Volatility Modulated Volterra Processes

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Synopsis

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- Intro: Turbulence and Finance; MultipowerVariation
- Volterra processes
- Volatility modulated Volterra Processes (*VMVP*)
- Ambit processes
- 1-dim MA BM setting: $Y = g * \sigma \bullet B$
- Concrete model type
- Realised Variation Ratio

Introduction

Modelling framework: in Finance

The basic framework for stochastic volatility modeling in finance is that of Brownian semimartingales

$$Y_t = Y_0 + \int_0^t \sigma_s \mathrm{d}B_s + \int_0^t a_s \mathrm{d}s$$

where σ and a are cadlag processes and B is Brownian motion, with σ expressing the volatility. In general, Y, σ, B and a will be multidimensional.

Introduction

Modelling framework: Turbulence (Phenomenological approach)

Whereas Brownian semimartingales are 'cumulative' in nature, for free turbulence it is physically natural to model timewise velocity dynamics by stationary processes:

At time t and at a fixed position x in the turbulent field, the velocity vector is specified as $V_t = \mu + Y_t$ with

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$$
\begin{aligned}
Y_t(x) = & \int_{-\infty}^{t} \int_{\mathbb{R}^3} g(t-s, x-\xi) \sigma_s(\xi) W(\mathrm{d}\xi \mathrm{d}s) \\
& + \int_{-\infty}^{t} \int_{\mathbb{R}^3} q(t-s, x-\xi) a_s(\xi) \mathrm{d}\xi \mathrm{d}s.
\end{aligned}
$$

where W is white noise, with σ expressing the intermittency (= volatility). In general, Y, g, σ, W, q, and a will be multidimensional.

Introduction

Multipower Variations For any stochastic process $Y = \{Y_t\}_{t\geq 0}$ (or $Y = \{Y_t\}_{t\in\mathbb{R}}$) the quadratic variation (QV) process $[Y]$ and the bipower variation (BV) process $\{Y\}$ are, respectively, the limits in probability, when they exist, of the *realised quadratic variation* (RQV) $[Y_\delta]$ and the *realised bipower variation* (RBV) $\{Y_\delta\}$.

To define RVR and RBP, for any $\delta > 0$ let Y_δ denote the δ-discretisation of Y, i.e. $(Y_\delta)_t = Y_{\lfloor t/\delta \rfloor \delta}$, and recall that for a standard normal variable u we have

$$\mu_1 = \mathrm{E}\{|u|\} = \sqrt{2/\pi}.$$

Furthermore, for positive integers n and $\delta = n^{-1}$, let

$$\Delta_j^n Y = Y_{j\delta} - Y_{(j-1)\delta}.$$

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Then RVR and RBP are given, respectively, by

$$[Y_\delta]_t = \sum_{j=1}^{\lfloor nt \rfloor} \left(\Delta_j^n Y \right)^2$$

and

$$\{Y_\delta\}_t = \frac{\pi}{2} [Y_\delta]_t^{[1,1]}$$

with

$$[Y_\delta]_t^{[1,1]} = \sum_{j=2}^{\lfloor t/n \rfloor} \left| \Delta_{j-1}^n Y \right| \left| \Delta_j^n Y \right| .$$

Introduction

General multipower:

$$\left\{ Y_{\delta}^{[\mathbf{r}]} \right\}_t = c_{\mathbf{r}} \left[Y_{\delta} \right]_t^{[\mathbf{r}]}$$

where

$$\left[Y_{\delta} \right]_t^{[\mathbf{r}]} = \sum_{j=k+1}^{\lfloor nt \rfloor} \left| \Delta_{j-k}^{n} Y \right|^{r_k} \cdots \left| \Delta_{j}^{n} Y \right|^{r_0}.$$

More generally,

$$\sum_{j=k+1}^{\lfloor nt \rfloor} f_1 \left(\Delta_{j-k}^{n} Y \right) \cdots f_k \left(\Delta_{j}^{n} Y \right)$$

Introduction

Applications In Finance

$$\delta^{-\frac{1}{2}}\left([Y_\delta]_t - \sigma_t^{2+}, \{Y_\delta\}_t - \sigma_t^{2+}\right) \overset{L-stably}{\longrightarrow} N_2\left((0,0)\,, 2\begin{bmatrix} 1 & 1 \\ 1 & 1+\vartheta \end{bmatrix}\sigma_t^{4+}\right)$$

where $\vartheta = \pi^2/4 + \pi - 5\,(\doteq 0.609)$.

Feasible results.

Introduction

Applications in Turbulence

Volterra processes

Brownian Volterra processes (*BVP*):

$$Y_t = \int_{-\infty}^{\infty} K_t(s)\,\mathrm{d}B_s + \int_{-\infty}^{\infty} Q_t(s)\,\mathrm{d}s,$$

Here K and Q are deterministic functions, sufficiently regular to give suitable meaning to the integrals.

Backward type:

$$Y_t = \int_{-\infty}^{t} K_t(s)\,\mathrm{d}B_s + \int_{-\infty}^{t} Q_t(s)\,\mathrm{d}s.$$

Volterra processes

Lévy Volterra processes (*LVP*):

$$Y_t = \int_{-\infty}^{\infty} K_t(s)\,\mathrm{d}L_s + \int_{-\infty}^{\infty} Q_t(s)\,\mathrm{d}s$$

Here L denotes a Lévy process on $\mathbb{R}$ and K and Q are deterministic kernels, satisfying certain regularity conditions.

Backward type:

$$Y_t = \int_{-\infty}^{t} K_t(s)\,\mathrm{d}L_s + \int_{-\infty}^{t} Q_t(s)\,\mathrm{d}s.$$

Volterra processes

Stochastic integration in this kind of setting is discussed for *BVP* in [Hu03], [Dec05], [DecSa06] and for *LVP* in [BeMar07].

When is Y a semimartingale? In that case what is the character of its spectral representation?

Andreas Basse [Bas07a], [Bas07b], [Bas07c], for Brownian case.

Volterra processes

Tempo-spatial Volterra processes:

$$Y_t(x) = \int_{-\infty}^{\infty} \int_{\Xi} K_t(\xi, s; x) \, L^{\#}(\mathrm{d}\xi \mathrm{d}s) + \int_{-\infty}^{\infty} \int_{\Xi} Q_t(\xi, s; x) \, \mathrm{d}\xi \mathrm{d}s$$

Here K and Q are deterministic functions, Ξ is a region in $\mathbb{R}^d$ and $L^{\#}$ is a homogeneous Lévy basis on $\Xi \times \mathbb{R}$.

Backward type:

$$Y_t(x) = \int_{-\infty}^{t} \int_{\Xi} K_t(\xi, s; x) \, L^{\#}(\mathrm{d}\xi \mathrm{d}s) + \int_{-\infty}^{t} \int_{\Xi} Q_t(\xi, s; x) \, \mathrm{d}\xi \mathrm{d}s$$

Volatility modulated Volterra processes

Volatility modulated Volterra Processes (*VMVP*):

$$Y_t(x) = \int_{-\infty}^{\infty} \int_{\Xi} K_t(\xi, s; x)\, \sigma_s(\xi)\, L^{\#}(\mathrm{d}\xi \mathrm{d}s) + \int_{-\infty}^{\infty} \int_{\Xi} Q_t(\xi, s; x)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s$$

where σ is a positive stochastic process, embodying the volatility or intermittency. (K and Q deterministic, σ and a stochastic.)

Backwards moving average type:

$$\begin{aligned} Y_t(x) &= \int_{-\infty}^{t} \int_{\Xi} g(\xi - x, t - s)\, \sigma_s(\xi)\, L^{\#}(\mathrm{d}\xi \mathrm{d}s) \\ &\quad + \int_{-\infty}^{t} \int_{\Xi} q(\xi - x, t - s)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s \end{aligned}$$

Inference on the volatility

A central issue in these settings is how to draw inference on the volatility process σ.

In cases where the processes are semimartingales, the theory of multipower variations provides effective tools for this. ([BNGJPS07], [BNGJS06] and references given there)

However, *VMVP* processes are generally not of semimartingale type and the question of how to proceed then is largely unsolved and poses mathematically challenging problems.

Inference on the volatility

It is further of interest to consider cases where processes expressing possible jumps or noise in the dynamics are added.

Some of these problems are presently under study in joint work with Jose-Manuel Corcuera, Neil Shephard, Jürgen Schmiegel and Mark Podolski.

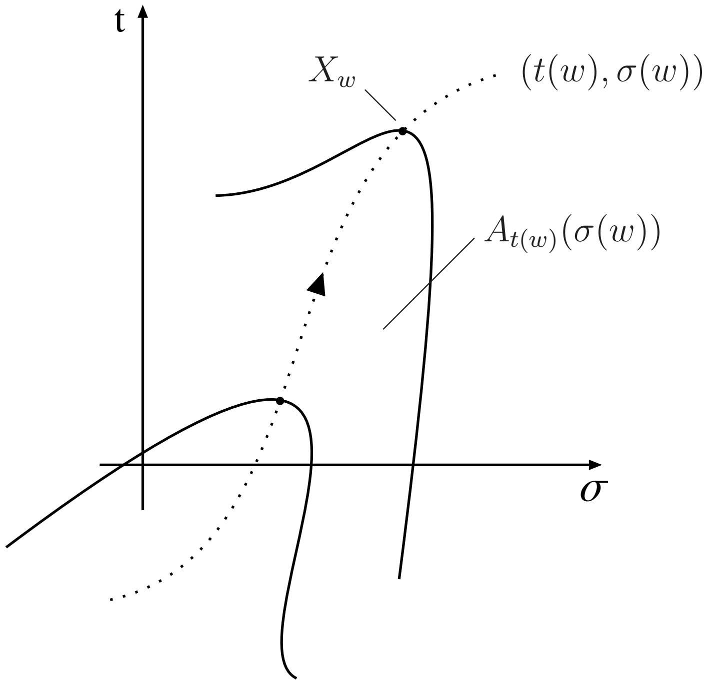
Ambit processes

Ambit processes: ([BNSch07a])

$$Y_t(x) = \mu + \int_{A_t(\sigma)} g(t-s, |\xi - x|)\, \sigma_s(\xi)\, W(\mathrm{d}\xi, \mathrm{d}s) + \int_{D_t(\sigma)} q(t-s, |\xi - x|)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s$$

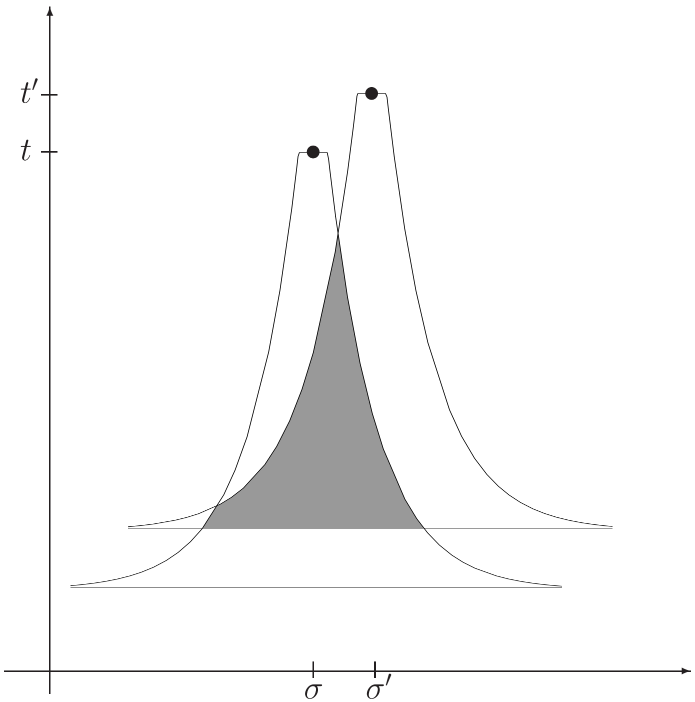
Here $A_t(\sigma)$ and $D_t(\sigma)$ are termed *ambit sets*.

Ambit processes



Ambit processes

Ambit processes



Two overlappng ambit sets

1-dim. BM MA setting

Recall: Modelling time series by stochastic processes of the form $V = \mu + Y$ with

$$Y_t = \int_{-\infty}^{t} g\,(t-u)\,\sigma_u \mathrm{d}B_u + \int_{-\infty}^{t} h(t-u)a_s \mathrm{d}u. \qquad (1)$$

Here B is Brownian motion, the kernels h and g are deterministic, positive and square integrable functions on $(0,\infty)$, presumed known, and σ is a stationary process which expresses the time-dependent variation or *volatility* of the process Y.

Moreover, a and σ are stochastic processes satisfying the same assumptions as are usual for Brownian semimartingales; in particular, σ is square integrable.

1-dim. BM MA setting

Concretely we (BN+Schmiegel) think of this as a modelling framework for the time-wise behaviour of the main component of the velocity vector (i.e. the component in the mean direction of the fluid motion) in a turbulent field.

Question: To what extent is the integral of the squared volatility over the interval $[0, t]$, i.e.

$$\sigma_t^{2+} = \int_0^t \sigma_u^2 \mathrm{d}u,$$

consistently estimable by a suitably normalised version of the realised quadratic variation of Y when the limiting scheme considered is that Y is observed at the time points $j\delta$, $j = 1, ..., n$, where $\delta = t/n$, and $n \to \infty$ with t fixed?

1-dim. BM MA setting

Conjecture: (of work in progress by BN+Corcuera+Podolskij)

The theory of multipower variation can be extended to processes of the form (1) under conditions on g of which the essential one is that the function

$$R(t) = \int_0^\infty g(t+u)\, g(u)\, \mathrm{d}u$$

satisfies the following (given on *next slide*) three assumptions **(A1)-(A3)** where $\bar{R} = 2\left(\|g\|^2 - R\right)$ and $0 < \gamma < \frac{5}{4}$:

Note The conjecture holds true for power variation when σ is a constant, as follows from [GuyLe89].

1-dim. BM MA setting

(A1) $\bar{R}(t) = t^{\gamma} L_0(t)$ for some slowly varying (at 0) function L_0, which is continuous on $(0, \infty)$.

(A2) $\bar{R}''(t) = t^{\gamma-2} L_2(t)$ for some slowly varying (at 0) function L_2, which is continuous on $(0, \infty)$.

(A3) There exists a $b \in (0, 1)$ with

$$\limsup_{x \to 0} \sup_{y \in [x, x^b]} \left| \frac{L_2(y)}{L_0(x)} \right| < \infty.$$

Conjecture: Some first considerations

Recall:

$$Y_t = \int_{-\infty}^{t} g\,(t-u)\,\sigma_u \mathrm{d}B_u + \int_{-\infty}^{t} h(t-u)a_s \mathrm{d}u.$$

The influence of the 'drift term', that is the second integral, will disappear under the limiting procedure we have in mind, so henceforth that term is assumed not to be present, and we write the expression for Y briefly as

$$Y = g * \sigma \bullet B.$$

Conjecture: Some first considerations

To ensure that Y is well defined we assume that $g(t-u)\sigma_u$ is square integrable with respect to u on $(-\infty, t]$, for all $t \in \mathbb{R}$.

Furthermore, we suppose that g is differentiable on $(0,\infty)$ and that for any $\varepsilon > 0$ and any t the integral $\int_{-\infty}^{t-\varepsilon} \dot{g}^2(t-u)\sigma_u^2 \mathrm{d}u$ exists and g is Lipschitz of order 2 on $[\varepsilon, \infty)$.

Conjecture: Some first considerations

Suppose for the moment that $\sigma = 1$ identically. Then $Y = g * B$ and this process has autocovariance and autocorrelation functions

$$R(t) = \int_0^\infty g(t+u)g(u)\,\mathrm{d}u$$

and

$$r(t) = \int_0^\infty \bar{g}(t+u)\bar{g}(u)\,\mathrm{d}u$$

where $\bar{g} = g/\|g\|$ and

$$\|g\|^2 = \int_0^\infty g^2(u)\,\mathrm{d}u.$$

We let

$$\bar{r}(t) = 1 - r(t) \quad \text{and} \quad \bar{R}(t) = 2\|g\|^2\,\bar{r}(t).$$

Quadratic variation of $Y = g * \sigma \bullet B$

Let $\Delta_j^n Y = Y_{j\delta} - Y_{(j-1)\delta}$ and for any $q > 0$, let $V(Y, q)_t^n$ be the *realised* q*-th order power variation* of Y, i.e.

$$V(Y, q)_t^n = n^{q/2-1} \sum_{j=1}^{n} \left| \Delta_j^n Y \right|^q .$$

For $q = 2$ this is the *realised quadratic variation*, which will be the basis for estimating σ_t^{2+}.

We let

$$\bar{V}(Y, 2)_t^n = \frac{\delta}{2 \left\| g \right\|^2 \bar{r}\left(\delta\right)} V(Y, 2)_t^n.$$

Quadratic variation of $Y = g * \sigma \bullet B$

Special restrictive setting: We suppose that the process σ is independent of the Brownian motion B, and we will argue conditionally on σ.

Remark: Under **(A1)-(A3)** the variance of $\bar{V}(Y,2)_t^n$ will go to 0 as $\delta \to 0$. What remains in order to establish consistency is then that

$$\mathrm{E}\left\{\bar{V}(Y,2)_t^n | \sigma\right\} \xrightarrow{p} \sigma_t^{2+}$$

Quadratic variation of $Y = g * \sigma \bullet B$

Behaviour of $\mathrm{E}\left\{V(Y,2)_t^n | \sigma\right\}$

Note that

$$\begin{aligned} Y_{t+\delta} - Y_t &= \int_t^{t+\delta} g\left(\delta + t - u\right)\sigma_u \mathrm{d}B_u \\ &+ \int_{-\infty}^{t} \left(g\left(\delta + t - u\right) - g\left(t - u\right)\right)\sigma_u \mathrm{d}B_u. \end{aligned}$$

Hence, for arbitrary $\varepsilon > 0$,

$$\begin{aligned} \delta \mathrm{E}\left\{V(Y,2)_t^n | \sigma\right\} &= \int_0^{\delta} \delta \sum_{j=1}^{n} \sigma_{j\delta - v}^2 g^2\left(v\right) \mathrm{d}v \\ &+ \int_0^{\infty} \delta \sum_{j=1}^{n} \sigma_{(j-1)\delta - v}^2 \left(g\left(\delta + v\right) - g\left(v\right)\right)^2 \mathrm{d}v \end{aligned}$$

Quadratic variation of $Y = g * \sigma \bullet B$

After some calculation we find (*key relation*)

$$\mathrm{E}\left\{\bar{V}(Y,2)_t^n|\sigma\right\} = \sigma_t^{2+} + \bar{R}\left(\delta\right)^{-1} A(\delta)$$

where

$$A\left(\delta\right) = A_0\left(\delta\right) + A_1\left(\delta;\varepsilon\right) + A_2\left(\delta;\varepsilon\right)$$

with

Quadratic variation of $Y = g * \sigma \bullet B$

$$A_0(\delta) = \int_0^\delta \left(\delta \sum_{j=1}^{n} \sigma^2_{j\delta - v} - \sigma^{2+}_t \right) g^2(v)\, \mathrm{d}v$$

and, for any $\varepsilon > 0$,

$$A_1(\delta;\varepsilon) = \int_0^\varepsilon \left(\delta \sum_{j=1}^{n} \sigma^2_{(j-1)\delta - v} - \sigma^{2+}_t \right) \left(g(\delta + v) - g(v) \right)^2 \mathrm{d}v$$

$$A_2(\delta;\varepsilon) = \int_\varepsilon^\infty \left(\delta \sum_{j=1}^{n} \sigma^2_{(j-1)\delta - v} - \sigma^{2+}_t \right) \left(g(\delta + v) - g(v) \right)^2 \mathrm{d}v.$$

Quadratic variation of $Y = g * \sigma \bullet B$

Let

$$c_0(\delta) = \int_0^\delta g^2(v)\,\mathrm{d}v \quad \text{and} \quad c(\delta) = \int_0^\infty (g(\delta + v) - g(v))^2\,\mathrm{d}v.$$

and note that $c_0(\delta) + c(\delta) = \bar{R}(\delta)$.

Furthermore, let

$$\hat{\sigma}^{2+}_{s|t} = \delta \sum_{j=1}^{n} \sigma^2_{(j-1)\delta - s}$$

and note that

$$\hat{\sigma}^{2+}_{s|t} \to \int_{-s}^{t-s} \sigma^2_u \mathrm{d}u.$$

Quadratic variation of $Y = g * \sigma \bullet B$

It follows that for any $\varepsilon > 0$

$$
\begin{aligned}
|\mathrm{E}\{\bar{V}(Y,2)^n_t|\sigma\} - \sigma^{2+}_t| \leq & \sup_{0\leq v\leq\delta} |\hat{\sigma}^{2+}_{v|t} - \sigma^{2+}_t| \frac{c_0(\delta)}{\bar{R}(\delta)} \\
& + \sup_{0\leq v\leq\varepsilon} |\hat{\sigma}^{2+}_{v|t} - \sigma^{2+}_t| \frac{c(\delta)}{\bar{R}(\delta)} \\
& + \sup_{\varepsilon<v<\infty} |\hat{\sigma}^{2+}_{v|t} - \sigma^{2+}_t| C(\varepsilon) \frac{\delta^2}{\bar{R}(\delta)}
\end{aligned}
$$

Conclusion

The upshot of these considerations is that if $c_0(\delta)$ and $c(\delta)$ are of the same asymptotic order as $\delta \to 0$, with this common order being smaller than that of δ^2, then

$$\bar{V}(Y,2)_t^n \xrightarrow{p} \sigma_t^{2+}.$$

More boldly, one may surmise that it will be possible to derive a feasible asymptotic normal limit result for inference on σ_t^{2+} under some additional assumption on the behaviour of g at 0.

A class of moving average models

Particular case: Suppose that $\sigma = 1$ and

$$g(t) = t^{\nu-1} e^{-\alpha t} \tag{3}$$

with $\nu > \frac{1}{2}$ and $\alpha > 0$.

Remark The derivative $\dot{g}$ of g is not square integrable if $\frac{1}{2} < \nu < 1$ or $1 < \nu \leq \frac{3}{2}$; hence, in these cases Y is not a semimartingale. For $\nu = 1$ the process Y is a semimartingale, in fact a modulated version of the Gaussian Ornstein-Uhlenbeck process. Note also that when $\nu > \frac{3}{2}$ then Y is of finite variation and hence, trivially, a semimartingale.

A class of moving average models

Remark Suppose that the volatility process is constant, $\sigma_t = \sigma$. In this case ([GuyLe89])

$$\bar{V}(Y, 2)_t^n \overset{p}{\longrightarrow} t\,\sigma^2.$$

In fact, considerably more is true: [GuyLe89] derived associated (nonfeasible) limit law results

It follows from those results that the limit distribution is normal if $\frac{1}{2} < \nu < \frac{5}{4}$, with rate $\delta^{-3/2}\bar{r}(\delta)$, while it belongs to the second order Wiener chaos, with rate $\delta^{2\nu-3}$, for $\frac{5}{4} < \nu < \frac{3}{2}$.

Extension to the power variations $V(Y, q)_t^n$, $q > 0$, are also given in [GuyLe89].

A class of moving average models

The following analysis uses a number of, mostly well known, properties of modified Bessel functions of the third type K_ν (not given explicitly here).

Steps in analysis:

A class of moving average models

(i) Properties of the autocorrelation function r of $Y = g * B$: Exact formulae for the autocorrelation function r and its derivatives.

(ii) Asymptotic properties of $\bar{r}(t) = 1 - r(t)$ for $t \to 0$.

(iii) Verification that **(A1)-(A3)** are satisfied

(iv) Asymptotics of $c_0(\delta)$ and $c(\delta)$ for $\delta \to 0$

(v) Example illustrating the asymptotics of $\mathrm{E}\left\{V(Y,2)_t^n\right\}$ for a very special choice of σ_t that allows explicit calculations.

A class of moving average models

Formulae for r and its derivatives

The autocorrelation function r of $Y = g * B$ has the form

$$r(t) = \frac{(2\alpha)^{2\nu-1}}{\Gamma(2\nu-1)} e^{-\alpha t} \int_0^\infty (t+u)^{\nu-1} u^{\nu-1} e^{-2\alpha u} \mathrm{d}t.$$

By formulae for the Bessel functions of type K we find

$$r(t) = \check{K}_{\nu-\frac{1}{2}}(\alpha t).$$

A class of moving average models

Suppose for notational simplicity that $\alpha = 1$, and let

$$c(\nu) = 2^{-\nu+1}\Gamma(\nu)^{-1}.$$

Then, we find, for $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$ we find

$$\bar{r}'(t) = -\frac{c\left(\nu - \frac{1}{2}\right)}{c\left(\frac{3}{2} - \nu\right)} t^{2\nu-2}\check{K}_{\frac{3}{2}-\nu}(t)$$

$$\bar{r}''(t) = c\left(\nu - \frac{1}{2}\right) t^{2\nu-3}\left\{\bar{K}_{\frac{5}{2}-\nu}(t) - \bar{K}_{\frac{3}{2}-\nu}(t)\right\}$$

$$\bar{r}'''(t) = -c\left(\nu - \frac{1}{2}\right) t^{2\nu-4}\left\{\bar{K}_{\frac{7}{2}-\nu}(t) - 3\bar{K}_{\frac{5}{2}-\nu}(t)\right\}$$

A class of moving average models

Behaviour of $\bar{r} = 1 - r$ near 0

Using formulae for the Bessel functions of type K we find that for $t \to 0$ the complementarry autocorrelation function $\bar{r}(t) = 1 - r(t)$ behaves as

$$\bar{r}(t) \sim \begin{cases} 2^{-2\nu+1}\frac{\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}(\alpha t)^{2\nu-1} + O\left(t^2\right) & \text{for } \frac{1}{2} < \nu < \frac{3}{2} \\ \frac{1}{2}(\alpha t)^2 \left|\log t\right| & \text{for } \nu = \frac{3}{2} \\ \frac{1}{4\left(\nu-\frac{3}{2}\right)}(\alpha t)^2 + O\left(t^{2\nu-1}\right) & \text{for } \frac{3}{2} < \nu \end{cases}$$

A class of moving average models

Verification of assumptions (A1)-(A3):

Conditions **(A1)-(A3)** are satisfied (with $\gamma = 2\nu - 1$ and $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$, i.e. $\gamma \in (0, 2)$)

A class of moving average models

On (A1): The complementary autocorrelation function $\bar{r}$ is of the form

$$\bar{r}(t) = t^{2\nu-1} L_0(t)$$

with

$$L_0(t) = t^{-2\nu+1}\left(1 - \check{K}_{\nu-\frac{1}{2}}(\alpha t)\right)$$

and

$$L_0(t) \to 2^{-2\nu+1} \frac{\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\nu+\frac{1}{2}\right)} \quad \text{for} \quad t \to 0.$$

It follows that L_0 is slowly varying at 0, and hence assumption **(A1)** is met.

A class of moving average models

On (A2): Note that

$$\bar{r}''(t) = t^{2\nu-3}L_2(t)$$

with

$$L_2(t) = c\left(\nu - \frac{1}{2}\right)\left\{\bar{K}_{\frac{5}{2}-\nu}(t) - \bar{K}_{\frac{3}{2}-\nu}(t)\right\},$$

where L_2 is slowly varying at 0 with

$$L_2(t) \to -2^3(\nu-1)\frac{\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\nu-\frac{1}{2}\right)} \quad \text{for} \quad t \to 0.$$

A class of moving average models

The latter follows from the rewrite

$$\begin{aligned}\bar{r}''(t) &= t^{2\nu-3} c\left(\nu - \frac{1}{2}\right) c\left(\frac{3}{2} - \nu\right)^{-1} \left\{(3-2\nu)\check{K}_{\frac{5}{2}-\nu}(t) - \check{K}_{\frac{3}{2}-\nu}(t)\right\} \\ &= t^{2\nu-3} 2^2 \frac{\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\nu-\frac{1}{2}\right)} \left\{(3-2\nu)\check{K}_{\frac{5}{2}-\nu}(t) - \check{K}_{\frac{3}{2}-\nu}(t)\right\}.\end{aligned}$$

Thus **(A2)** holds.

A class of moving average models

On (A3): Finally, we find

$$\begin{aligned}
L_2'(t) &= c\left(\nu - \frac{1}{2}\right)\left\{\bar{K}'_{\frac{5}{2}-\nu}(t) - \bar{K}'_{\frac{3}{2}-\nu}(t)\right\} \\
&= c\left(\nu - \frac{1}{2}\right) t \left\{\bar{K}_{\frac{1}{2}-\nu}(t) - \bar{K}_{\frac{3}{2}-\nu}(t)\right\} \\
&= c\left(\nu - \frac{1}{2}\right) t \left\{t^{-2\nu+1}\bar{K}_{\nu-\frac{1}{2}}(t) - \bar{K}_{\frac{3}{2}-\nu}(t)\right\} \\
&= c\left(\nu - \frac{1}{2}\right) t^{-2\nu+2} \\
&\quad \cdot \left\{c(\nu - \frac{1}{2})^{-1} t^{-2\nu+1}\check{K}_{\nu-\frac{1}{2}}(t) - c(\nu - \frac{3}{2})^{-1} t^{2\nu-1}\check{K}_{\frac{3}{2}-\nu}(t)\right\}
\end{aligned}$$

Hence (for $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$) $L_2(t)$ is increasing near 0. Consequently

A class of moving average models

$$\limsup_{x\to 0} \sup_{y\in[x,x^b]} \left| \frac{L_2(y)}{L_0(x)} \right| \leq \limsup_{x\to 0} \left| \frac{L_2\left(x^b\right)}{L_0(x)} \right| ;$$

Here, as $x \to 0$,

$$L_0(x) \to 2^{-2\nu+1} \frac{\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}.$$

while

$$L_2\left(x^b\right) \to c\left(\nu-\frac{1}{2}\right)\left\{c\left(\frac{5}{2}-\nu\right)^{-1} - c\left(\frac{3}{2}-\nu\right)^{-1}\right\}.$$

Therefore also condition **(A3)** is satisfied.

A class of moving average models

Asymptotic behaviour of $c_0(\delta)$ and $c(\delta)$, taking $\alpha = 1$,

$$c_0(\delta) = \frac{1}{2\nu - 1}\delta^{2\nu-1} + O\left(\delta^{2\nu+n-1}\right)$$

$$c(\delta) \sim \begin{cases} \frac{1}{2\nu-1}\left(2^{-2(\nu-1)}\frac{\Gamma(\nu)\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}\right)} - 1\right)\delta^{2\nu-1} + O\left(\delta^{2}\right) & \text{for } \frac{1}{2} < \nu < \frac{3}{2} \\ \frac{1}{2}\delta^{2}|\log\delta| & \text{for } \nu = \frac{3}{2} \\ 2^{-2\nu}\frac{\Gamma(2\nu-1)}{\nu-\frac{3}{2}}\delta^{2} + O\left(\delta^{2\nu-1}\right) & \text{for } \frac{3}{2} < \nu \end{cases}$$

A class of moving average models

Key Example

Consider now the special case where σ is given by

$$\sigma_u = e^{(\psi - 1)u}.$$

This particular choice allows explicit calculation of $\mathrm{E}\left\{V(Y,2)_t^n\right\}$. After some calculation one finds

A class of moving average models

$$\frac{\delta}{2\left\|g\right\|^{2}\bar{r}\left(\delta\right)}\mathrm{E}\left\{V(Y,2)_{t}^{n}\right\} = \left(\delta\sum_{j=1}^{\lfloor t/\delta\rfloor} e^{-2(1-\psi)j\delta}\right)\psi^{-(2\nu-1)}\mathbf{A}\left(\delta\right)$$

$$\sim \psi^{-(2\nu-1)}\mathbf{A}\left(\delta\right)\sigma_{t}^{2+}$$

where

$$\mathbf{A}\left(\delta\right) = e^{-(1-\psi)\delta}\frac{\bar{r}\left(\psi\delta\right)}{\bar{r}\left(\delta\right)} + \frac{\left(1-e^{-(1-\psi)\delta}\right)^{2}}{\bar{r}\left(\delta\right)}.$$

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When $\frac{1}{2} < \nu \leq \frac{3}{2}$ we have

$$\mathbf{A}\left(\delta\right) \sim \psi^{2\nu-1} + O\left(\delta^{2}\right)$$

and hence

$$\frac{\delta}{2\left\|g\right\|^{2}\bar{r}\left(\delta\right)}\mathrm{E}\left\{V(Y,2)_{t}^{n}\right\} \to \sigma_{t}^{2+}.$$

On the other hand, if $\nu > \frac{3}{2}$ we obtain

$$\frac{\delta}{2\left\|g\right\|^{2}\bar{r}\left(\delta\right)}\mathrm{E}\left\{V(Y,2)_{t}^{n}\right\} \to \psi^{-2}\left(\psi^{-2\nu+1} + 4\left(\nu - \frac{3}{2}\right)\left(1-\psi\right)^{2}\right)\sigma_{t}^{2+}.$$

A class of moving average models

Remark For the concrete model considered here, i.e.

$$Y_t = \int_{-\infty}^{t} (t-u)^{\nu-1} e^{-\alpha(t-u)} \sigma_u \mathrm{d}B_u,$$

let

$$X_t = \int_{-\infty}^{t} e^{\alpha s} (t-s)^{1-\nu} Y_s \mathrm{d}s.$$

Then (Fubini!?), for $\frac{1}{2} < \nu < \frac{3}{2}$,

$$\begin{aligned} X_t &= \int_{-\infty}^{t} e^{\alpha s} (t-s)^{1-\nu} \int_{-\infty}^{s} (s-u)^{\nu-1} e^{-\alpha(s-u)} \sigma_u \mathrm{d}B_u \mathrm{d}s \\ &= \int_{-\infty}^{t} e^{\alpha u} \sigma_u \mathrm{d}B_u \int_{u}^{t} (t-s)^{1-\nu} (s-u)^{\nu-1} \mathrm{d}s \\ &= B(1-\nu, \nu) \int_{-\infty}^{t} e^{\alpha u} \sigma_u \mathrm{d}B_u. \end{aligned}$$

Realised Variation Ratio

Recall that RVR and RBP are given, respectively, by

$$[Y_\delta]_t = \sum_{j=1}^{\lfloor nt \rfloor} \left(\Delta_j^n Y \right)^2$$

and $\{Y_\delta\}_t = \frac{\pi}{2} [Y_\delta]_t^{[1,1]}$ with

$$[Y_\delta]_t^{[1,1]} = \sum_{j=2}^{\lfloor t/n \rfloor} \left| \Delta_{j-1}^n Y \right| \left| \Delta_j^n Y \right| .$$

Realised Variation Ratio

Let

$$[Y_\delta]' = \sum_{j=2}^{n} \left(\Delta_{j-1}^n Y\right)^2, \quad \text{and} \quad [Y_\delta]'' = \sum_{j=2}^{n} \left(\Delta_j^n Y\right)^2.$$

and define the realised variation ratio (RVR) by

$$\{Y_\delta] = \frac{\{Y_\delta\}}{\frac{[Y_\delta]'+[Y_\delta]''}{2}}.$$

The probability limit of this ratio, when it exists, is the variation ratio VR), denoted $\{Y]$, i.e.

$$\{Y] = p\text{-}\lim\{Y_\delta].$$

Realised Variation Ratio

The RVR as a diagnostic tool for model checking.

Realised Variation Ratio

Note The variation ratio may well exist even in cases where the quadratic variation and the bipower variation are both infinite or both zero. This is the case, in particular, for $Y = g * \sigma \bullet B$ with $g(t) = t^{\nu-1}e^{-\alpha t}$, infinite occurring for $\frac{1}{2} < \nu < 1$ and zero for $1 < \nu < \frac{3}{2}$. (Another simple example of this is $Y_t = t$ or $Y_t = 1$ for then $\{Y\} = [Y] = 0$ while $\{Y] = \frac{\pi}{2}$.)

Realised Variation Ratio

We have

$$\{Y_\delta\} = \frac{\pi}{4}\left([Y_\delta]' + [Y_\delta]''\right) - \frac{\pi}{4}\sum_{j=2}^{n}\left(\left|\Delta_j^n Y\right| - \left|\Delta_{j-1}^n Y\right|\right)^2.$$

From this equation it follows that

$$0 \leq \{Y_\delta] \leq \frac{\pi}{2} \quad \text{and hence} \quad 0 \leq \{Y] \leq \frac{\pi}{2}.$$

Realised Variation Ratio

It also follows that RVR is close to $\frac{\pi}{2}$ if the correlation between $\operatorname{cor}\left\{\left|\Delta_{j-1}^{n}Y\right|,\left|\Delta_{j}^{n}Y\right|\right\}$ is close to 1 for all j. This, in turn, holds if $\operatorname{cor}\left\{\Delta_{j-1}^{n}Y,\Delta_{j}^{n}Y\right\}$ is close to 1 or -1 for all j.

Realised Variation Ratio

To see the latter, recall that for arbitrary standard normal variables u and v with correlation coefficient ρ

$$\mathrm{E}\left\{|uv|\right\} = \frac{2}{\pi}\left(\rho \arcsin \rho + \sqrt{1-\rho^2}\right).$$

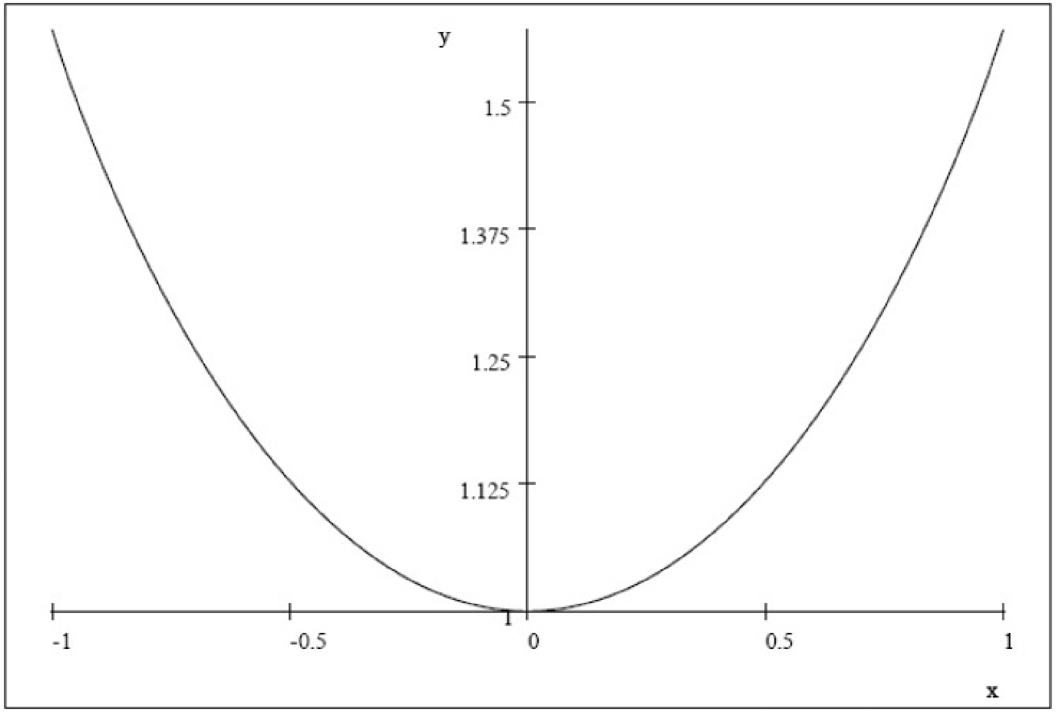
It follows that

$$\operatorname{cor}\left\{|u|, |v|\right\} = \frac{2}{\pi}\left(\rho \arcsin \rho + \sqrt{1-\rho^2} - 1\right). \tag{4}$$

Note that $\operatorname{cor}\left\{|u|, |v|\right\}$ does not depend on the sign of ρ and that it is an increasing function of $|\rho|$.

Realised Variation Ratio

The Figure below shows a plot of the function $\rho \arcsin \rho + \sqrt{1-\rho^2}$.



Realised Variation Ratio

Under certain conditions we will have that, for $\delta \to 0$,

$$\{Y_\delta] \sim \frac{\mathrm{E}\left\{\{Y_\delta\}\right\}}{\mathrm{E}\left\{\frac{[Y_\delta]' + [Y_\delta]''}{2}\right\}}.$$

Suppose in particular that $Y = g * \sigma \bullet B$ with $g(t) = t^{\nu - 1} e^{-\alpha t}$. Then

$$\{Y_\delta] \sim \rho(\delta) \arcsin \rho(\delta) + \sqrt{1 - \rho(\delta)^2}$$

with

$$\rho(\delta) = \frac{\bar{r}(2\delta)}{2\bar{r}(\delta)} - 1.$$

Realised Variation Ratio

Example Suppose that $\bar{r}$ is of the form

$$\bar{r}(t) = c_\gamma t^\gamma + o(t^\gamma)$$

for $t \to 0$ and some $\gamma \neq 1$, and where c_γ is a positive constants (i.e. as is the case for $g(t) = t^{\nu-1} e^{-\alpha t}$ with $\frac{1}{2} < \nu < \frac{3}{2}$ and $\gamma = 2\nu - 1$).

Then

$$\begin{aligned} \{Y_\delta] &\sim \frac{c_\gamma 2^\gamma \delta^\gamma + o(t^\gamma)}{2 c_\gamma \delta^\gamma + o(t^\gamma)} - 1 \\ &= 2^{\gamma-1} - 1 + o(1). \end{aligned}$$

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