Analysis of Ruin Probability under investment for non Markovian interarrival times

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Main points

Establish a connection between high order integro-differential equations and ruin probability

Analyze asymptotic properties of the ruin probability in non-Markovian cases.

Introduce a framework for analyzing investment strategies in relation to the Risk process.

Basic Modeling Assumptions

Claims occurring at random times $T_k = \tau_1 + \tau_2 ... + \tau_k$ with interarrival times τ_k having a density $f_\tau(t)$.

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Investment of capital and premium into a risky asset satisfying a SDE

$$dZ = \mu(Z)dt + \sigma(Z)dW.$$

Solution with $Z_0 = u$ denoted by Z_t^u. It is assumed that $Z_t^u > 0$ if $u > 0$.

Model for Risk Process - Renewal Jump-Diffusion Process

Initial capital $u, U^u(0) = u$

$$U^u(t) = \begin{cases} Z_{t-T_k}^{U^u(T_k)} & \text{for } T_k \le t < T_{k+1} \\ U^u(T_K^-) - X_k & \text{for } t = T_k \end{cases}$$

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In these examples, the Risk process is a Markov Process

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1. Gamma dist with integer (shape) parameter n

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What is required for the framework developed here is that $f_\tau(t)$, the density of the interrarival times, satisfy a constant coefficient ode of order n, and, if $n > 1$,

$$f_\tau^{(k)}(0) = 0, \text{for} \quad k = 0, 1, ..., n-2.$$

$$0 = \mathcal{L}(\frac{d}{dt}) f_\tau(t) \equiv \sum_{k=0}^{n} \alpha_k \frac{d^k f}{dt^k}$$

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Example: Sparre-Andersen model, no investments.

Basic Questions:

Let $T_u = \inf\{s > 0 : U_s^u \leq 0\}$ the first passage time trough 0. Determine an equation for $\psi(u) = P(T_u < \infty)$ (Ruin Probability)

Find the asymptotic behavior of the ruin probability as the initial capital $u \to \infty$.

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No investment - Classical Result - Cramer-Lundberg

Assume claim size distribution is $F_X(x) = 1 - e^{-x/\mu}$ and $c/(\lambda\mu) > 1$. Then

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In general, if $h(r) = \mathbf{E}(e^{rX}) - 1$, and ν is the positive solution of the Lundberg equation

$$\lambda h(r) = cr,$$

then, with $K = (\lambda\mu - c)/(c + \lambda\mathcal{F}'(-\nu))$, $\mathcal{F}$ the Laplace Stieltjes transform of $F_X(x)$,

$$\psi(u) \sim K e^{-\nu u}$$

Investment - Exponential size claim distribution (Frolova, Kabanov, Pergamenshikov) Geometric Brownian model for risky asset, $dZ = aZdt + \sigma ZdW$, If

$$\rho = \frac{2a}{\sigma^2} > 1$$

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Investment - General size claim distribution (Constantinescu - MS Thesis) Assume $h(r)$ moment generating function of the claim size X is defined in a neighborhood of the origin. Assume

$$\rho = \frac{2a}{\sigma^2} > 1$$

then, as $u \to \infty$

$$\psi(u) \sim u^{1-\rho}.$$

Basic ingredients of proof

(i) Markov property of the Ruin process. This determines an integro-differential equation for the ruin probability $\psi(u)$.

$$(c + au)\frac{d}{du}\psi + \frac{1}{2}\sigma^2 u^2 \frac{d^2}{du^2}\psi - \lambda\psi = \lambda \int_0^\infty \psi(u-x)dF_X(x).$$

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(ii) $\widehat{\psi}(s) =$ Laplace transform of ψ satisfies a forced second order ode with $\mathcal{F}(s)$ the Laplace Stieltjes transform of $F_X(x)$

$$\frac{s^2\sigma^2}{2}\frac{d^2}{ds^2}\widehat{\psi} + (2s\sigma^2 - as)\frac{d}{ds}\widehat{\psi} + (cs - \lambda + \lambda\mathcal{F}(s) + \sigma^2 - a)\widehat{\psi} = c\psi(0) - \frac{\lambda}{s}(1 - \mathcal{F}(s))$$

(iii) Perturbation theory to characterize behavior of $\widehat{\psi}$ near 0.

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(iv) Karamata Tauberian Theorems to relate behavior at the origin for $\widehat{\psi}(s)$ into behavior at infinity of $\psi(u)$.

Generator of the Renewal Jump Diffusion Process

The discrete time ruin process $U^u_k = U^u(T_k)$ is a Markov process with generator

$$Tg(u,0) - g(u,0) = \mathbf{E}(g(Z_{\tau_1}, X_1) | Z_0 = u, X_0 = 0) - g(u,0)$$

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Let A denote the infinitesimal generator of Z, eg

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Denote by $\mathcal{L}^*(\frac{d}{dt})$ the formal adjoint of $\mathcal{L}$ i.e.

$$\mathcal{L}^*(\frac{d}{dt}) = \sum_{j=0}^{n} (-1)^j \alpha_j \frac{d^j}{dt^j}$$

Theorem: For h sufficiently smooth (eg $h \in \mathcal{C}_0^\infty \cap \mathcal{D}_{A^n}$) set $g(u,x) = h(u-x)$. Then

$$\mathcal{L}^*(A)Th(u) = \alpha_0 \, \mathbf{E}g(u, X_1) = \alpha_0 \int_0^\infty h(u-x)dF_X(x).$$

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Pf: Take $n = 2$ so

$$\mathcal{L}(\frac{d}{dt}) = \frac{d^2}{dt^2} + \alpha_1 \frac{d}{dt} + \alpha_0$$

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$$\begin{aligned} \alpha_0 Tg(u,0) &= \int_0^\infty \int_0^\infty \mathbf{E}(g(Z_t, x) | Z_0 = u) \alpha_0 f_\tau(t) dF_X(x) dt \\ &= -\sum_{j=1}^{2} \alpha_j \int_0^\infty \int_0^\infty \frac{d^j f_\tau}{dt^j} \mathbf{E}(g(Z_t, x) | Z_0 = u) dF_X(x) dt \end{aligned}$$

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&= \int_0^\infty g(u,x) f_\tau'(0)dF_X(x)
\end{aligned}$$

$$-\int_0^\infty \int_0^\infty \sum_{j=1}^{2} (-1)^j \alpha_j \frac{d^j}{dt^j} \mathbf{E}(g(Z_t,x)|Z_0 = u) f_\tau(t)dF_X(x)dt$$

Let $T_t^\sharp g(u,x) = \mathbf{E}(g(Z_t,x)|Z_0 = u)$. Note that for $t > 0$, and $h \in \mathcal{D}_{A^n}$, $T_t^\sharp(Ag) = A(T_t^\sharp g)$ and

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Now use the regularity of h to justify that

$$\int_0^\infty \int_0^\infty A(A(\mathbf{E}(g(Z_t,x)|Z_0 = u)))f_\tau(t)dF_X(x)dt$$
$$= A^2\left[\int_0^\infty \int_0^\infty \mathbf{E}(g(Z_t,x)|Z_0 = u)f_\tau(t)dF_X(x)dt\right]$$
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In particular, if h is T harmonic, i.e. $(T-I)h(u) = 0$, then

$$\mathcal{L}^*(A)(h) = \alpha_0 \int_0^\infty h(u-x)dF_X(x). \quad \text{(IDE)}$$

For example, since the non ruin probability, $\Phi(u) = P_u(U_k > 0 \ \forall k)$ is T harmonic, it satisfies the integrodifferential equation (IDE). As a consequence, the ruin probability $\psi(u)$ also satisfies the same (IDE).

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Exponential holding times

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Now, use $\Phi = 1 - \psi$, to get equation for ψ.

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Gamma$(2, \lambda)$ holding times, no investments

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Then, equation for ruin probability

$$\left(-c\frac{d}{du} + \lambda\right)^2 \psi = \lambda^2 \int_0^\infty \psi(u-x) dF_X(x).$$

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Asymptotic behavior of the ruin probability. $\rho = 2a/\sigma^2$.

$$\psi(u) \sim \begin{cases} u^{1-\rho} & 1 < \rho < 2 \\ u^{-2\sqrt{\lambda/\sigma^2}} & \rho = 1 \\ u^{-\alpha} & \rho < 1 \end{cases} \quad \alpha = \sqrt{\left(\frac{1}{2}(1-\rho)\right)^2 + \frac{4\lambda}{\rho}} - \frac{1}{2}(1-\rho)$$