Stochastic multiscale analysis and reconstruction of time series of stochastic cascade processes

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Overview

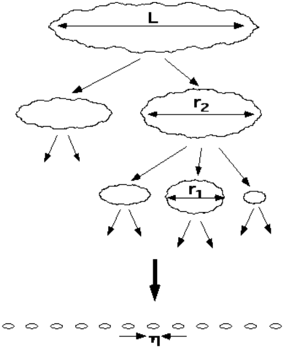
1. Stochastic cascade processes
   - Description by Fokker-Planck equation
   - Obtaining the Fokker-Planck equation
   - Verification
   - Summary on stochastic cascade process
2. Multiscale reconstruction
   - Idea of a hypothetical cascade
   - Reconstruction procedure
   - Example
   - Prediction of pdf
3. Conclusions

Motivation of stochastic cascades

Picture for . . .

Turbulence: Large vortices generate smaller ones

Finance: Market information is distributed hierarchically



⇒ stochastic cascade process evolving in r

Description by Fokker-Planck equation

Characterization of disorder by joint n-scale statistics of the
velocity increment $v_r(x) := u(x+r) - u(x)$
or log return $P_r(t) := \log P(t+\tau) - \log P(t)$

General processes: All n-scale joint PDF required
$$p(v_n, r_n; v_{n-1}, r_{n-1}; \ldots; v_0, r_0)$$

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Simplification 1: Markov process – Complete information in conditional 2-point PDF $p(v_i, r_i | v_j, r_j)$.

Simplification 2: Included noise is Gaussian distributed.

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⇒ Description by Fokker-Planck equation

$$-\frac{d}{dr} p(v, r | v_0, r_0) = \left\{ -\frac{\partial}{\partial v} D^{(1)}(v, r) + \frac{\partial^2}{\partial v^2} D^{(2)}(v, r) \right\} p(v, r | v_0, r_0)$$

Obtaining the Fokker-Planck equation

Drift and Diffusion coefficients $D^{(1)}$ and $D^{(2)}$

$$D^{(k)}(v,r) = \lim_{\Delta r \to 0} \frac{1}{k!\Delta r} M^{(k)}(v,r,\Delta r)$$

with $$M^{(k)}(v,r,\Delta r) = \left\langle (v_{r+\Delta r} - v_r)^k \right\rangle \Big|_{v_r = v}$$

Obtaining the Fokker-Planck equation

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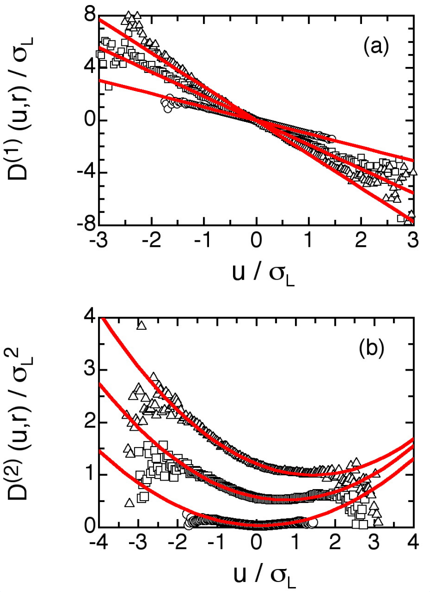
The Drift and Diffusion coefficients $D^{(1)}$ and $D^{(2)}$ can be obtained directly from experimental data.

Kolmogorov, Math. Ann. **104**, 415 (1931)

Kramers-Moyal coefficients



Examples for

$(a) \quad D^{(1)}(u_r, r)$

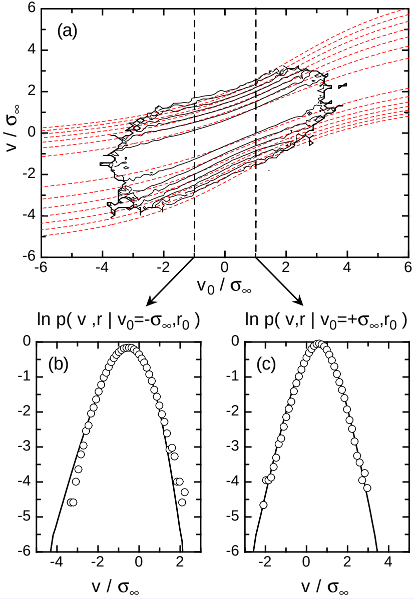
and

$(b) \quad D^{(2)}(u_r, r)$

from a Helium free jet

Renner, JFM **433** (2001)

Verification of the Fokker-Planck equation



Reconstructed

and empirical

conditional pdf

$p(v_r, r|v_{r_0}, r_0)$

for $r_0 = L$ and $r = 0.6L$

Summary on stochastic cascade process

- Stochastic description by Fokker-Planck equation captures complete n-scale statistics
- Successful application to and new insights for a number of examples:
  - turbulent flows,
  - traffic flow,
  - surface topographies,
  - financial markets,
  - ...

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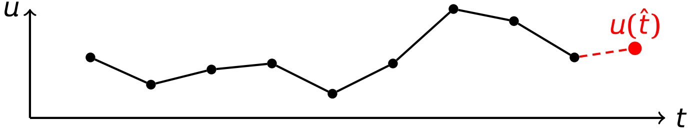
Remaining question:

How to create synthetic time series of stochastic cascade processes?

Multiscale reconstruction: Idea

Problem: Both Fokker-Planck and Langevin equation describe the scale domain but not the time domain.



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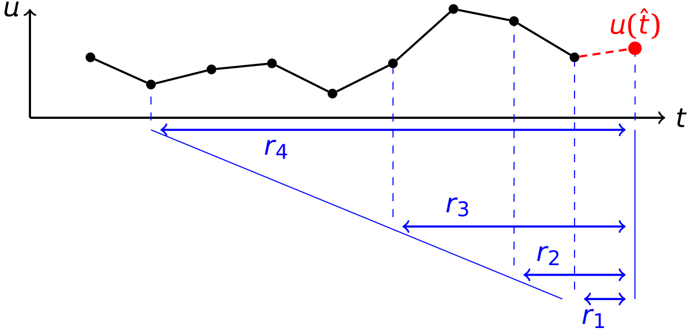

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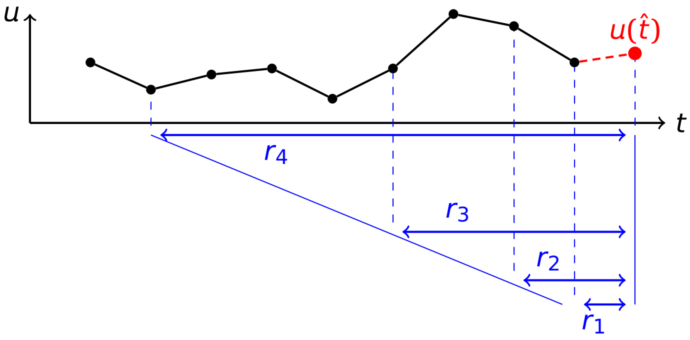


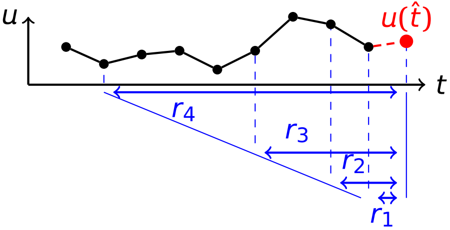
... and derive $p\left(u(\hat{t})\right)$ from n-scale pdf

Reconstruction procedure

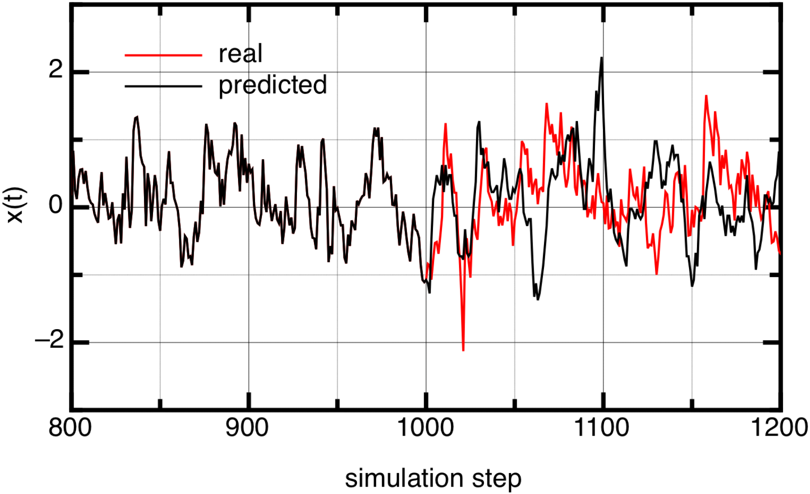
1. Use right-bounded increments $v_r(t) = u(t) - u(t-r)$
2. For the next time step $\hat{t}$, determine the n-scale joint pdf (writing $v_{r_i}(\hat{t}) = v_i$)

$$p(v_1; v_2; \ldots; v_n) = p(v_1|v_2) \cdot p(v_2|v_3) \cdot \ldots \cdot p(v_{n-1}|v_n) \cdot p(v_n)$$
$$\Rightarrow p\left(u(\hat{t})\right)$$

3. Draw a sample from that distribution $p\left(u(\hat{t})\right)$



Reconstruction procedure: Example



Nawroth, Phys. Lett. A **360** (2006)

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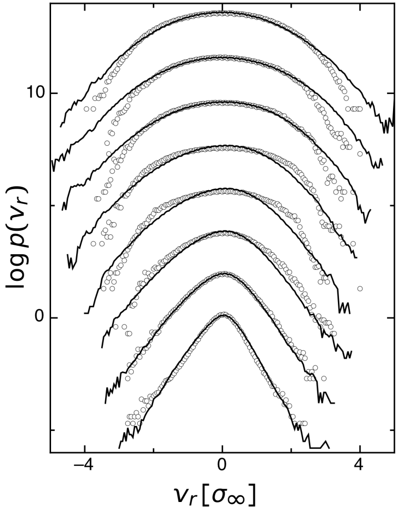
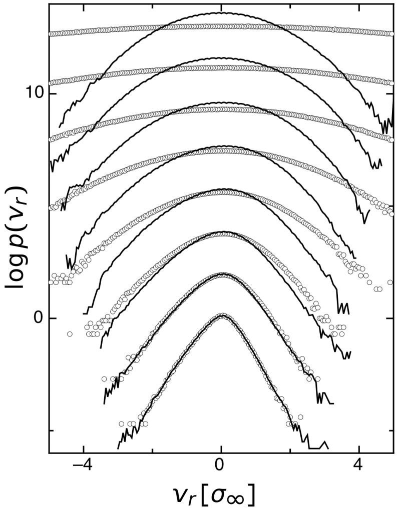
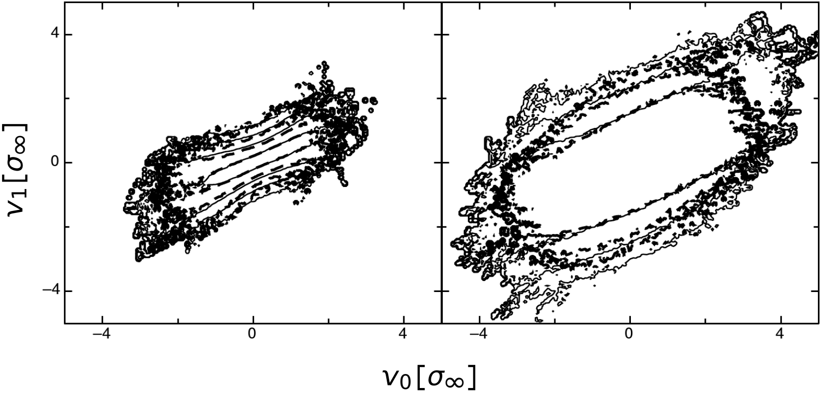


Multiscale reconstruction



Single scale reconstruction

Reconstruction procedure: Example



Pdf $p(v_1|v_0)$ for small and large separation $r_0 - r_1$

Conclusions

- Many Stochastic cascade processes (such as turbulence and financial markets) can be comprehensively described by a Fokker-Planck equation
- Result: Knowledge of complete n-scale joint pdf
- Recent results allow synthetic reconstruction of time series
- Method estimates complete pdf for next value, including prediction of standard deviation/volatility/gusts
- Open questions:
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Thank you for listening!