# Uniqueness of Solutions to the Stochastic Navier-Stokes, the Invariant Measure and Kolmogorov's Theory

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# Outline

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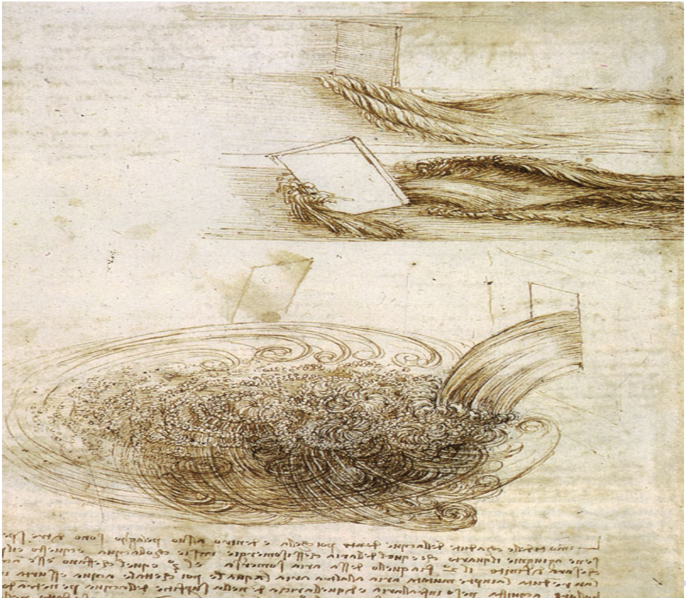
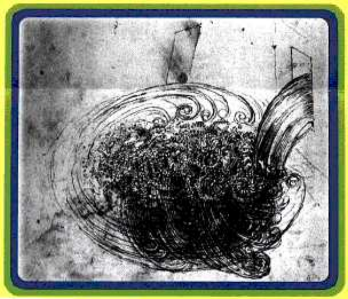
# A Drawing of an Eddy

## Studies of Turbulence by Leonardo da Vinci



# Leonardo's Observations





"Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the *principal current*, the other to the *random and reverse motion*."

# Outline

# The Mean of the Solution to the Navier-Stokes Equations



- The flow satisfies the Navier-Stokes Equation

$$u_t + u \cdot \nabla u = \nu \Delta u + \nabla \{ \Delta^{-1} [trace(\nabla u)^2] \}$$

- However, in turbulence the fluid flow is not deterministic
- Instead we want a statistical theory
- In applications the flow satisfies the Reynolds Averaged Navier-Stokes Equation (RANS)

$$\overline{u}_t + \overline{u} \cdot \nabla \overline{u} = \nu \Delta \overline{u} - \nabla \overline{p} + (\overline{u} \cdot \nabla \overline{u} - \overline{u \cdot \nabla u})$$

# An Invariant Measure



- What does the bar $\bar{u}$ mean?
- In experiments and simulations it is an ensamble average

$$\bar{u} = \langle u \rangle$$

- Mathematically speaking the mean is an expectation, if $\phi$ is any bounded function on $H$

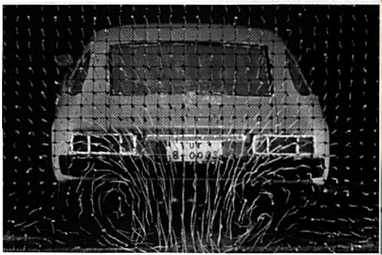
$$E(\phi(u)) = \int_H \phi(u) d\mu(u) \tag{1}$$

- The mean is defined by the invariant measure $\mu$ on the function space $H$ where $u$ lives
- We must prove the existence of a unique invariant measure to make mathematical sense of the mean $\bar{u}$

# Turbulence around Cars and Aircraft



- Thermal currents and gravity waves in the atmosphere also create turbulence encountered by low-flying aircraft
- Turbulent drag prevents the design of more fuel-efficient cars, aircrafts and ships



# Turbulence in Design and Disease



- Turbulence is harnessed in combustion engines in cars and jet engines for effective combustion and reduced emission of pollutants
- The flow around automobiles and downtown buildings is controlled by turbulence and so is the flow in a diseased artery

# Outline

# Noise-driven Instabilities



- Let $U(x_1)j_1$ denote the mean flow (Leonardo's principal current), taken to be in the $x_1$ direction
- If $U' < 0$ the flow is unstable
- The largest wavenumbers ($k$) can grow the fastest
- The initial value problem is ill-posed
- There is always white noise in the system that will initiate this growth
- If U is large the white noise will continue to grow

# How does Noise get inserted?



- Let $U_0 j_1$ denote fast mean flow in one direction
- The equation linearized about the initial flow $U = U_0 j_1 + U'(x_1, -\frac{x_2}{2}, -\frac{x_3}{2})^T$, where $T$ denotes the transpose, and $u_{\text{old}} = U + u$, becomes

$$
\begin{aligned}
u_t &+ U_0 \partial_{x_1} u + U' \begin{pmatrix} u_1 \\ -\frac{u_2}{2} \\ -\frac{u_3}{2} \end{pmatrix} + U' \begin{pmatrix} x_1 \\ -\frac{x_2}{2} \\ -\frac{x_3}{2} \end{pmatrix} \cdot \nabla u \\
&+ U \cdot \nabla U = \nu \Delta u + \sum_{k \neq 0} c_k^{1/2} d\beta_t^k e_k \qquad (2) \\
u(x,0) &= 0
\end{aligned}
$$

- Each $e_k = e^{2\pi i k \cdot x}$ comes with its own independent Brownian motion $\beta_t^k$, $c_k << 1$ are small coefficients representing small (white) noise

# The Exponentially-Growing Noise



The formula for the solution of the Navier-Stokes equation linearized about the initial flow $U(x)$ is

$$u(x,t) = \sum_{k\neq 0} \int_0^t e^{-(4\nu\pi^2|k|^2 + 2\pi i U_0 k_1)(t-s)} \times$$

$$\begin{pmatrix} e^{-U'(t-s)} & 0 & 0 \\ 0 & e^{\frac{U'}{2}(t-s)} & 0 \\ 0 & 0 & e^{\frac{U'}{2}(t-s)} \end{pmatrix} c_k^{1/2} d\beta_t^k e_k + O(|U'|)$$

if $|U'| << 1$ is small. This is clearly noise that is growing *exponentially* in time, in the $x_1$ direction, if $U' < 0$

# Saturation by the Nonlinearities

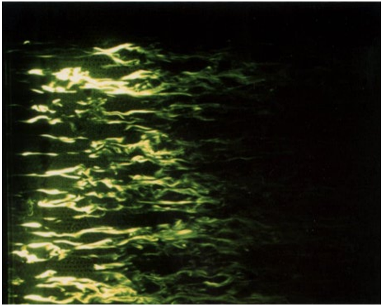


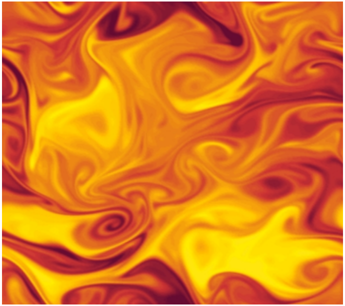
- This growth of the noise does not continue forever
- The exponential growth is saturated by the nonlinear terms in the Navier-Stokes equation
- The result is large noise that now drives the turbulent fluid
- Thus for fully developed turbulence we get the Stochastic Navier-Stokes driven by the large noise

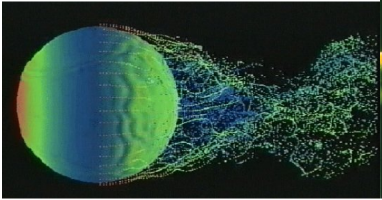
$$\begin{aligned} du &= (\nu\Delta u - u\cdot\nabla u + \nabla\{\Delta^{-1}[trace(\nabla u)^2]\})dt \\ &+ \sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k \end{aligned}$$

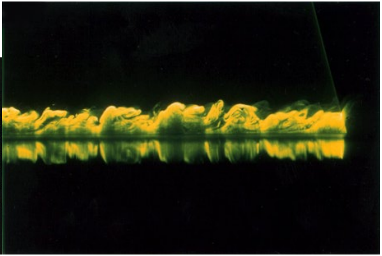
- Determining how fast the coefficients $h_k^{1/2}$ decay as $k \to \infty$ is now a part of the problem

# Turbulent Flow and Boundary Layers









# Outline

# The Iteration



- Now we find a solution $u+U$ of the Navier-Stokes equation by Picard iteration
- Starting with $u_0$ as the first iterate

We use the solution of the linearized equation

$$u_o(x,t)=\sum_{k\neq 0} h_k^{1/2}\int_0^t e^{-(4\pi^2\nu|k|^2+2\pi iU_1k_1)(t-s)}d\beta_s^k e_k(x)$$

where $U_1$ is now the mean flow in fully-developed turbulence

$$u(x,t)=u_o(x,t)+\int_{t_o}^t K(t-s)*[-u\cdot\nabla u+\nabla\Delta^{-1}(trace(\nabla u)^2)]ds \tag{3}$$

# Probabilistic Leray's Theory



In the deterministic case, by taking the inner product of $u$ with the Navier-Stokes equation one can show that the $L^2$ and similary the $H^1$ norm are bounded

$$\begin{aligned} E(|u|_2^2)(t) &\leq E(|u|_2^2(0))e^{-2\nu\lambda_1 t} + \frac{1-e^{-2\nu\lambda_1 t}}{2\nu\lambda_1}\sum_{k\neq 0} h_k \\ E(\int_0^t |\nabla u|_2^2(s)ds) &\leq \frac{1}{2\nu}E(|u|_2^2(0)) + \frac{t}{2\nu}\sum_{k\neq 0} h_k \qquad (4) \end{aligned}$$

It is well known, in the deterministic case, that such a priori bounds allow one to prove the existence of weak solutions but do not give the uniqueness of such solutions

# The Kolmogorov-Obukhov Scaling



In 1941, Kolmogorov formulated his famous scaling theory of the inertial range in turbulence, stating that the second-order structure function, scales as

$$S_2(x) = \langle |u(y+x) - u(y)|^2 \rangle \sim (\epsilon |x|)^{2/3},$$

where $y, y+x$ are points in a turbulent flow field, $u$ is the component of the velocity in the direction of $x$, $\epsilon$ is the mean rate of energy dissipation, and the angle brackets denote an (ensamble) average. A Fourier transform yields the Kolmogorov-Obukhov power spectrum in the inertial range

$$E(k) = C\epsilon^{2/3} k^{-5/3},$$

where C is a constant, and $k$ is the wave number. These results form the basis of turbulence theory

# Kolmogorov's Condition

Kolmogorov (1941) found a necessary condition for the existence of a statistically stationary state

- The pressure and inertial terms must be able to drive each other as they were white noise

$$u \cdot \nabla u = \frac{\eta}{\sigma}$$

- Let $u \sim x^\alpha$ and $\sigma \sim x^\alpha$

$$2\alpha - 1 = -\alpha, \quad \alpha = \frac{1}{3}$$

- Using the pressure gives the same result!

$$\nabla\{\Delta^{-1}[trace(\nabla u)^2]\} = \frac{\eta}{\sigma}$$

- This holds in the Sobolev space with index $\frac{11}{6}$ of Hölder continuous functions with index $\frac{1}{3}$

# Oscillations of the Fluid Velocity



- The iteration scheme converges quickly because of rapid oscillations driven by the fast uniform flow U

$$\hat{w} = \frac{1}{2}\int_0^T (w(s) - w(s + \frac{1}{2kU_1}))e^{-2\pi i k_1 U_1 s} ds$$
$$= -\frac{1}{4k_1 U_1}\int_0^T \frac{\partial w}{\partial s}(s) e^{-2\pi i k_1 U_1 s} ds + O(\frac{1}{(kU)^2})$$

- This will give an priori estimate for $\|u\|$ and contraction
- We also need an estimate on $(Ku)_t$

$$|\frac{\partial (Ku)}{\partial t}|_2 \leq |u_0|_2 + |u|_\infty |\nabla u|_2 + |\nabla u|_4^2$$

# Swirl = Uniform flow + Rotation



- We are in trouble when the Fourier coefficients $\hat{w}(0, k_2, k_3, t)$ do not depend on $k_1$
- To deal with these component we have to start with some rotation with angular velocity $\Omega$ and axis of rotation $j_1$
- But this opens the possibility of a resonance between the uniform flow $U_1 j$ and the rotation with angular velocity $\Omega j$
- We rule out such resonances, the values of $U_1$ and $\Omega$ are chosen so that $U_1 j_1$ and $\Omega j_1$ are not in resonance

# In which Sobolev Space do we need to work?

- This is the same questions as
  - How rough can the noise be?
  - What Sobolev space dominates $|\nabla u|_4$?
- Sobolev's inequality

$$|\nabla u|_4 \leq C\|u\|_{\frac{n}{2}-\frac{n}{4}+1}$$

- In 3-d the index is $\frac{7}{4}$
- By the Sobolev imbedding theorem, the solutions are Hölder conditions with exponent $\frac{1}{4} < \frac{1}{3}$
- Thus we should work in the Sobolev space $W^{(11/6,2)}$, containing Hölder continuous functions with index $\frac{1}{3}$

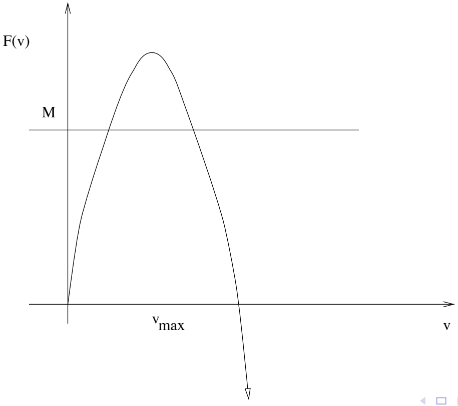
# The a Priori Estimate



- An a priori estimate

$$\frac{d}{dt}\|u\|^2 - \frac{C}{U_1}\|u\|^6 \leq \nu\|\nabla u\|^2 + \|f\|^2$$

- If $\|f\|^2 < M$ and $u(x,0)$ is small, then $\|u\|^2(t) < M$



# Onsager's Conjecture Dimensions



We let $\mathcal{L}^2_{(m,p)}$ denote the space of functions in $W^{(m,p)}$ whose Sobolev norm lies in $L^2(\Omega, P)$

## Theorem

*If*

$$E(\|u_o\|^2_{(\frac{11}{6}^+,\,2)}) \leq \frac{1}{2}\sum_{k\neq 0}\frac{(1+(2\pi|k|)^{(11/3)^+})}{(2\pi|k|)^2}h_k < \frac{CU_1}{24} \qquad (5)$$

*where the uniform flow $U_1$ and the angular velocity $\Omega$ are sufficiently large, so that*

$$\text{ess sup}_{t\in[0,\infty)} E(\|u\|^2_{(\frac{11}{6}^+,\,2)})(t) < C(U_1 + |\Omega|) \qquad (6)$$

*holds, then the integral equation (3) has unique global solution $u(x,t)$ in the space $C([0,\infty); \mathcal{L}^2_{(\frac{11}{6}^+,\,2)})$, $u$ is*

# What are the properties of these solutions are?



- They are stochastic processes that are continuous both in space and in time
- But they are not smooth in space, they are Hölder continuous with exponent $\frac{1}{3}$
- However, there is no blow-up in finite time
- Instead the solutions roughens
- The solutions start smooth, $u(x, 0) = 0$, but as the noise gets amplified they roughen, until they have reached the characteristic roughness $\chi = \frac{1}{3}$ in the statistically stationary state
- Neither $\nabla u$ nor $\nabla \times u$ are continuous in general

# Outline

# Doob's Theorem on Invariant Measure



A Markovian semigroup $P_t$ is said to be $t_0$-regular if all transition probabilities $P_{t_0}(x, \cdot), x \in H$ are mutually equivalent. In 1948 J. L. Doobs proved the following

## Theorem

*Let $P_t$ be a stochastically continuous Markovian semigroup and $\mu$ an invariant measure with respect to $P_t$, $t \geq 0$. If $P_t$ is $t_0$ regular for some $t_0 > 0$, then*

- *$\mu$ is strongly mixing and for arbitrary $x \in H$ and $\Gamma \in \sigma(H)$*
$$\lim_{t \to \infty} P_t(x, \Gamma) = \mu(\Gamma)$$
- *$\mu$ is the unique invariant probability measure for the semigroup $P_t$, $t \geq 0$*
- *$\mu$ is equivalent to all measures $P_t(x, \cdot)$, for all $x \in H$ and $t > t_0$*

# Enter the Probabilists

- In 1996 Sinai proves the existence of an invariant measure for the stochastically-driven one-dimensional Burger's equation
- In 1994 Flandoli proves the existence of an invariant measure for the stochastically-driven 2-dimensional Navier-Stokes equation
- In 1996 Flandoli and Maslowski prove the uniqueness for the stochastically-driven 2-dimensional Navier-Stokes equation
- In 1996 G. Da Prato and J. Zabczyk publish the "green" book "Ergodicity for Infinite Dimensional System" where the general theory is explained and applied to many different systems
- 1998 Mattingly shows that finitely many stochastic components also give a unique invariant measure for the stochastically-driven 2-d Navier-Stokes

# A Unique Invariant Measure

- We must prove the existence of a unique invariant measure to prove the existence of Kolmogorov's statistically invariant stationary state
- A measure on an infinite-dimensional space $H$ is invariant under the transition semigroup $R(t)$, if

$$R_t \int_H \varphi(x) d\mu = \int_H \varphi(x) d\mu$$

- Here $x = u(t)$ is the solution of N-S and $R_t$ is induced by the Navier-Stokes flow
- We define the invariant measure by the limit

$$\lim_{t \to \infty} E(\phi(u(t))) = \int_H \phi(u) d\mu(u) \tag{7}$$

# Existence and Uniqueness of the Invariant Measure



- This limit exist if we can show that the sequence of associated probability measures are tight
- If the N-S semigroup maps bounded function on $H$ onto continuous functions on $H$, then it is called Strongly Feller
- Irreducibility says that for an arbitrary $b$ in a bounded set

$$P(sup_{\{t<T\}}\|u(t)-b\|<\epsilon)>0$$

- Tightness amounts to proving the existence of a bounded and compact invariant set for the N-S flow
- Strongly Feller is a generalization of a method developed by McKean (2002) in "Turbulence without pressure"
- Irreducibility is essentially a problem in stochastic control theory

# Thightness



## Lemma

*The sequence of measures*

$$\frac{1}{T}\int_0^T P(u_o,\cdot)dt$$

*is tight.*

$$\frac{1}{T}\int_0^T \mathbb{P}(\|u(t)\|^2_{(\frac{11}{6}^+,2)} < R)dt > 1-\epsilon$$

for $T \geq 1$. By Chebychev's inequality

$$\frac{1}{T}\int_0^T \mathbb{P}(\|u(t)\|^2_{(\frac{11}{6}^+,2)} \geq R)dt \leq \frac{1}{R}C(U_1+|\Omega|) < \epsilon$$

for $R$ sufficiently large.

# Strong Feller

## Lemma

*The Markovian semigroup $P_t$ generated by the integral equation (3) is strongly Feller.*

Following McKean (2002),

$$\phi_t(u) - \phi_t(v) = \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} BM \int_0^1 \nabla\phi(h)\cdot(u - v)(y,t)w(x = x_t, t)\,drdxdy$$

where $w(x,t) = \frac{\partial u(x,t)}{\partial u(y,0)}$ and $h = v + (u-v)r$. Thus

$$|\phi_t(u) - \phi_t(v)| \leq C(|U_1|^2 + \|u\|^2_{(\frac{11}{6}^+,\,2)})|\phi|_\infty \|(u-v)\|_{(\frac{11}{6}^+,\,2)}$$

# Irreducibility

## Lemma

*The Markovian semigroup $P_t$ generated by the integral equation (3) is irreducible.*

Consider the deterministic equation

$$y_t + U_1 y_x = \nu \Delta y - y \cdot \nabla y + Qh(x,t)$$
$$y(x,0) = 0, \quad y(x,T) = b(x)$$

where $Q : H^{-1} \to L^2(\mathbb{T}^3)$ and kernel $Q$ is empty. By Gronwall's inequality

$$E(\|u-y\|^2_{(\frac{11}{6}^+,2)}) \leq E(\|u_o(x,t) - y_o(x,t)\|^2_{(\frac{11}{6}^+,2)})\sqrt{T} \leq \epsilon$$

# Outline

# The Infinite Dimensional Brownian Motion



Consider the heat equation driven by noise

$$dw = \nu \Delta w dt + \sum_{k \neq 0} h_k^{1/2} d\beta_t^k e_k$$

The solution is

$$w = \int_0^t K(t-s, x_s - y) dx_s(y)$$

where $K$ is the heat kernal and $x_t = \sum_{k \neq 0} h_k^{1/2} \beta_t^k e_k$ is the infinite-dimensional Brownian motion. The invariant measure for $w$ is $\mu_P = \mu_G * \mu_{wG}$, where $\mu_G$ is a Gaussian and $\mu_{wG}$ a weighted Gaussian measure.

# Girsanov's Theorem



Now consider the Stochastic Navier-Stokes Equation

$$du = (\nu \Delta u - u \cdot \nabla u)dt + \sum_{k \neq 0} h_k^{1/2} d\beta_t^k e_k$$

and let

$$dy_t = u(t,x)dt + dx_t$$

Then

$$M_t = e^{\int_0^t u(s,x_s)\cdot dx_s - \frac{1}{2}\int_0^t |u(s,x_s)|_2^2 ds}$$

is a Martingale since $u$ satisfies Novikov's Condition

$$E(e^{\frac{1}{2}\int_0^t |u(s,x_s)|_2^2 ds}) < \infty$$

# The Absolute Continuity of the Invariant Measure



The solution of the Stochastic Navier-Stokes equation can be written as

$$u = \int_0^t K(t-s, x_s - y) M_s dx_s(y)$$

and it follows from Girsanov's theorem that the invariant measure of $u$ is

$$d\mu_Q = M_t d\mu_P$$

or the Radon-Nikodym derivative is

$$\frac{d\mu_Q}{d\mu_P} = M_t$$

We conclude that $\mu_Q$ is absolutely continuous with respect to $\mu_P$

$$\mu_Q << \mu_P$$

# Kolmogorov's Conjecture



- We want to prove Kolmogorov's statistical theory of turbulence
- Kolmogorov's statistically stationary state (3-d)

$$S_2(x,t) = \int_H |u(x+y,t) - u(y,t)|_2^2 d\mu(u) \sim |x|^{2/3}$$

**Theorem**

*In three dimensions there exists a statistically stationary state, characterized by a unique invariant measure, and possessing the Kolmogorov scaling of the structure functions*

$$S_2(x) \sim |x|^{2/3}$$

# Proof of Kolmogorv's Conjecture



**Proof.**

For $x$ small the statement follows by Hölder continuity.
For $x$ larger we write $d\mu_Q$ in terms of $d\mu_P$. But the scaling of $s_2(w)$ can be computed explicitly. Thus

$$c|x\cdot(L-x)|^{2/3} \leq S_2(x) \leq C|x\cdot(L-x)|^{2/3}$$

where $c$ and $C$ are constants, $x\in\mathbb{T}^3$ and $L$ is a three vector with entries the sizes of the faces of $\mathbb{T}^3$ □

As bonus we get the relationship between the Lagrangian $y_t$ and Eulerian $x_t$ motion of a fluid particle

$$y_t = \int_0^t u(s,x_s)ds + x_t$$

# Outline

# Conclusions

- Starting with sufficiently fast constant flow in some direction there exist turbulent solutions
- These solutions are Hölder continuous with exponent $\frac{1}{3}$ (Onsager's conjecture)
- There exist a unique invariant measure corresponding to these solutions
- This invariant measure give a statistically stationary state where the second structure function of turbulence scales with exponent $\frac{2}{3}$ (Kolmogorov's conjecture)
- In one and two dimensions the same works but the scaling exponents are respectively $3/2$ (Hack's Law) and 2 (Batchelor-Kraichnan Theory)

# Applications

- Better closure approximations for RANS
- Better subgrid models for LES
- To do this we must be able to approximate the invariant measure
- Applications to Rayleigh-Bénard experiments (Ahlers, UCSB) and Taylor-Couette (LANL)
- For RB we want to compute the heat transport
- Apply to compute the $CO_2$ production in river basins, and the Pacific boundary layer around the equator (RB), to help NCAR (Tibbia) with weather and climate predictions

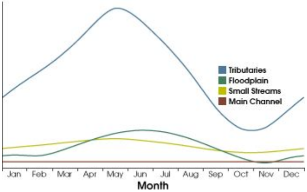
# The Artist by the Water's Edge

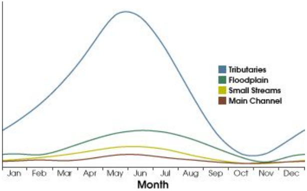
## Leonardo da Vinci Observing Turbulence



# Turbulence and Global Warming

## Flooded Area (left) $CO_2$ Released (right) From Melack et al.




- The area covered by the Amazon River and its tributaries more than triples over the course of a year.
- The rivers in the Amazon Basin carry a large amount of dissolved carbon dioxide gas. As the river system floods each year, a huge amount of this carbon dioxide is released into the atmosphere
- By identifying the carbon dioxide being transferred from the rivers of the Amazon Basin to the atmosphere, scientists are trying to understand the role the Amazon plays in the global carbon cycle

# Numerical Simulations made Accurate



- There is no universal way now of approximating the eddy viscosity (closure problem)

$$\overline{u} \cdot \nabla \overline{u} - \overline{u \cdot \nabla u} \sim \nu_{\text{eddy}} |\overline{u}| \Delta \overline{u}$$

- Knowing $\mu$ we will be able to solve the closure problem to any desired degree of accuracy
- The same applies to LES (Large Eddy Simulations)
- Currently LES use a Gaussian as a cut-off function or the simulation is cut off below a certain spatial scale
- We will be able to find the correct cut-off function (subgrid models) for any desired accuracy