Pathwise Stationary Solutions of SPDEs and Infinite Horizon BDSDEs

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1. The problem

The problem is to find the stationary solution of the following SPDEs

$$\begin{aligned} dv(t,x) &= [\mathcal{L}v(t,x) + f(x, v(t,x), \sigma^*(x)Dv(t,x))]dt \\ &\quad + g(x, v(t,x), \sigma^*(x)Dv(t,x))dB_t, \qquad (1) \\ v(0,x) &= h(x). \end{aligned}$$

Here B_s is a two-sided Q-Brownian motion on a separable Hilbert space U ($Q \in L(U)$ is a nonnegative and symmetric trace class operator),

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

with $(a_{ij}(x)) = \sigma\sigma^*(x)$.

2. The concept of stationary solutions

Consider a *random dynamical system* on a measurable space $(S, \mathcal{B})$ over a metric DS $(\Omega, \mathcal{F}, P, \{\theta(t)\}_{t\in\mathrm{T}})$ with time

T

$$\phi : \mathrm{T} \times \Omega \times S \rightarrow S, \quad (t, \omega, h) \rightarrow \phi(t, \omega, h),$$

A *stationary solution* is a $\mathcal{F}$-measurable random variable $Y^* : \Omega \rightarrow X$ such that

$$\phi(t, \omega, Y^*(\omega)) = Y^*(\theta_t \omega), \quad t \geq 0 \ a.s.$$

This is the corresponding notion of a steady or equilibrium state in deterministic dynamical systems $\phi : S \rightarrow S$.

Example 1 Simplest ever nontrivial example:

As a random perturbation to the deterministic equation

$$\frac{dy}{dt} = -y, \quad y(0) = h,$$

we consider the Ornstein-Uhlenbeck process

$$dy = -ydt + dB_t, \quad y(0) = h.$$

Variation of constant formula gives the following solution:

$$\phi_t^{\omega} h = he^{-t} + \int_0^t e^{-(t-s)} dB_s(\omega).$$

It is easy to check that

$$Y^*(\omega) = \int_{-\infty}^{0} \mathrm{e}^{s} dB_s(\omega).$$

is the stationary solution of the equation and for any $h \in R^1$, as $t \to \infty$

$$\begin{aligned} & |\phi_t^{\omega} h - Y^*(\theta_t \omega)| \\ = \ & \mathrm{e}^{-t} |h - \int_{-\infty}^{0} \mathrm{e}^{s} dB_s(\omega)| \\ \to \ & 0. \end{aligned}$$

It is well known that

- $$\begin{array}{rcl} Y^*(\theta(t)\omega) & = & \phi(t,\omega)Y^*(\omega) \\ & \Updownarrow & \\ \mu(dx,d\omega) & = & \delta_{Y^*(\omega)}(dx)P(d\omega) \\ \text{is} & \text{an} & \text{invariant measure} \;\; P.a.s. \end{array}$$

- Every ergodic invariant measure μ of a RDS on R^1 is a random Dirac measure.

In general, this is not true. However, the following is also well known:

- Every invariant measure is Dirac by considering the extended probability space:

$$(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, (\bar{\theta}(t))_{t\in\mathbb{T}} = (\Omega \times S, \mathcal{F} \times \mathscr{B}(S), \mu, (\Theta(t))_{t\in\mathbb{T}})$$

and

$$\bar{\phi}(t, \bar{\omega}) = \phi(t, \omega).$$

But, by considering the extended probability space, one regards the dynamical system as noise as well, so the dynamics is different.

Remarks

(i) The stationary solution is not a fixed point in the deterministic sense, but a random moving fixed point or equilibrium of the stochastic system in the state space. It describes the invariance over time along the measurable and P-preserving transformation $\theta_t\colon \Omega \to \Omega$.

(ii) For SPDEs, a stationary solution consists of infinitely many random moving surfaces on the configuration space due to the random external force pumped to the systems constantly.

(iii) Random periodic solution, see a forthcoming paper of Zhao and Zheng.

3. Previous work

The existence of stationary solutions of SPDEs is one of the basic problems: no general methods.

- Sinai (1991, 1996), Stochastic Burgers equations with additive C^3 noise, Feynman-Kac formula and Hopf-Cole transformation, so good regularity is needed.

- E, Khanin, Mazel and Sinai, Annals of Mathematics (2000), Stochastic inviscid Burgers equations with additive C^3 noise (minimizing method)

• Mattingly, 2D Stochastic Navier-Stokes equation with additive noise, CMP (1999)

• Caraballo, Kloeden and Schmalfuss (2004), stochastic evolution equations with small Lipschitz constant and linear noise.

• Mohammed, Zhang and Zhao, Memoirs of AMS, Vol 196 (2008), pp.1-105 (to appear), Stochastic evolution equations and SPDEs with discrete spectrum, integral

equation with linear or additive noise, stable/unstable manifolds.

A basic assumption in invariant manifold theory: there exists a stationary solution.

- Duan, Lu and Schmalfuss, Annals of probability (2003).

- Mohammed, Zhang and Zhao, (2008).

4. BDSDEs (backward doubly stochastic differential equations)–a new tool to the weak solutions of SPDEs

a joint work with Q Zhang (JFA, vol. 252 (2007), 171-219.)

First observe the following trivial time reversal: let $\hat{B}_t = B_{T-t} - B_T$ and $u(t) = v(T-t)$, then

$$\begin{aligned} du(t,x) &= -[\mathcal{L}u(s,x) + f(x,u(t,x),\sigma^*(x)Du(t,x))]dt \\ &\quad +g(x,u(t,x),\sigma^*(x)Du(t,x))d^\dagger\hat{B}_t. \\ u(T,x) &= v(0,x) = h(x) \end{aligned} \tag{2}$$

Fix notation:

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(\hat{B}_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ be two mutually independent standard Brownian motion processes with values on U and R^d. Let $\mathcal{N}$ denote the class of P-null sets. For each $t \geq 0$, we define

$$\begin{aligned} \mathcal{F}_{t,T} &= \mathcal{F}^{\hat{B}}_{t,T} \otimes \mathcal{F}^{W}_{0,t} \bigvee \mathcal{N}, t \leq T; \\ \mathcal{F}_{t} &= \mathcal{F}_{t,\infty}, t \geq 0. \end{aligned}$$

Here for any process $\{\eta_t\}$, $\mathcal{F}^{\eta}_{s,t} = \sigma\{\eta_r - \eta_s; 0 \leq s \leq r \leq t\}$, $\mathcal{F}^{\eta}_{t,\infty} = \bigvee_{T\geq 0} \mathcal{F}^{\eta}_{t,T}$.

Let $\rho(x) = (1+|x|)^q$, $q > 3$. Denote $I = [t, T]$ or $[0, \infty)$, $S^{2,-K}(I; L^2_\rho(R^d, R^1))$ the set of jointly measurable and adapted continuous random processes $\{\psi_t, t \geq 0\}$ with values on $L^2_\rho(R^d, R^m)$ satisfying

$$E[\sup_{t \in I} \mathrm{e}^{-Kt} \|\psi_t\|^2_{L^2_\rho(R^d,R^m)}] < \infty;$$

and $M^{2,-K}(I; L^2_\rho(R^d, R^m))$ the set of jointly measurable and adapted random processes $\{\psi_t, t \geq 0\}$ with values on $L^2_\rho(R^d, R^m)$ satisfying

$$E[\int_I \mathrm{e}^{-Ks} \|\psi_s\|^2_{L^2_\rho(R^d,R^m)} ds] < \infty.$$

An useful fact is that, for $0 \leq T' \leq T$ and arbitrary a, b satisfying $0 \leq a \leq b \leq T$, if a process $h.$ with values on $\mathcal{L}^2_{U_0}(L^2_\rho(R^d, R^1))$ satisfies $\int_a^b \|h_s\|^2_{\mathcal{L}^2_{U_0}} ds < \infty$ and $h(s)$ is $\mathcal{F}^{\hat{B}}_{s,T'} = \mathcal{F}^{B}_{T-T',T'-s}$ measurable,

$$\int_t^{T'} h_s d^\dagger \hat{B}_s = -\int_{T-T'}^{T-t} h_{T-s} dB_s \text{ a.s..} \tag{3}$$

The following BDSDE with for a finite dimensional Brownian motion $\hat{B}$ was first introduced by Pardoux and Peng

(1994): for $0 \leq t \leq s \leq T$,

$$
\begin{aligned}
Y_s &= \xi + \int_s^T f(r, Y_r, Z_r)dr \\
&\quad - \int_s^T \langle g(r, Y_r, Z_r), d^\dagger \hat{B}_r \rangle - \int_s^T \langle Z_r, dW_r \rangle.
\end{aligned}
\tag{4}
$$

Under some conditions (mainly pointwise Lipschitz),

Theorem 1 *(Pardoux and Peng (1994)) For any given $\mathcal{F}_T$-measurable $\xi \in L^2(dP)$, Eq.(4) has a unique solution*

$$(Y_\cdot, Z_\cdot) \in S^{2,0}([t, T]; \mathrm{R}^1) \times M^{2,0}([t, T]; \mathrm{R}^d).$$

Let $(X_s^{t,x})_{0\leq s\leq T}$ be defined by

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dW_s, \quad s > t \\ X_s^{t,x} = x, \;\; 0 \leq s \leq t. \end{cases}$$

and consider Eq.(4) in the following form for $t \leq s \leq T$

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \\ &\quad - \int_s^T \langle g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}), d^\dagger \hat{B}_r\rangle \\ &\quad - \int_s^T \langle Z_r^{t,x}, dW_r\rangle. \end{aligned} \tag{5}$$

Pardoux and Peng also proved that under some strong smoothness conditions of h, b, σ, f and g, $u(t,x) = Y_t^{t,x}$, $(t,x) \in [0,T] \times \mathrm{R}^d$, is independent of $\omega^2 \in \Omega$ and is the unique classical solution of the backward SPDE (2).

Definition 1 *A pair of processes* $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2,0}([t,T]; L_\rho^2(\mathrm{R}^d;\mathrm{R}^1)) \bigotimes M^{2,0}([t,T]; L_\rho^2(\mathrm{R}^d;\mathrm{R}^d))$ *is called a solution of Eq.(5) if for any* $\varphi \in C_c^0(\mathrm{R}^d;\mathrm{R}^1)$*,*

$$\begin{aligned}
&\int_{\mathrm{R}^d} Y_s^{t,x}\varphi(x)dx \\
= \ &\int_{\mathrm{R}^d} h(X_T^{t,x})\varphi(x)dx + \int_s^T \int_{\mathrm{R}^d} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxdr \\
&- \int_s^T \int_{\mathrm{R}^d} g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\varphi(x)dxd^\dagger \hat{B}(r) \\
&- \int_s^T \langle \int_{\mathrm{R}^d} Z_r^{t,x}\varphi(x)dx, dW_r\rangle \quad P-\text{a.s.}.
\end{aligned}$$

(H.1). h is $\mathcal{F}_{T,\infty}^{\hat{B}} \otimes \mathcal{B}_{\mathrm{R}^d}$ measurable and $E[\int_{\mathrm{R}^d} |h(x)|^2 \rho^{-1}(x)dx] < \infty$;

(H.2). f, g are $\mathcal{B}_{\mathrm{R}^d} \otimes \mathcal{B}_{\mathrm{R}^1} \otimes \mathcal{B}_{\mathrm{R}^d}$ measurable and Lipschitz;

(H.3). $\int_0^T \int_{\mathrm{R}^d} |f(s,x,0,0)|^2 \rho^{-1}(x)dxds < \infty$ and $\int_0^T \int_{\mathrm{R}^d} \|g(s,x,0,0)\|^2_{\mathcal{L}^2_{U_0}(\mathrm{R}^1)} \rho^{-1}(x)dxds < \infty$;

(H.4). $b \in C^2_{l,b}(\mathrm{R}^d; \mathrm{R}^d)$, $\sigma \in C^3_{l,b}(\mathrm{R}^d; \mathrm{R}^d \times \mathrm{R}^d)$.

Properties of X

(i) $X_{\cdot}^{t,\cdot} \in M^{p,-K}([0,\infty); L_{\rho}^{p}(\mathrm{R}^d; \mathrm{R}^d))$ for $2 \le p < q-1$.

(ii) A stochastic flow of diffeomorphism $X_s^{t,\cdot} : \mathrm{R}^d \to \mathrm{R}^d$ and denote by $\hat{X}_s^{t,\cdot}$ the inverse flow (See e.g. Kunita). Denote by $J(\hat{X}_s^{t,x})$ the determinant of the Jacobi matrix of $\hat{X}_s^{t,x}$. Then

$$\int_{\mathrm{R}^d} u(X_s^{t,x})\varphi(x)dx = \int_{\mathrm{R}^d} u(x)\varphi(\hat{X}_s^{t,x})J(\hat{X}_s^{t,x})dx.$$

Lemma 1 *(generalized equivalence of norm principle) If $s \in [t, T]$, $\psi : \Omega \times \mathrm{R}^d \to \mathrm{R}^1$ is independent of $\mathcal{F}_{t,s}^W$ and $\psi\rho^{-1} \in L^1(\Omega \otimes \mathrm{R}^d)$, then there exist two constants $c > 0$ and $C > 0$ such that*

$$
\begin{aligned}
cE[\int_{\mathrm{R}^d} |\psi(x)|\rho^{-1}(x)dx]
&\le E[\int_{\mathrm{R}^d} |\psi(X_s^{t,x})|\rho^{-1}(x)dx] \\
&\le CE[\int_{\mathrm{R}^d} |\psi(x)|\rho^{-1}(x)dx].
\end{aligned}
$$

Theorem 2 *Under Conditions (H.1)–(H.4), Eq.(5) has a* *unique solution.*

Note, $X_r^{s,X_s^{t,x}} = X_r^{t,x}$

$$
\begin{aligned}
Y_r^{s,X_s^{t,x}} &= h(X_T^{t,x}) + \int_r^T f(X_u^{t,x}, Y_u^{s,X_s^{t,x}}, Z_u^{s,X_s^{t,x}})du \\
&\quad - \int_r^T g(X_u^{t,x}, Y_u^{s,X_s^{t,x}}, Z_u^{s,X_s^{t,x}})d^\dagger \hat{B}_u \\
&\quad - \int_r^T \langle Z_u^{s,X_s^{t,x}}, dW_u \rangle.
\end{aligned}
$$

For $t \leq s \leq r \leq T$, note $(Y_r^{s,\cdot}, Z_r^{s,\cdot})$ is $\mathcal{F}_{r,\infty}^{\hat{B}} \otimes \mathcal{F}_{s,r}^{W}$ measurable so is independent of $\mathcal{F}_{t,s}^{W}$. Thus by Lemma 1,

$$
\begin{aligned}
& E[\int_s^T \int_{\mathrm{R}^d} (|Y_r^{s,X_s^{t,x}}|^2 + |Z_r^{s,X_s^{t,x}}|^2)\rho^{-1}(x)dxdr \\
\leq\ & C_p E[\int_s^T \int_{\mathrm{R}^d} (|Y_r^{s,x}|^2 + |Z_r^{s,x}|^2)\rho^{-1}(x)dxdr < \infty.
\end{aligned}
$$

Therefore by the uniqueness of the solution of Eq. (5),

Proposition 1 *Assume Conditions (H.1)-(H.4). If we define* $u(t,x) = Y_t^{t,x}$, $v(t,x) = Z_t^{t,x}$, *then* $u(s, X_s^{t,x}) = Y_s^{t,x}$, $v(s, X_s^{t,x}) = Z_s^{t,x}$ *for a.e.* $s \in [t,T]$, $x \in \mathrm{R}^d$ *a.s..*

By the generalized equivalence of norm principle again,

$$
\begin{aligned}
& E\int_0^T\int_{R^d}(u(s,x)^2+v(s,x)^2)\rho^{-1}(x)dxds \\
\leq\ & CE\int_0^T\int_{R^d}(u(s,X_s^{t,x})^2+v(s,X_s^{t,x})^2)\rho^{-1}(x)dxds \\
=\ & CE\int_0^T\int_{R^d}((Y_s^{t,x})^2+(Z_s^{t,x})^2)\rho^{-1}(x)dxds<\infty.
\end{aligned}
$$

Theorem 3 *u is the unique weak solution of the SPDE (2), $(u,\sigma^*\nabla u)\in M^{2,0}([0,T];L_\rho^2(R^d,R^1))\otimes M^{2,0}([0,T];L_\rho^2(R^d,R^d))$ and $\sigma^*\nabla u(s,X_s^{t,x})=Z_s^{t,x}$.*

5. The BDSDE on infinite horizon and stationarity –its solution gives the stationary solution of SPDEs

Consider for $s \geq 0$, $K > 0$

$$
\begin{aligned}
& \mathrm{e}^{-Ks} Y_s \\
= \;& \int_s^\infty \mathrm{e}^{-Kr} f(X_r^{t,\cdot}, Y_r, Z_r) dr + \int_s^\infty K \mathrm{e}^{-Kr} Y_r dr \\
& - \int_s^\infty \mathrm{e}^{-Kr} g(X_r^{t,\cdot}, Y_r, Z_r) d^\dagger \hat{B}_r - \int_s^\infty \mathrm{e}^{-Kr} \langle Z_r, dW_r \rangle,
\end{aligned}
$$

equivalently, for $0 \le t \le s \le T$,

$$\begin{cases} Y_s = Y_T + \int_s^T f(X_u^{t,\cdot}, Y_u, Z_u)du \\ \qquad\qquad - \int_s^T g(X_u^{t,\cdot}, Y_u, Z_u)d^\dagger \hat{B}_u - \int_s^T Z_u dW_u, \\ \lim_{T\to\infty} \mathrm{e}^{-KT} Y_T = 0 \quad a.s.. \end{cases} \tag{6}$$

At the moment, assume Eq. (6) has a unique solution $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2,-K}([0,\infty); L_\rho^2(\mathrm{R}^d, \mathrm{R}^1)) \otimes M^{2,-K}([0,\infty); L_\rho^2(\mathrm{R}^d, \mathrm{R}^d))$. Define

$$\hat{\theta}_r W(s) = W(r+s) - W(r), \quad \hat{\theta}_r \hat{B}(s) = \hat{B}(r+s) - \hat{B}(r).$$

For arbitrary measurable ϕ defined on $(\Omega, \mathcal{F}, P)$, we define

$$(\hat{\theta} \circ \phi)(\omega) = \phi(\hat{\theta}(\omega)).$$

First we have (Kunita, Arnold),

$$\hat{\theta}_r \circ X_s^{t,x} = X_{s+r}^{t+r,x} \quad \text{a.s..}$$

Therefore

$$\begin{aligned} \hat{\theta}_r \circ f(X_s^{t,x}, y, z) &= f(X_{s+r}^{t+r,x}, y, z), \\ \hat{\theta}_r \circ g(X_s^{t,x}, y) &= g(X_{s+r}^{t+r,x}, y) \end{aligned}$$

for arbitrary $r, s \geq 0$ and fixed y, z. Thus

$$\begin{cases} \hat{\theta}_r \circ Y_s = \hat{\theta}_r \circ Y_T + \int_{s+r}^{T+r} f(X_u^{t+r,\cdot}, \hat{\theta}_r \circ Y_{u-r}, \hat{\theta}_r \circ Z_{u-r})du \\ \qquad\qquad - \int_{s+r}^{T+r} g(X_u^{t+r,\cdot}, \hat{\theta}_r \circ Y_{u-r}, \hat{\theta}_r \circ Z_{u-r})d^{\dagger}\hat{B}_u \\ \qquad\qquad - \int_{s+r}^{T+r} \hat{\theta}_r \circ Z_{u-r} dW_u, \\ \lim_{T\to\infty} \mathrm{e}^{-K(T+r)}(\hat{\theta}_r \circ Y_T) = 0 \quad \text{a.s..} \end{cases}$$

Then by uniqueness of the solution of the BDSED on infinite horizon we have

$$\hat{\theta}_r \circ Y_s^{t,\cdot} = Y_{s+r}^{t+r,\cdot}, \quad \hat{\theta}_r \circ Z_s^{t,\cdot} = Z_{s+r}^{t+r,\cdot} \quad \text{a.s..}$$

In particular,

$$\hat{\theta}_r \circ Y_t^{t,\cdot}(\hat{\omega}) = Y_{t+r}^{t+r,\cdot}(\hat{\omega})$$
$$\text{for any } r \geq 0, t \geq 0 \quad \text{a.s..}$$

Note that $v(t, \cdot) = u(T - t, \cdot)$, hence $v(t, x)(\omega) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ is the solution of Eq.(1) for arbitrary $T > 0$. Here $\hat{B}_s(\hat{\omega}) = B_{T-s}(\omega) - B_s(\omega)$. In fact

- $Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ does not depend on T.

For this, if we take $T' \geq T$, then we can show that

$$Y_{T-t}^{T-t,\cdot}(\hat{\omega}) = Y_{T'-t}^{T'-t,\cdot}(\tilde{\omega})$$

when $0 \leq t \leq T$, where $\hat{\omega}(s) = B_{T-s} - B_T$, $0 \leq s < \infty$, and $\tilde{\omega}(s) = B_{T'-s} - B_{T'}$, $0 \leq s < \infty$. Let $\hat{\theta}.$ and $\tilde{\theta}.$ are the shifts of $\hat{\omega}(\cdot)$ and $\tilde{\omega}(\cdot)$ respectively. Since by (7) we have

$$Y_{T-t}^{T-t,\cdot}(\hat{\omega}) = \hat{\theta}_{T-t} Y_0^{0,\cdot}(\hat{\omega}) = Y_0^{0,\cdot}(\hat{\theta}_{T-t}\hat{\omega})$$
$$Y_{T'-t}^{T'-t,\cdot}(\tilde{\omega}) = \tilde{\theta}_{T'-t} Y_0^{0,\cdot}(\tilde{\omega}) = Y_0^{0,\cdot}(\tilde{\theta}_{T'-t}\tilde{\omega}),$$

we just need to show that $\hat{\theta}_{T-t}\hat{\omega} = \tilde{\theta}_{T'-t}\tilde{\omega}$. In fact we have

$$\begin{aligned}
(\hat{\theta}_{T-t}\hat{\omega})(s) &= \hat{\omega}(T-t+s) - \hat{\omega}(T-t) \\
&= (B_{T-(T-t+s)} - B_T) \\
&\quad -(B_{T-(T-t)} - B_T) \\
&= B_{t-s} - B_t.
\end{aligned}$$

The right hand side of the above formula does not depend on T, therefore

$$\hat{\theta}_{T-t}\hat{\omega}(s) = \tilde{\theta}_{T'-t}\tilde{\omega}(s) = B_{t-s} - B_t.$$

That is to say $Y^{T-t,\cdot}_{T-t}(\hat{\omega})$ does not depend on the choice of T.

On probability space $(\Omega, \mathcal{F}_\infty^B \otimes \mathcal{F}_\infty^W, P)$, we define $\theta_t = (\hat{\theta}_t)^{-1}$, $t \geq 0$. We can see that θ_t is a shift w.r.t. $\{B_t\}$. Since $v(t, \cdot)(\omega) = u(T-t, \cdot)(\hat{\omega}) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ a.s.,

$$\begin{aligned} \theta_r v(t, \cdot)(\omega) &= \hat{\theta}_{-r} u(T-t, \cdot)(\hat{\omega}) \\ &= u(T-t-r, \cdot)(\hat{\omega}) = v(t+r, \cdot)(\omega), \end{aligned}$$

for $r \geq 0$ and $T > t + r$ a.s.. In particular, let $Y^*(\omega) = v_0(\omega) = Y_T^{T,\cdot}(\hat{\omega})$. Then above formula implies that a.s.

$$\begin{aligned} \theta_r Y^*(\omega) &= Y^*(\theta_r \omega) = v(r, \omega) \\ &= v(r, v_0(\omega), \omega) = v(r, Y^*(\omega), \omega), \ r \geq 0. \end{aligned}$$

That is to say $v(t,\omega) = Y^*(\theta_t\omega) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ is a stationary solution of the SPDE (1).

Proposition 2 *Under a monotonicity condition, Eq.(6) has a unique solution*

$$\begin{aligned}(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \quad \in \quad & S^{2,-K}([0,\infty); L_\rho^2(\mathrm{R}^d, \mathrm{R}^1)) \\ & \times M^{2,-K}([0,\infty); L_\rho^2(\mathrm{R}^d, \mathrm{R}^d)),\end{aligned}$$

and $u(t,.) = Y_t^{t,\cdot}$ is a weak solution of (2) and $u(t,.)$ is continuous almost surely with respect to t in $L_\rho^2(\mathrm{R}^d, \mathrm{R}^1)$.

6. A result for stochastic Burgers equation (joint with Liu)

We shown the stationary point $Y^*(\omega)$ of the differentiable random dynamical system $U : R \times L^2[0,1] \times \Omega \to L^2[0,1]$ generated by the stochastic Burgers equation with $L^2[0,1]$-noise and large viscosity, is the unique solution of the following equation

$$Y^*(\omega) = \frac{1}{2} \int_{-\infty}^{0} T_\nu(-s) \frac{\partial (Y^*(\theta(s,\omega))^2}{\partial x} ds + \int_{-\infty}^{0} T_\nu(-s) dW_s(\omega).$$