A RATE OF CONVERGENCE FOR LAGRANGIAN AVERAGED NAVIER-STOKES EQUATIONS

ED WAYMIRE

BASED ON JOINT WORK WITH
LARRY CHEN
RON GUENTHER
SUN-CHUI KIM
ENRIQUE THOMANN

Incompressible Navier-Stokes on a periodic domain

$$D = [-L, L]^3, L > 0$$

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = \nu \Delta v - \nabla p + g$$

$$\nabla \cdot v = 0, \quad v(x, 0) = v_0(x), \quad x \in D$$

$$v = (v_1, v_2, v_3), \qquad x = (x_1, x_2, x_3)$$

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REGULARIZATION
WHAT & WHY ?

$$\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v = \nu \Delta v - \nabla p + g$$

- SPATIAL FILTER
  $$u = G * v$$

- MATHEMATICAL TECHNIQUE: Leray existence theory for NS.

- COMPUTATIONAL TECHNIQUE: Flow structure may exist below grid in high Reynolds NS.

Gallavotti Principle: Maintain Kelvin Circulation Theorem

Time rate of change of momentum (per unit mass) around a closed material loop moving with the regularized fluid velocity should be an integral over viscous and external forces acting on the fluid.

Leray's regularization did not satisfy this principle.

Foias, Holm, Titi (2002) DERIVED LANSalpha

$$\frac{d}{dt}\oint_{\gamma(u)} v \cdot dx = \oint_{\gamma(u)} (\nu\Delta v + g)dx$$

Incompressible LANSalpha on a periodic domain

$$D = [-L, L]^3, L > 0$$

LANSalpha $\quad \alpha \geq 0$

$$\frac{\partial \mathsf{v}^{(\alpha)}}{\partial t} + \nabla \cdot (\mathsf{u}^{(\alpha)} \otimes \mathsf{v}^{(\alpha)}) + (\nabla \mathsf{u}^{(\alpha)})^{\mathsf{T}} \mathsf{v}^{(\alpha)} = \nu \Delta \mathsf{v}^{(\alpha)} - \nabla p + \mathsf{g}$$

$$\nabla \cdot \mathsf{v}^{(\alpha)} = 0, \qquad (1 - \alpha^2 \Delta) \mathsf{u}^{(\alpha)} = \mathsf{v}^{(\alpha)}$$

AVERAGING OPERATOR

$$u = G * v = (I - \alpha^2 \Delta)^{-1} v$$

CURRENT THEORY (BRIEF SURVEY)

- Foias, Holm, Titi (2002), Marsden, Shkoller (2003): Existence and regularity theory based on energy estimates.
- Kolmogorov Scaling and Attractor Dimension Estimates.
- Foias, Holm, Titi (2002), Linshutz, Titi (2007): Convergence of subsequences as $\alpha \downarrow 0$.
- Computational numerical experiments.

Fourier Expansion:

$$v(x,t) = \sum_{k \in \mathbf{Z}^3} \hat{v}(k,t) e^{i\beta k \cdot x}$$

Aspect Ratio: $\beta = \dfrac{2\pi}{2L}$

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$$\frac{\partial \hat{v}(k,t)}{\partial t} + i\beta \left(k \sum_l \hat{u}(l,t) \otimes \hat{v}(k-l,t) + \sum_l l\hat{u}(l,t) \cdot \hat{v}(k-l,t) \right)$$

$$= -\nu |\beta k|^2 \hat{v} - i\beta k \hat{p}(k,t) + \hat{g}.$$

WHERE

$$\hat{u}(k,t) = \frac{\hat{v}(k,t)}{1 + \alpha^2 |\beta k|^2}$$

A SIMPLER PROBABILISTIC DRESS
- FOR ILLUSTRATION -

$$\frac{dv}{dt} = \Delta v + g \qquad \frac{\partial \hat{v}}{\partial t} = -|k|^2 \hat{v} + \hat{g}$$

$$\hat{v}(k,t) = \hat{v}_0(k) e^{-|k|^2 t} + \int_0^t e^{-|k|^2 s} \hat{g}(k, t-s) ds$$

$$P(S > t) = e^{-|k|^2 t}$$

$$\hat{v}(k,t) = \hat{v}_0(k) e^{-|k|^2 t} + \int_0^t |k|^2 e^{-|k|^2 s} \frac{\hat{g}(k, t-s)}{|k|^2} ds$$

$$X(k,t) = \begin{cases} \hat{v}_0(k) & \text{if} \quad \mathrm{S} > \mathrm{t} \\ \frac{\hat{g}(k,t-S)}{|k|^2} & \text{if} \quad \mathrm{S} \leq \mathrm{t} \end{cases}$$

$$\chi(\mathsf{k},t) = \frac{\hat{\mathsf{v}}(\mathsf{k},t)}{h(\mathsf{k})}, \qquad \varphi(\mathsf{k},t) = \frac{\hat{\mathsf{g}}(\mathsf{k},t)}{\nu|\beta\mathsf{k}|^2 h(\mathsf{k}) q_3}$$

$$\begin{aligned}
\chi(\mathsf{k},t) &= \exp[-\nu|\beta\mathsf{k}|^2 t]\chi_0(\mathsf{k}) \\
&+ \sum_{l=0}^{2} q_l \int_0^t \nu|\beta\mathsf{k}|^2 \exp[-\nu|\beta\mathsf{k}|^2 s] \sum_{\mathsf{j},\mathsf{n}} \mathsf{m}_l^{(\alpha)}(\mathsf{j},\mathsf{n}) Q_l(\chi(\mathsf{j},t-s),\chi(\mathsf{n},t-s);\mathsf{j},\mathsf{n}) W(\mathsf{j},\mathsf{n};\mathsf{k})\, ds \\
&+ q_3 \int_0^t \nu|\beta\mathsf{k}|^2 \exp[-\nu|\beta\mathsf{k}|^2 s]\, \varphi(\mathsf{k},t-s) ds \qquad (2.1
\end{aligned}$$

MAIN INGREDIENTS

Multipliers: $m_l^{(\alpha)}(j,n)$

(Branching) Quadratic Forms: $Q_l(\cdot,\cdot)$

Wave Number Transition Probabilities: $W(j,n:k)$

Offspring Type Probabilities: q_l

EXPECTED VALUE OF WHAT ?

$$\mathbb{X}^{(\alpha)}(\mathsf{k}_{\langle \mathsf{v} \rangle}, t) =$$

$$\begin{cases} \chi_0(\mathsf{k}_{\langle \mathsf{v} \rangle}) & \text{if } S_{\langle \mathsf{v} \rangle} \geq t \\ \\ \varphi(\mathsf{k}_{\langle \mathsf{v} \rangle}, t - S_{\langle \mathsf{v} \rangle}) & \text{if } S_{\langle \mathsf{v} \rangle} < t, \text{ and } \kappa_{\langle \mathsf{v} \rangle} = 3 \\ \\ \mathsf{m}_l^{(\alpha)}(\mathsf{k}_{\langle \mathsf{v}1 \rangle}, \mathsf{k}_{\langle \mathsf{v}2 \rangle}) Q_l \left(\mathbb{X}^{(\alpha)}(\mathsf{k}_{\langle \mathsf{v}1 \rangle}, t - S_{\langle \mathsf{v} \rangle}), \mathbb{X}^{(\alpha)}(\mathsf{k}_{\langle \mathsf{v}2 \rangle}, t - S_{\langle \mathsf{v} \rangle}); \mathsf{k}_{\langle \mathsf{v}1 \rangle}, \mathsf{k}_{\langle \mathsf{v}2 \rangle} \right) \\ \quad \text{if } S_{\langle \mathsf{v} \rangle} < t, \text{ and } \kappa_{\langle \mathsf{v} \rangle} = l \neq 3. \end{cases}$$

$$\mathcal{F}_{h,T} = \left\{ v \in L^2 : \sup_{0\le t\le T, \mathbf{k}\ne \mathbf{0}} \frac{|\hat{v}(k,t)|}{h(\mathbf{k})} < \infty \right\}$$

Theorem 3.1 *Assume that* $\hat{v}_0(k), \hat{g}(k,s)$ *and* $h(k)$ *are such that* $\mathbf{E}(|\chi^{(\alpha)}(k,t)|)$ *is finite for all* $k \in \mathbf{Z}^3$, $0 \le t \le T$. *Then* $\hat{v}^{(\alpha)}(k,t) = h(k)\mathbf{E}(\chi^{(\alpha)}(k,t))$ *is a mild solution of the LANS*α *equation.*

SLEDGE HAMMER APPROACH: MAKE

$$|X^{(\alpha)}(k,t)| \le 1$$

$$\mathsf{m}_0^{(\alpha)}(\mathsf{j},\mathsf{n}) = \mathsf{m}(\mathsf{k})\frac{1}{q_0(1+\alpha^2|\beta\mathsf{j}|^2)} \leq \frac{\mathsf{m}(\mathsf{k})}{q_0},$$

$$\mathsf{m}_l^{(\alpha)}(\mathsf{j},\mathsf{n}) = \mathsf{m}(\mathsf{k})\frac{\alpha^2|\beta\mathsf{j}|^l|\frac{\beta\mathsf{k}}{2}|^{2-l}}{(1+\alpha^2|\beta\mathsf{j}|^2)(1+\alpha^2|\beta\mathsf{n}|^2 q_l}.$$

$$m(k) = \frac{h * h(k)}{h(k)\nu|\beta k|}$$

LEMMA

The following inequality holds for any $\alpha, \beta > 0$ and $\mathsf{k} \in \mathbf{Z}^3$.

$$\frac{\alpha^2|\beta \mathsf{k}||\beta \mathsf{j}|}{(1+\alpha^2|\beta \mathsf{j}|^2)(1+\alpha^2|\beta \mathsf{k} - \beta \mathsf{j}|^2)} \leq 1.$$

$$m(k) = \frac{h * h(k)}{h(k)\nu|\beta k|}, \quad W(j,n;k) = \frac{h(j)j(n)}{h * h(k)}\delta_k(j+n)$$

$$Q_0(\mathsf{a},\mathsf{b};\mathsf{j},\mathsf{n}) = -i(\mathsf{e}_\mathsf{k}\cdot\mathsf{a})\pi_\mathsf{k}(\mathsf{b}),$$

$$Q_1(\mathsf{a},\mathsf{b};\mathsf{j},\mathsf{n}) = -i\pi_\mathsf{k}(\mathsf{e}_\mathsf{n})(\mathsf{a}\cdot\mathsf{b}),$$

$$Q_2(\mathsf{a},\mathsf{b};\mathsf{j},\mathsf{n}) = i\pi_\mathsf{k}(\mathsf{e}_\mathsf{n})(\mathsf{e}_\mathsf{j}\cdot\mathsf{e}_\mathsf{n})(\mathsf{a}\cdot\mathsf{b}).$$

$$|Q_l(\mathsf{a},\mathsf{b};\mathsf{j},\mathsf{n})| \leq |\mathsf{a}||\mathsf{b}|.$$

SMALL BALL APPROACH: CHOOSE A RADIUS R:

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NOTE ON ROLE OF MAJORIZING CONSTANTS:

$$\mathcal{F}_h = \mathcal{F}_{ch}, \quad c > 0 \qquad || \cdot ||_{ch} = \frac{1}{c} || \cdot ||_h$$

MAJORIZING KERNEL: $h * h(k) \leq C|k|h(k)$

$$ch * ch(k) \leq cC|k|ch(k)$$

SMALL BALL APPROACH: CHOOSE A RADIUS R:

NOTE ON ROLE OF MAJORIZING CONSTANTS:

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MAJORIZING KERNEL: $h * h(k) \leq C|k|h(k)$

$$Rh * Rh(k) \leq RC|k|Rh(k)$$

$$|\hat{v}_0(\mathsf{k})| \leq Rh(\mathsf{k}), \qquad |\hat{g}(\mathsf{k}, t)| \leq \nu|\beta\mathsf{k}|^2 Rh(\mathsf{k})q_3,$$

$$m(k) = \frac{h * h(k)}{h(k)\nu|\beta k|}$$

$$\mathsf{m}_l^{(\alpha)}(\mathsf{k}, \mathsf{j}) \leq 1, \quad l = 0, 1, 2.$$

RECALL SLEDGE HAMMER CONDITION

(FOLLOWING IS A COROLLARY)

Theorem 3.2 *Let h be a standardized majorizing kernel. Take $q_3 = \frac{1}{2}$, and $q_0 = q_1 = q_2 = \frac{1}{6}$. Let $B_R \subseteq \mathcal{F}_h$ denote the ball of radius R centered at 0, where $R = \frac{\cancel{(2L)^3}\nu\beta}{6}$. If the $\mathsf{v}_0 \in B_R$ and $\Delta^{-1}g \in B_{\frac{\nu R}{2}}$ then the solution of each LANSα, $\hat{\mathsf{v}}_\alpha(\mathsf{k}, t)$ exists and is unique for all $t > 0$. Moreover, for each $\mathsf{k} \in \mathbf{Z}^3$ one has*

$$\lim_{\alpha \to 0} \mathsf{v}^{(\alpha)}(\mathsf{k}, t) = \mathsf{v}^{(0)}(\mathsf{k}, t).$$

RATE OF CONVERGENCE

Theorem 5.1 *Let $h \in l^1(\mathbf{Z}^3)$ be a standardized majorizing kernel satisfying the following further moment conditions:*

$$\sum_{j} |j| h(j) < \infty, \quad \sum_{j} |j|^l h(j) h(k-j) < \infty, \quad k \in \mathbf{Z}^3, l = 2, 3.$$

Take $q_0 = q_1 = q_2 = \frac{1}{6}$ and $q_3 = \frac{1}{2}$. Let $\gamma = \frac{\nu\beta^2}{2}$. Let $R = \frac{\nu\beta}{6}$ and suppose $v_0 \in B_R$, $\Delta^{-1} g \in B_{\frac{\nu R}{2}}$. Then LANS$\alpha$ has a unique global solution for all $\alpha \geq 0$. Moreover, there is a positive constant $A(T)$, not depending on α, such that

$$\int_0^T ||v^{(\alpha)}(\cdot, t) - v^{(0)}(\cdot, t)||_{L^2(T^3)} dt \leq A(T)\alpha.$$

MOMENTS OF ALL ORDERS:

Corollary 4.1 *The function*

$$h(k) = \frac{e^{-|k|}}{|k|}, \quad k \in \mathbf{Z}^3, k \neq \mathbf{0}, \quad h(\mathbf{0}) = 0,$$

defines a majorizing kernel. In fact, $h \in l_1$ *is normalizable to a probability.*

LEJAN-SZNITMAN -- BESSEL--HELMHOLTZ TYPE !

Proposition 4.1 *For measurable $h : \mathbf{R}^3 \to [0, \infty)$, define*

$$h *_c h(\xi) := \int_{\mathbf{R}^d} h(\xi - \eta) h(\eta) d\eta, \quad \xi \in \mathbf{R}^3,$$

and

$$h *_d h(\mathbf{k}) := \sum_{\mathbf{k} \in \mathbf{Z}^3} h(\mathbf{k} - \mathbf{j}) h(\mathbf{j}), \quad \mathbf{k} \in \mathbf{Z}^3.$$

Suppose

$$h *_c h(\xi) \leq c|\xi| h(\xi), \quad \xi \in \mathbf{R}^3.$$

Let $Q_{\mathbf{k}}(1)$ denote the unit cube centered at $\mathbf{k} \in \mathbf{Z}^3$. If there are constants c_1, c_2 such that

$$c_2 h(\mathbf{k}) \leq h(\eta) \leq c_1 h(\mathbf{k}), \quad \forall \eta \in Q_{\mathbf{k}}(1),$$

then

$$c_2^2 h *_d h(\mathbf{k}) \leq h *_c h(\mathbf{k}) \leq c_1^2 h *_d h(\mathbf{k}), \quad \mathbf{k} \in \mathbf{Z}^3.$$

In particular,

$$h *_d h(\mathbf{k}) \leq \frac{c}{c_2^2} |\mathbf{k}| h(\mathbf{k}), \quad \mathbf{k} \neq \mathbf{0}.$$

APPROACH TO RATES:

Gronwall inequality to an integral equation for the difference

$$\delta(\mathsf{k},t) = \hat{\mathsf{v}}^{(\alpha)}(\mathsf{k},t) - \hat{\mathsf{v}}^{(0)}(\mathsf{k},t), \qquad \mathsf{k} \in \mathbf{Z}^3, \quad \Delta(t) := \sup_{\mathsf{k}} |\delta(\mathsf{k},t)|, t \geq 0.$$

$$\tilde{\Delta}(t) = e^{\gamma t}\Delta(t), \quad t \geq 0. \quad \gamma = \frac{\nu\beta^2}{2}$$

$$\tilde{\Delta}(t) \leq M^* \left(\alpha^2 e^{\gamma t} + \int_0^t \frac{\tilde{\Delta}(s)ds}{\sqrt{\nu(t-s)}}\right), \quad t \geq 0.$$

INVERT ABEL TRANSFORM:

$$\tilde{\Delta}(t) \leq \alpha^2 M^* \left(e^{\gamma t} + \frac{1}{\sqrt{\nu}}c(t)\right) + \frac{M^{*2}\pi}{\nu}\int_0^t \tilde{\Delta}(s)ds.$$