μ is a homogeneous statistical solution P that is isotropic in x

The main results

Theorem 1 Given μ homogeneous measure on $\mathcal{H}^0(r)$ with finite energy density, there exists homogeneous statistical solution of the Navier-Stokes equations P with initial condition μ.

Theorem 2 Given μ isotropic measure on $\mathcal{H}^0(r)$ with finite energy density, there exists isotropic statistical solution of the Navier-Stokes equations $\widehat{P}$ with initial condition μ.

Draft of Theorem's 1 proof

Step 1. Introduce the finite-dimensional space of trigonometric polynomials of degree l and period $2l$:

$$\mathcal{M}_l = \{ \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3, |k| \leq l} a_k e^{ik\cdot x} : a_k \cdot k = 0,\ a_k = \overline{a}_{-k}\ \forall\ k\},$$

For all $l \in \mathbb{N}$ $\quad \mathcal{M}_l \subset \mathcal{H}^0(r)$.

Lemma 1. For a given initial homogeneous measure μ on $\mathcal{H}^0(r)$ with finite energy density there exists a sequence of homogeneous measures μ_l as $l \to \infty$ defined on $\mathcal{B}(\mathcal{H}^0(r))$ and supported on $\mathcal{M}_l$ such that

$$\int \exp\left(i(u,\varphi)\right)\mu_l(du) \to \int \exp\left(i(u,\varphi)\right)\mu(du)$$

as $l \to \infty$, and

$$\int |u|^2(x)\ \mu_l(du) \leq \int |u|^2(x)\ \mu(du)$$

Step 2. Galerkin's approximations

$$\partial_t u(t,x) - \Delta u + \pi_l(u,\nabla)u = 0, \text{ div} u = 0,$$

$$u(t,x) \in C(0,T;\mathcal{M}_l), \quad u|_{t=0} = u_0 \in \mathcal{M}_l$$

where π_l is the projection on $\mathcal{M}_l$. Let $S_l : \mathcal{M}_l \to C(0,T;\mathcal{M}_l)$ be operator mapping initial conditions to solutions of this problem (i.e. Galerkin's resolving operator).

Galerkin's approximations of homogeneous statistical solution for Navier-Stokes equations are the measures P_l that are defined with help of approximation μ_l from Lemma 1 of initial measure μ by the formula:

$$P_l(A) = \mu_l(S_l^{-1}A) \quad \forall A \in \mathcal{B}(C(0,T;\mathcal{M}_l))$$

Step 3. Passage to limit

$$P_l \to P \quad \text{weakly as} \quad l \to \infty$$

on $L_2(0,T;\mathcal{H}^0(r))$ where P is homogeneous statistical solution of Navier-Stokes equations.

Isotropic statistical solution

Examples of isotropic measures: for homogeneous measure μ define

$$\widehat{\mu}(A) = \int_{O(3)} R_\omega^* \mu(A) \, d\omega \quad \forall A \in \mathcal{B}(\mathcal{H}^0(r))$$

where $d\omega = H$ is the standard Haar measure on $O(3)$ normalized. Then $\widehat{\mu}$ is isotropic measure.

Draft of Theorem's 2 proof

Let P be a homogeneous statistical solution constructed in Theorem 1. Then

$$\widehat{P}(A) = \int_{O(3)} R_\omega^* P(A) \, d\omega \quad \forall A \in \mathcal{B}(\mathcal{H}^0(r))$$

is isotropic statistical solution of Navier-Stokes eqautions.

Approximations of isotropic statistical solution

Spaces for isotropic approximations:

$$\widehat{\mathcal{M}_l} = \bigcup_{\omega \in O(3)} R_\omega M(l)$$

where, recall,

$$\mathcal{M}_l = \{ \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3, |k| \le l} a_k e^{ik\cdot x} : a_k \cdot k = 0,\ a_k = \overline{a}_{-k}\ \forall\, k \},$$

Let $S_l : \mathcal{M}_l \to C(0, T; \mathcal{M}_l)$ be Galerkin's resolving operator. Given v in $\widehat{\mathcal{M}_l}$, there are ω in $O(3)$ and u in $\mathcal{M}_l$ such that $v = R_\omega u$. Extend S_l from $\mathcal{M}_l$ to $\widehat{\mathcal{M}_l}$ as

$$\widehat{S_l} v = R_\omega S_l u.$$

Lemma 2 This extension is well-defined.

Let μ be initial measure, μ_l be its periodic approximation,

$$\widehat{\mu}_l(A) = \int_{O(3)} R^*_\omega \mu_l(A)$$

be isotropic averaging of μ_l. We set

$$\widehat{P_l}(A) = \widehat{\mu_l}(\widehat{S_l}^{-1}A), \; A \in \mathcal{B}(L^2(0, T, \mathcal{H}^0(r)))$$

Lemma 3 The measure $\widehat{P_l}$ is homogeneous and isotropic. Moreover

$$\widehat{P_l}(A) = (P_l \times H)(a^{-1}A)$$

where $a(\omega, u) = R_\omega u$ and H is the Haar measure on $O(3)$, normalized.

Theorem 3 $\widehat{P}_l \to \widehat{P}$ weakly on $L_2(0, T; \mathcal{H}^0(r))$ as $l \to \infty$, where $\widehat{P}$ is the isotropic statistical solution, constructed in Theorem 2.