Continuous Time Random Walks in the Continuum Limit

Simulation of Atmospheric Wind Speeds

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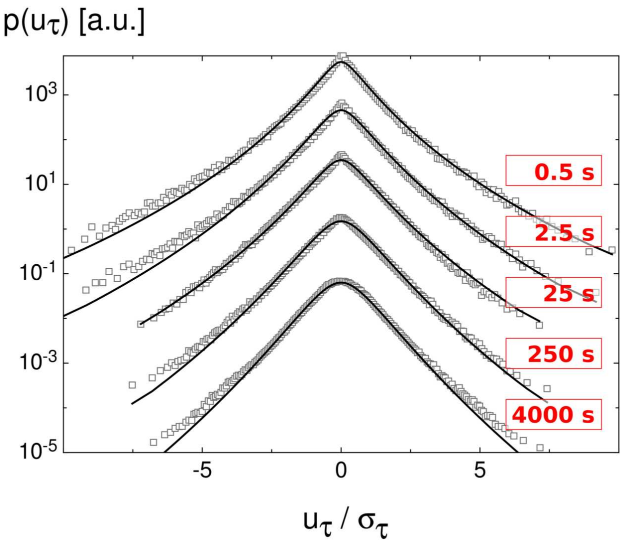
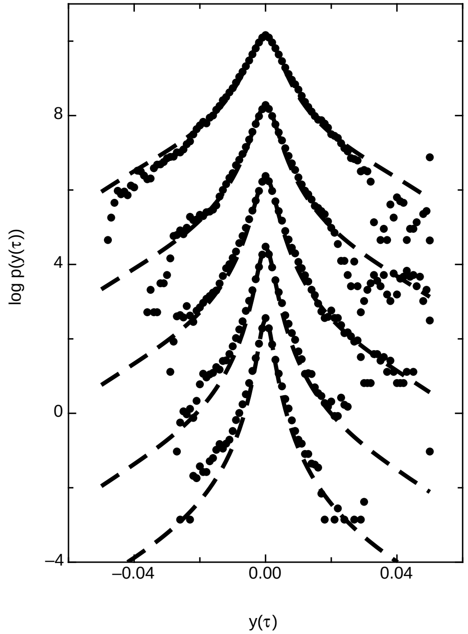
Turbulence and Finance:

Instationary Processes

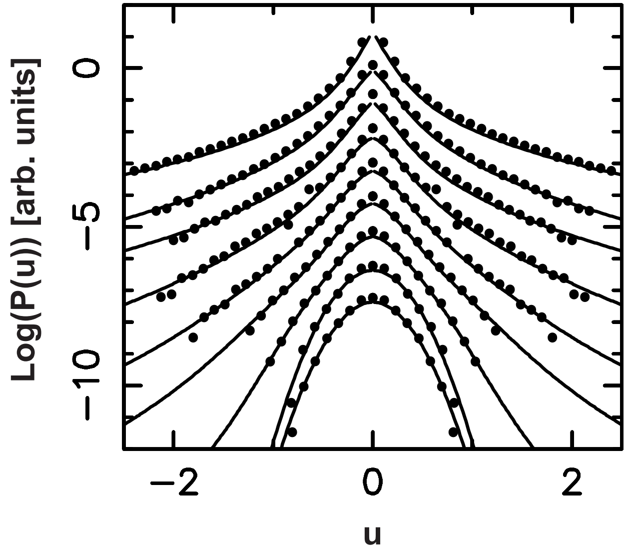
Motivation



[Böttcher, Bath & Peinke 2007]



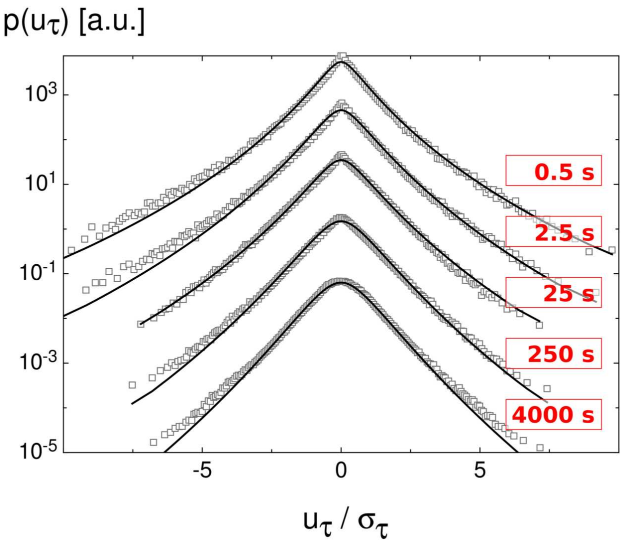
[Nawroth & Peinke 2003]



[Friedrich 2003]

[Mordant, Metz, Michel & Pinton 2001]

Motivation



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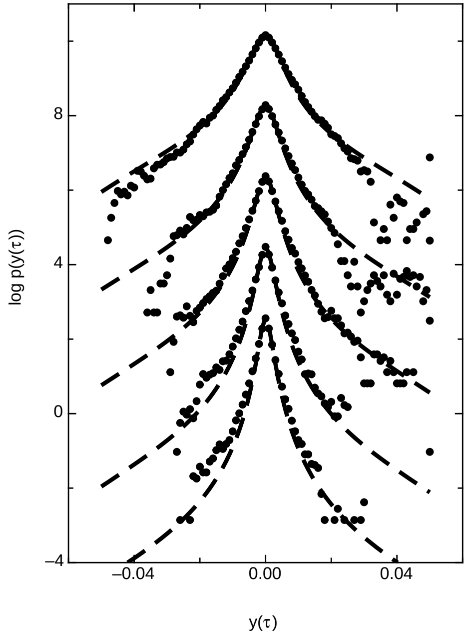

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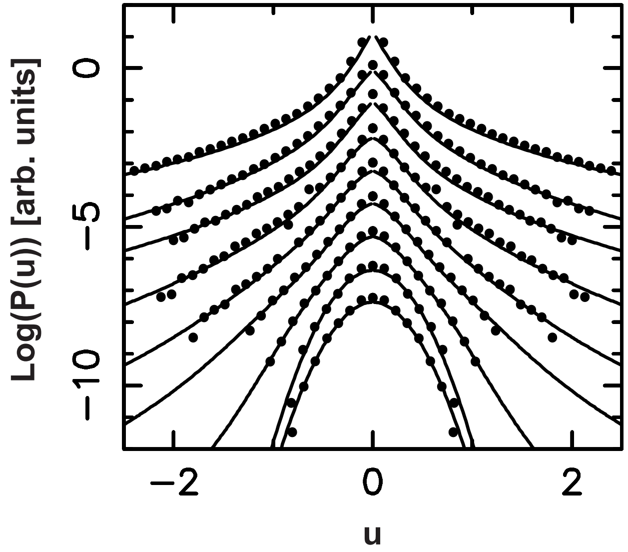
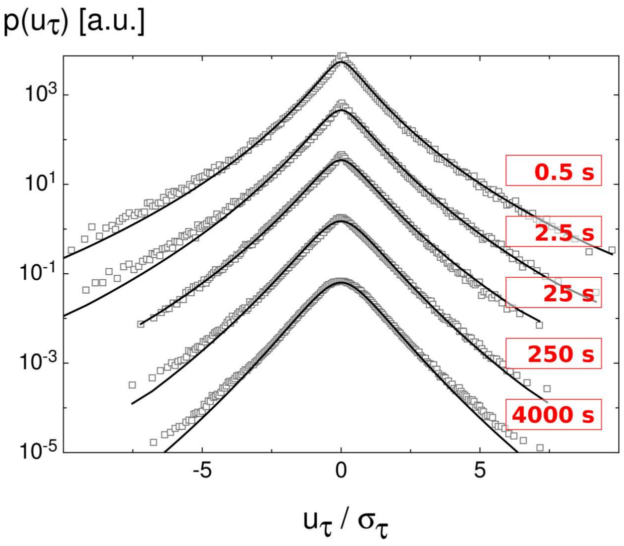


[Friedrich 2003]

[Mordant, Metz, Michel & Pinton 2001]

- CTRWs are potential generators of Lagrangian tracer dynamics
- Similar structure of increment statistis in **Finance** and **Atmospheric turbulence**

Motivation



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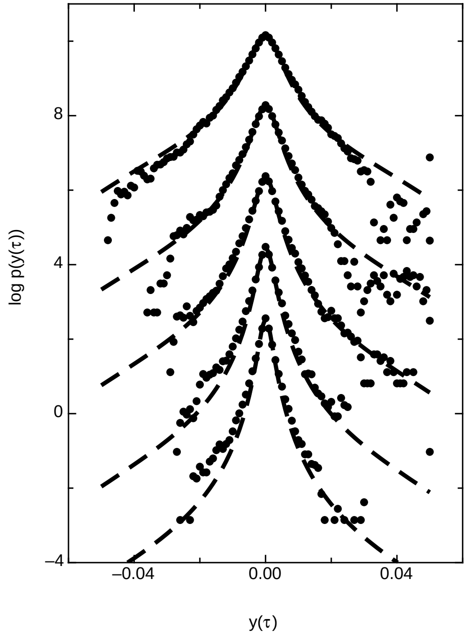

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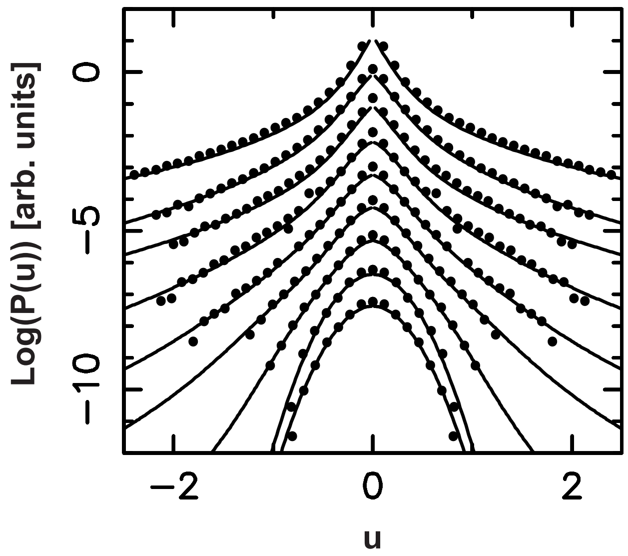

[Friedrich 2003]

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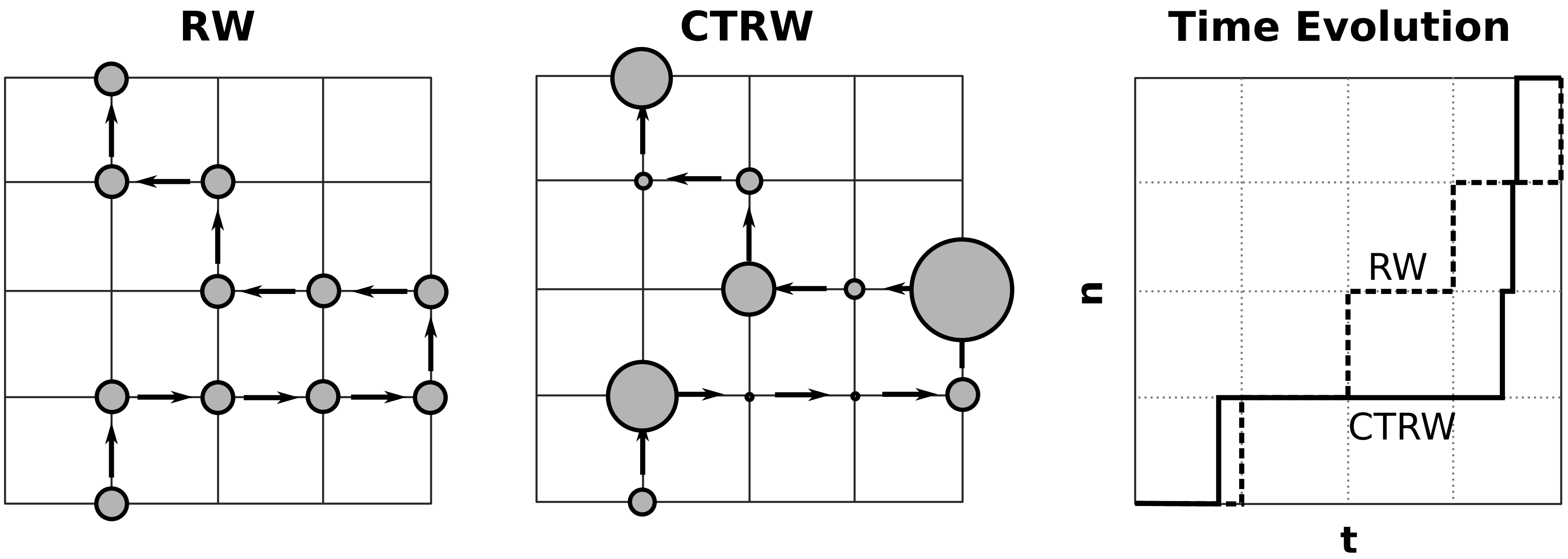
Outline:

- **Introduction to Continuous Time Random Walks (CTRWs)**
  - Initial definition
  - Continuous sample paths, application to finance
- **CTRW model for atmospheric turbulence**

Continuous time random walks (CTRWs)

Discrete Random Walks



[Metzler & Klafter 2000]

Continuous time random walk:

$$x_{i+1} = x_i + \eta_i$$

$$t_{i+1} = t_i + \tau_i$$

PDFs of jumping times are **continuous**, paths are **discontinuous**

Diffusion Limit of CTRWs

Evolution of CTRWs

- Sums of random variables $\Rightarrow$ Fourier / Laplace-Representation
- Montroll-Weiss equation:

$$\hat{\tilde{P}}(\boldsymbol{k}, u) = \frac{1 - \hat{P}_t(u)}{u\left[1 - \tilde{P}_x(\boldsymbol{k})\hat{P}_t(u)\right]} \quad . \tag{1}$$

Diffusion Limit of CTRWs

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Diffusion Limit ($\Rightarrow$ Long Time)

- Assumptions:
  - Jump PDF has finite variance
  - Asymptotical Power-Law decay $\sim x^{-(1+\alpha)}$ of waiting time PDF (heavy tailed)
- Fractional Diffusion equation

$$\frac{\partial}{\partial t}W(\boldsymbol{x},t) = \delta(\boldsymbol{k})\delta(t) + \frac{\sigma^2}{T^\alpha}\,{}_0D_t^{1-\alpha}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}W(\boldsymbol{x},t) \tag{3}$$

Diffusion Limit of CTRWs

With ${}_0D_t^{1-\alpha} f(t)$, $0 < \alpha \leq 1$: **Riemann-Liouville** integro-differential **fractional operator**

$$ {}_0D_t^{1-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^\infty dt' \frac{f(t)}{(t-t')^{1-\alpha}} \tag{4} $$

Connection to integer PDEs: Memory Kernel [Metzler & Klafter 2000, Barkai 2001]

$$ W(\boldsymbol{x}, t) = \int_0^\infty s\; A(s|t) W_1(\boldsymbol{x}, (K_\alpha / K_1)^{1/\alpha}\, s) \tag{5} $$

with

$$ A(s|t) = \frac{1}{\alpha} \frac{t}{s^{1+1/\alpha}} L_{\alpha,1}\left(\frac{t}{s^{1/\alpha}}\right) \tag{6} $$

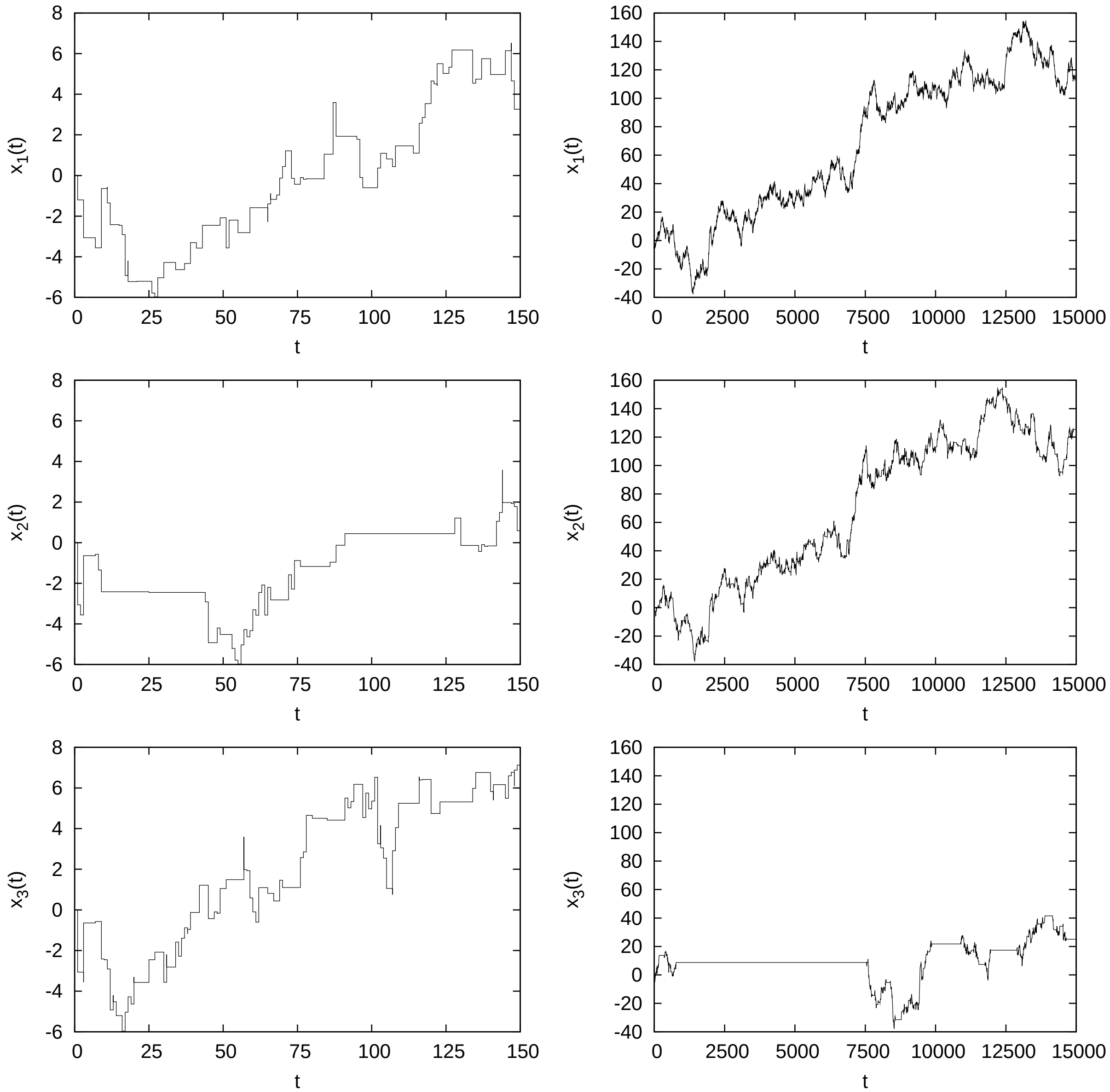
Long Time Limit: $t \gg 1$

Waiting time PDF $P(\tau) =$

$$\begin{cases} \tau \geq 0 & : \quad \sqrt{\frac{2}{\pi\sigma^2}} \exp\left(\frac{\tau^2}{2\sigma^2}\right) \\ \tau < 0 & : \quad 0 \end{cases}$$

$$\begin{cases} \tau \leq 100 & : \quad \mathcal{N}(0.8, 100) L_{0.8,1}(\tau) \\ \tau > 100 & : \quad 0 \end{cases}$$

$L_{0.8,1}(\tau)$



Continuous Trajectories at Finite Time

Adequate **scaling** of processes: Continuous (fractional) trajectories at **finite** time

$\Rightarrow$ Monte-Carlo simulation of fractional PDEs

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Initial work:

Heinsalu, Patriarca, Goychuk, Schmid & Hänggi 2006,

Fractional Fokker-Planck dynamics: Numerical algorithm and simulations

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Fractional Fokker-Planck dynamics: Numerical algorithm and simulations

Recent advancements:

- Magdziarz & Weron 2007,

  Fractional FP dynamics: Stochastic representation and computer simulation

- Gorenflo, Mainardi & Vivoli 2007,

  Continuous-time random walk and parametric subordination in fractional diffusion

- Kleinhans & Friedrich 2007,

  Continuous time random walks: Simulation of continuous trajectories

Continuum limit [Fogedby 1994]

According to Fogedby: **Continuum limit** of discrete equations,

$$\left.\begin{aligned} x_{i+1} &= x_i + \eta_i \\ t_{i+1} &= t_i + \tau_i \end{aligned}\right\} \quad \overset{i \to s}{\Longrightarrow} \quad \left\{\begin{aligned} \frac{\partial}{\partial s} x(s) &= \eta(s) \\ \frac{\partial}{\partial s} t(s) &= \tau(s) \end{aligned}\right.$$

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Notation:

$$\begin{aligned} dx_s &= dW_s \\ dt_s &= dL_s^\alpha \end{aligned}$$

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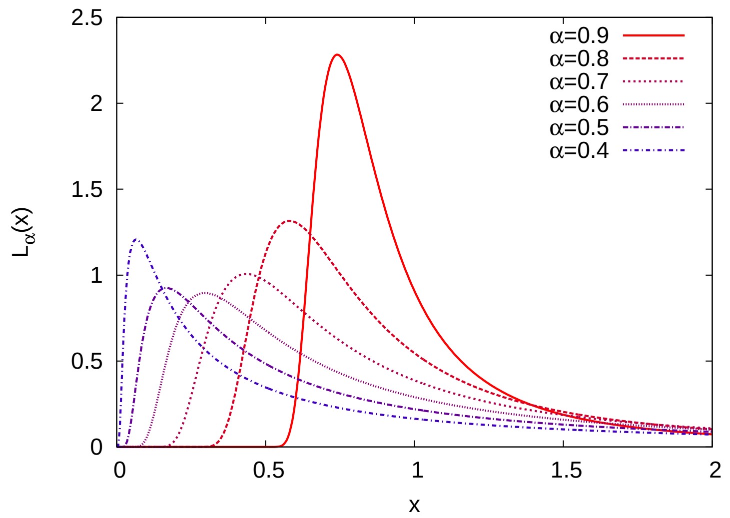
Here: $\eta(s)$ and $\tau(s)$ have to obey **stable distribution**

$$L_{\alpha,1}(x) = \frac{1}{\pi} \mathrm{Re} \left\{ \int_0^\infty dk \; \exp\left[-ikx - |k|^\alpha \exp\left(-i \frac{\pi\alpha}{2} \right) \right] \right\}$$



Result: **Fractional dynamics** at finite time

Intrinsic Time in Finance [Müller 1993]

FRACTALS AND INTRINSIC TIME – A CHALLENGE TO ECONOMETRICIANS

U. A. MÜLLER, M. M. DACOROGNA, R. D. DAVÉ,
O. V. PICTET, R. B. OLSEN AND J. R. WARD *

UAM.1993-08-16

June 28, 1995

Intrinsic Time in Finance [Müller 1993]

6 Intrinsic time: a time scale to model the volatility

The daily and weekly seasonal aspect of volatility has been modeled by the ϑ-time introduced in section 3. The volatility also exhibits non-seasonal, autoregressive clusters, as can be seen in Figures 5 and 6 and in the ARCH literature.

For modeling the volatility in all its aspects, the introduction of intrinsic time is proposed. The intrinsic time τ is defined as the cumulated sum of a market activity variable which is a statistical measure of very recent volatility. The τ value at the j'th time series observation is defined as

$$\tau_j \equiv \tau_{j-1} + k \, \frac{\vartheta_j - \vartheta_{j-1}}{\vartheta_r} \, v_r^{1/D} \, T \qquad (6.1)$$

The last two factors together are inverse scaling law, equation 2.1, applied to a variable v_r, which is the recent volatility (not annualized); ϑ_r is a range parameter (the ϑ-time-interval size of the price changes considered for computing the recent volatility v_r). In the implementation of (Dacorogna et al., 1992), this volatility is defined quite simply as an absolute price change:

$$v_r = |x(\vartheta_j) - x(\vartheta_j - \vartheta_r)| \qquad (6.2)$$

where $\vartheta_r = 1$ hour is chosen to reflect a short-term volatility. The factor k is calibrated in such a way that τ-time flows neither more slowly nor faster than physical time or ϑ-time in the

Numerical algorithm

Numerical Integration Scheme: (Itô)

$$
\begin{aligned}
x(s+\Delta s) &= x(s) + \Delta s F(x(s)) + (\Delta s)^{1/2}\,\eta(s) \\
t(s+\Delta s) &= t(s) + (\Delta s)^{1/\alpha}\,\tau_\alpha(s) \quad .
\end{aligned}
$$

Numerical algorithm

Numerical Integration Scheme: (Itô)

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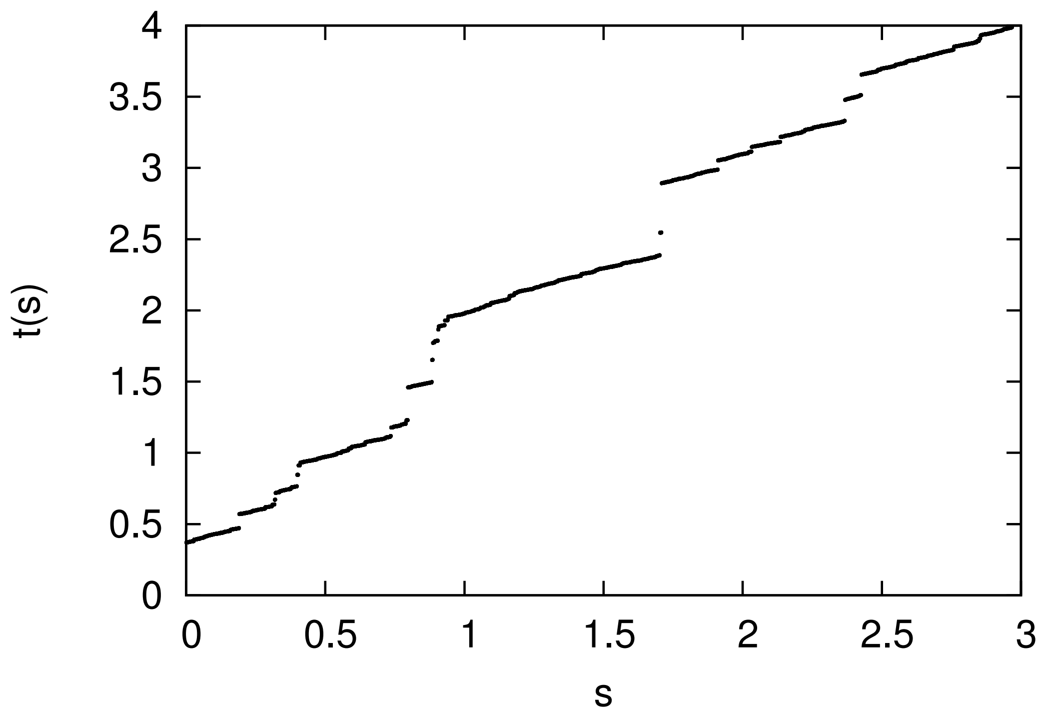
Discontinuous character of t(s):



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\end{aligned}
$$

Algorithm for simulation of $x(t)$:

- Initialisation of $x_s(0)$ and $t_s(0)$, set $s = 0$
- for every $j = 0$ to N:
  1. while $(t_s(s) < t_j)$:
     (a) calculate $x_s(s+\Delta s)$ and $t_s(s+\Delta s)$ from discrete equations (see above)
     (b) increase s by Δs
  2. set $x(t_j) := x_s(s)$

Numerical algorithm

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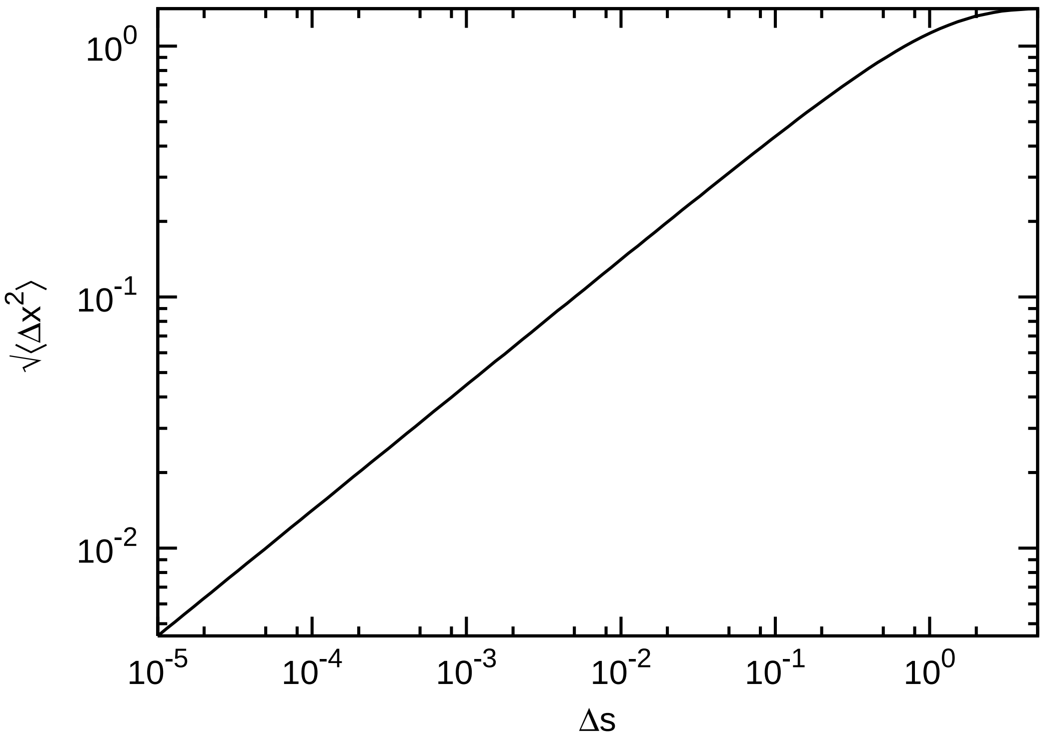
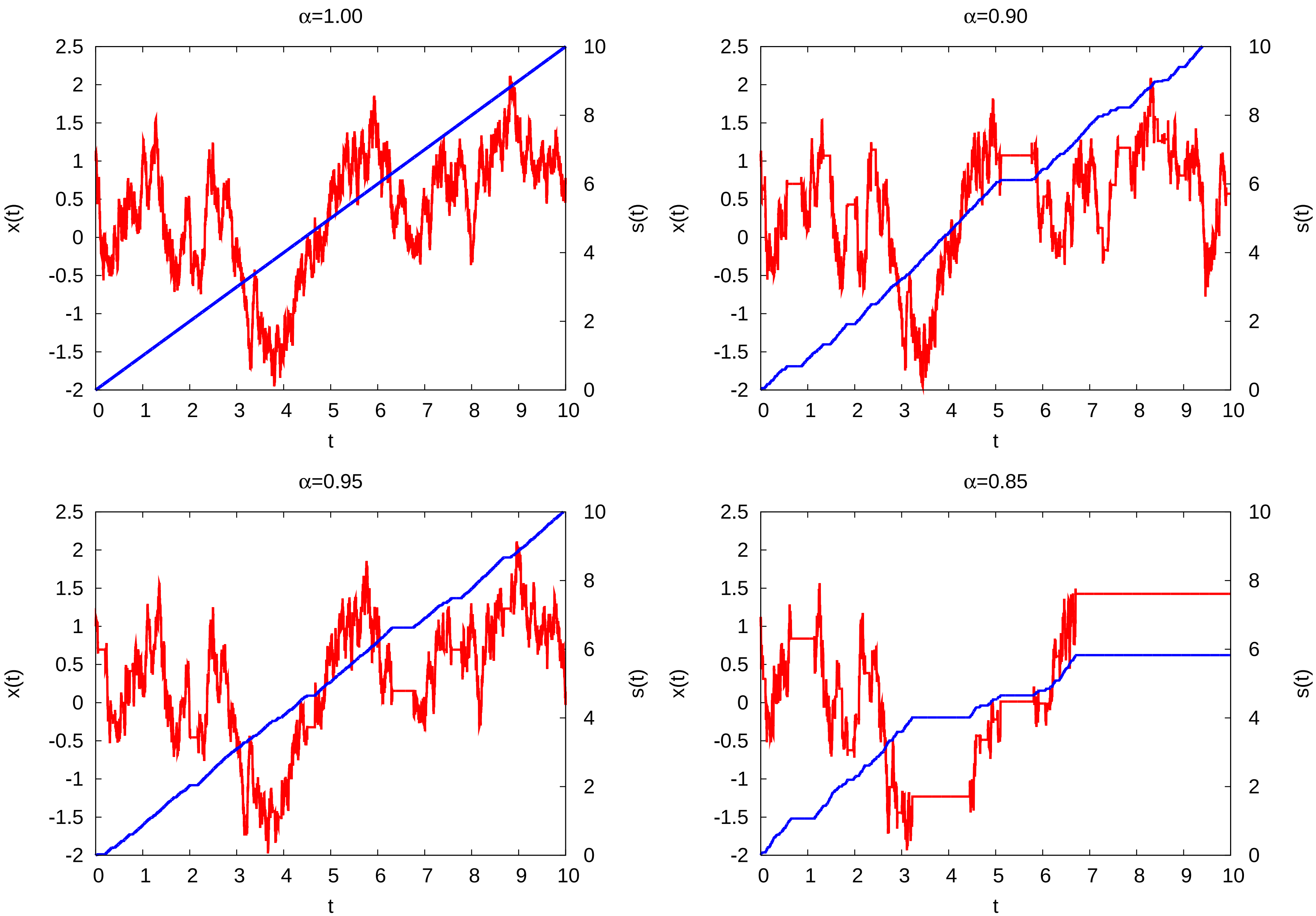
\end{aligned}
$$

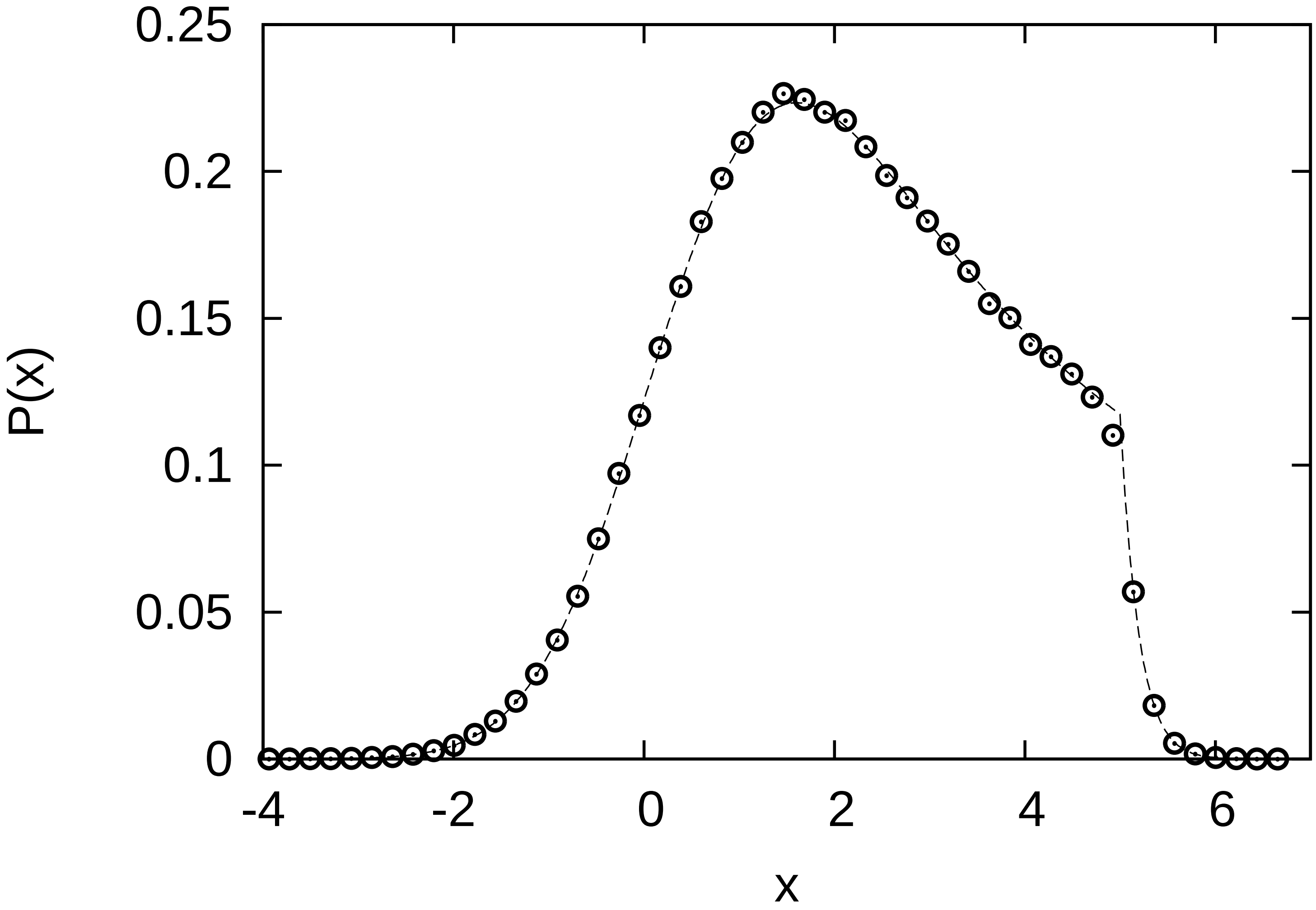
Appropriate Δs can be determined:



Numerical results



Validation: Fractional statistics



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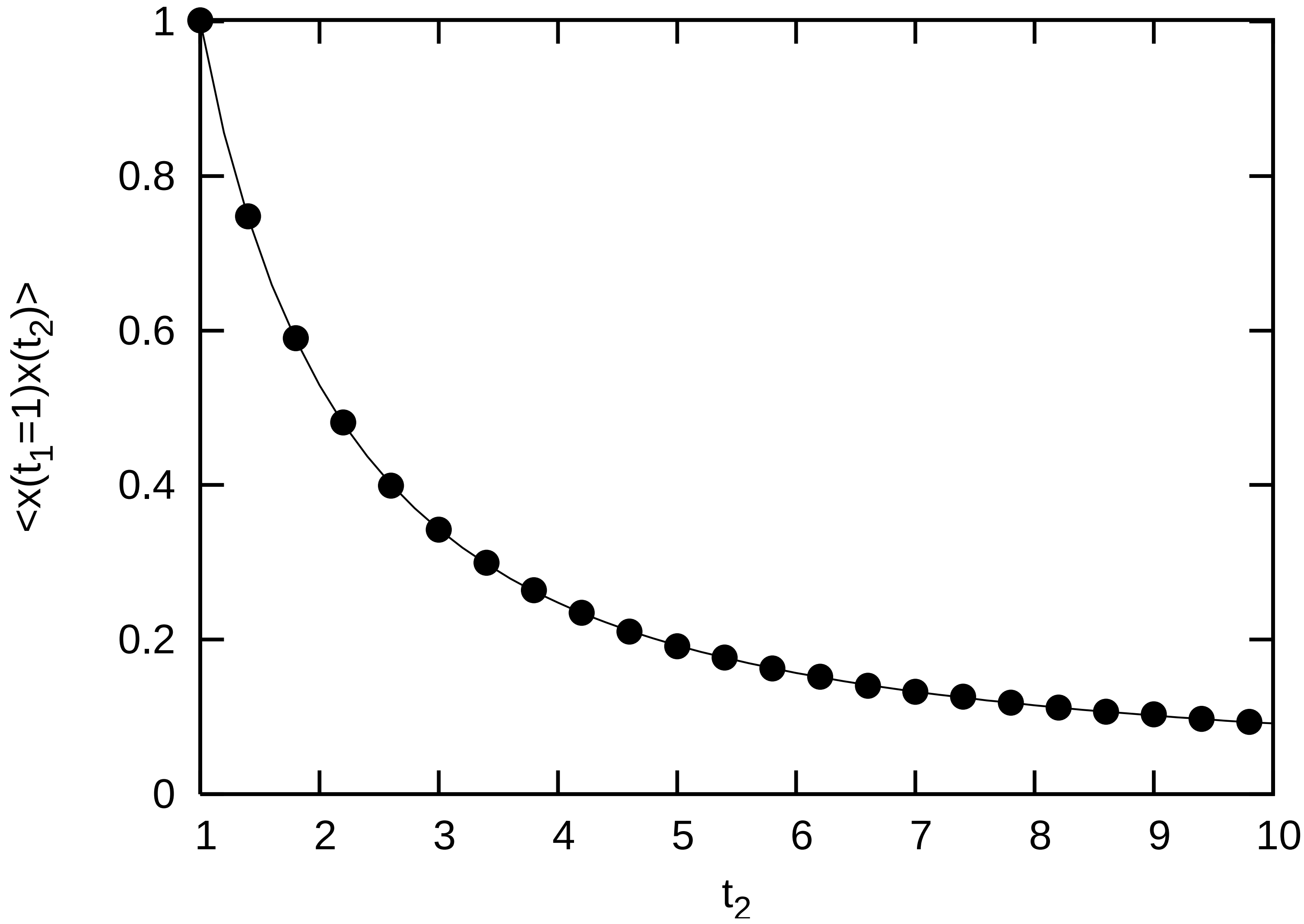
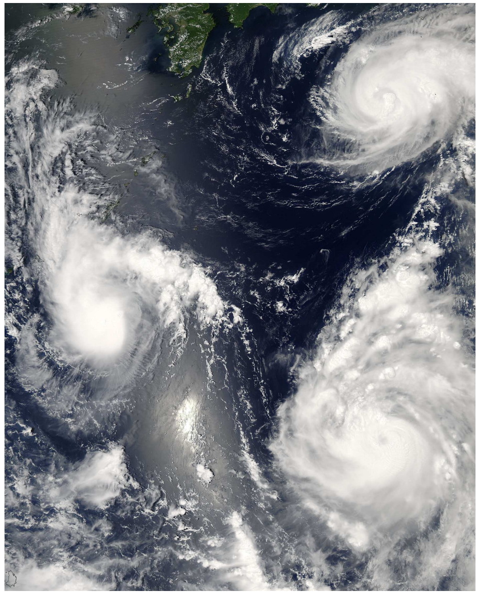


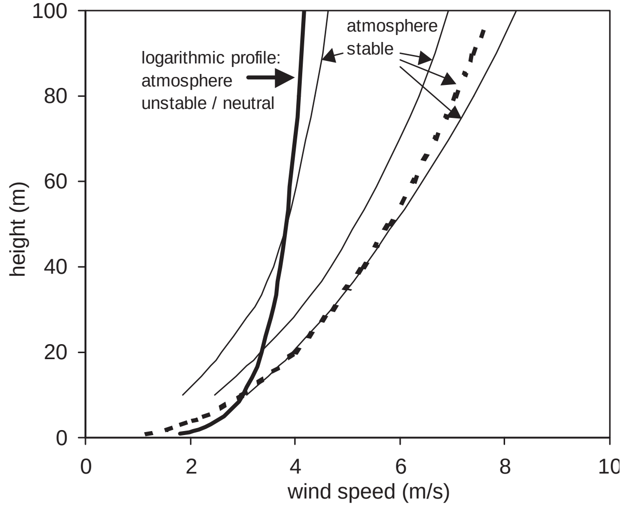
[Kleinhans & Friedrich 2007, Baule & Friedrich 2007]

Simulation of Atmospheric Wind Fields

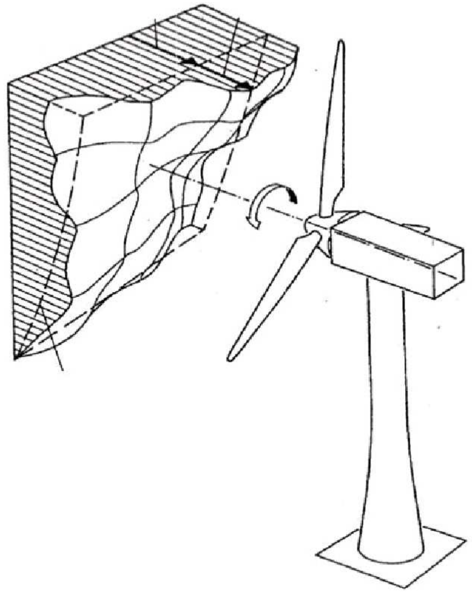
Atmospheric Wind Fields



[NASA visible earth]



[Berg (2004)]



[R. Gasch, Windkraftanlagen, Teubner, 1993.]

Turbulence intensity

Engineer: Character of wind field fully specified by (here: $T = 600s$)

- **Mean** wind speed $\langle u(t) \rangle_T$
- **Turbulence intensity** $TI := \frac{\langle u^2 \rangle_T - \left[\langle u(t) \rangle_T\right]^2}{\langle u(t) \rangle_T}$

Turbulence intensity

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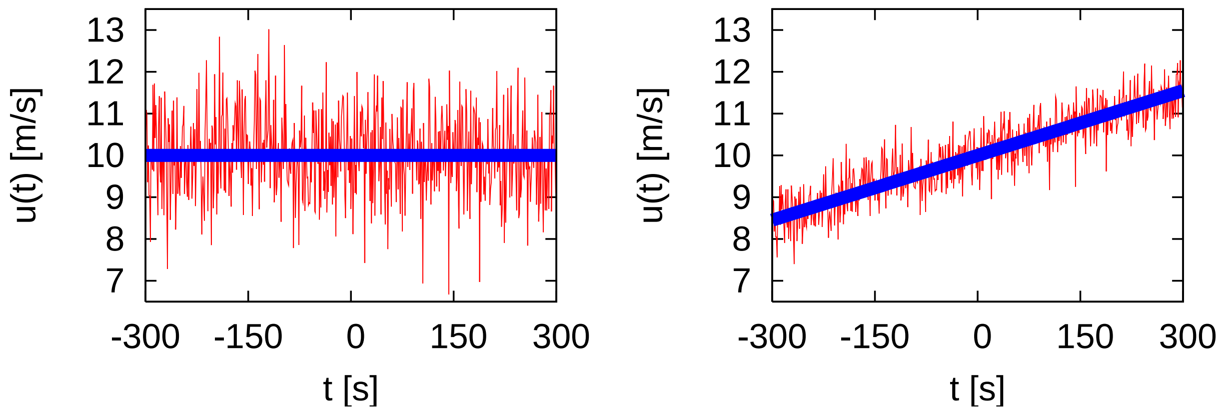
Example:

Temporal drift of wind velocity $u(t) = u_0 + at + u'(t)$ with $\langle u'(t) \rangle = \langle t u'(t) \rangle = 0$, $\left\langle u'^2(t) \right\rangle = \sigma^2$

- $TI = 100 \sqrt{a^2 \frac{T^2}{12} + \sigma^2} / u_0$
- $\Rightarrow$ Turbulence intensity very weak parameter



Two sample time series with turbulence intensity TI$= 10\%$

Windfield models: Overview

- DNS, Large Eddy Simulation (LES), Reynolds Averaged Navier Stokes (RANS)
- **Spectral Models** (IEC compliant)
  - Veers: Three-dimensional wind fiels simulation (1984)
  - Mann: Wind field simulation (1998)
  - some further works e.g. by Bierbooms, Nielsen, Larsen, Hansen . . .
  - (Fung: Kinematic Simulation (KS) (1992))
- **Fractal models**
  - Schertzer, Lovejoy: . . . anisotropic scaling multiplicative processes (1987)
  - Cleve: Fractional Brownian motion (2005)
- **Stochastic / Probabilistic**
  - Nawroth: Reconstruction of processes with Markov properties in scale
  - Schmiegel / Barndorff-Nielsen: Delay kernel
  - Cleve: Cascade model

Spectral surrogate data (1D) [Shinozuka, Deodatis 1991]

Aim: Simulation of $u'(t)$ with $\langle u' \rangle = 0$ and proper **energy spectrum**.

'Fourier-Stieltjes-Integral':

$$u'(t) = \int_0^\infty [\cos(\omega t) du(\omega) + \sin(\omega t) dv(\omega)]$$

Properties of the coefficients (for $\omega, \omega' \geq 0$):

Mean $\langle du(\omega) \rangle = \langle dv(\omega) \rangle = 0$

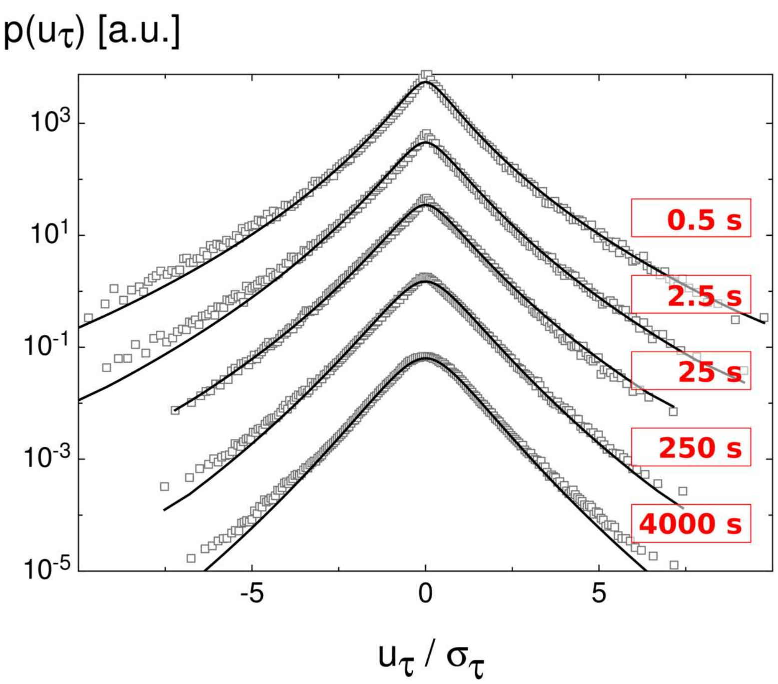
Autocorrelation $\langle du(\omega) du(\omega') \rangle = 2\delta(\omega' - \omega) S(\omega) d\omega$

$\langle dv(\omega) dv(\omega') \rangle = 2\delta(\omega' - \omega) S(\omega) d\omega$

Crosscorrelation $\langle du(\omega) dv(\omega') \rangle = 0$

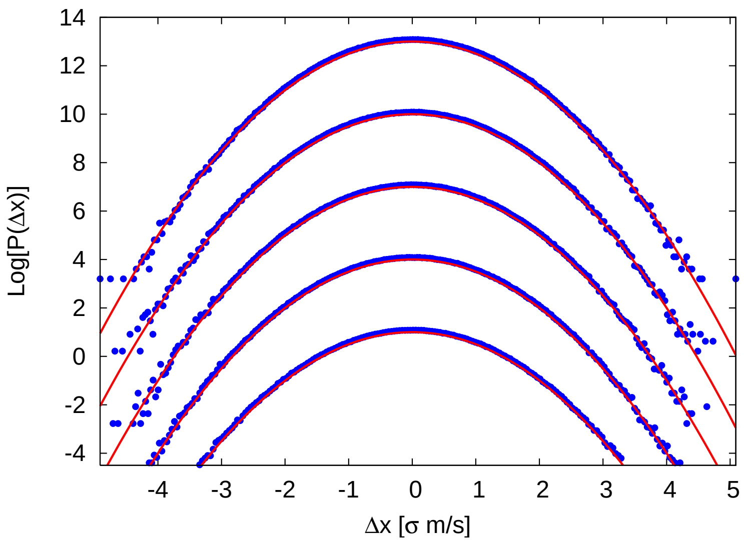
Intermittent velocity increments

Atmospheric measurement:



Spectral surrogates:

[Shinozuka, Deodatis 1991; Veers 1988; Mann 1998]



- Spectral surrogates do not reproduce intermittent statistics

  How could they?

- Advanced methods for fast simulation of atmospheric winds required

Motivation: Starting point

Requirements

- *Intermittent statistics* of atmospheric wind
- Simulation of a 2D (3D?) field in time
- Adaptability to measured wind data

Recent work:

- [Friedrich (2003)]

  Motivation of the applicability of CTRW's for Lagrangian tracers

- [Baule (2005)]

  Several tools for the application of CTRW processes.

- [Castaing (1990), Beck (2003), Böttcher (2005)]

  Analysis of turbulent data sets:

  Generation of intermittency by superposition of gaussian processes.

Model

Coupled Langevin equations

$$\begin{aligned}
\frac{\partial}{\partial t} x_{ref}(s) &= -\gamma_{ref}\left[x_{ref}(s) - x_0\right] + \sqrt{D_{ref}}\Gamma_{ref}(s) \\
\frac{\partial}{\partial s} x_i(s) &= -\gamma\left[x_i(s) - \alpha_i x_{ref}(s)\right] + \sum_j \sqrt{D_{ij}}\Gamma_j(s) \quad \forall\, i \\
\frac{\partial}{\partial s} t(s) &= \tau(s)
\end{aligned}$$

Properties

- x_0: Mean wind speed at reference height
- Reference process x_{ref}: Models wind variations on larger timescales ($\gamma_{ref} \ll \gamma$)

Model

Coupled Langevin equations

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\end{aligned}
$$

Properties

- x_0: Mean wind speed at reference height
- Reference process x_{ref}: Models wind variations on larger timescales ($\gamma_{ref} \ll \gamma$)
- α_i incorporate (logarithmic) wind profile
- Interaction of fluctuations decays exponentially, $D_{ij} \sim \exp(-\left|\boldsymbol{r_i} - \boldsymbol{r_j}\right|)$
- Kramers-like equations (11a,11a) can be treates analytically
- Stochastic process $t(s)$ naturally introduces intermittency

Fokker-Planck Equation

Evolution equation for PDF:

$$\frac{\partial}{\partial s}P(\boldsymbol{x}, x_{ref}, s) = \left[\boldsymbol{\nabla}_{\boldsymbol{x}}\gamma\left[\boldsymbol{x} + \boldsymbol{\alpha} x_{ref}\right] + \boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{x}}^T D + \frac{\partial}{\partial x_{ref}}\gamma\left[x_{ref} + x_0\right] + \frac{\partial^2}{\partial x_{ref}^2} D_{ref}\right] P(\boldsymbol{x}, x_{ref}, s) \quad (13)$$

- Time evolution of expectation values
- Estimation of stationary results

Here: Mean and Variance:

$$\langle x_i \rangle = \alpha_i x_0 \quad (14)$$

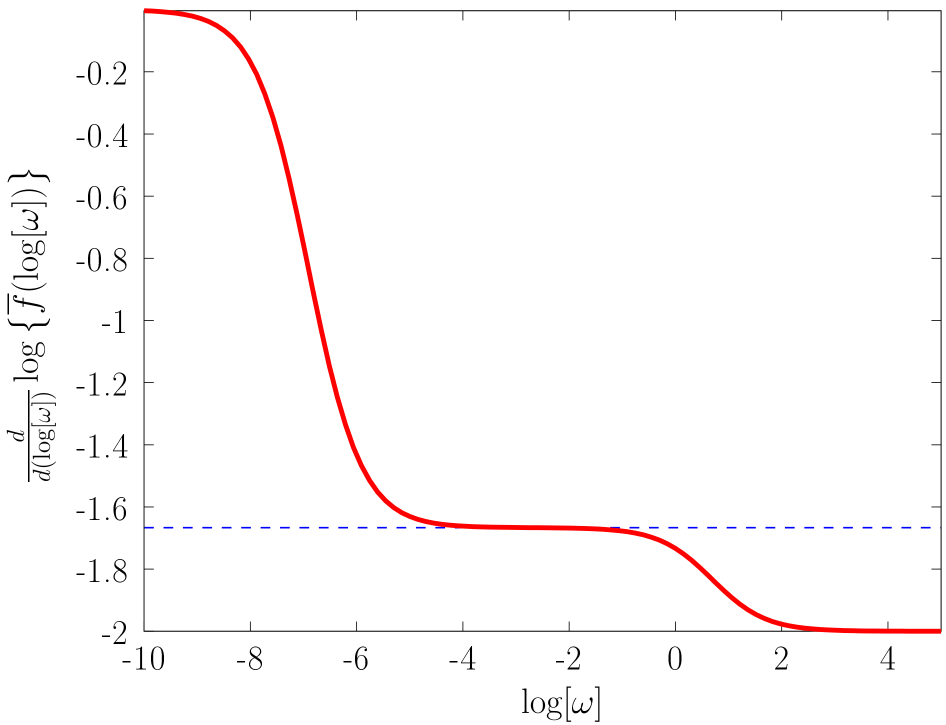
$$\left\langle (x_i - \langle x_i \rangle)^2 \right\rangle = \frac{D}{\gamma} + \alpha_i^2 \frac{\gamma}{\gamma + \gamma_{ref}} \frac{D_{ref}}{\gamma_{ref}} \quad (15)$$

Spectrum

Spectrum of (hidden) $\boldsymbol{x}(s)$:

($s \gg \gamma, \gamma_{ref}$ und $i \equiv k$)

$$\overline{f}(\omega) = \left\{ D + \alpha_i^2 \frac{\gamma^3}{(\gamma + \gamma_{ref})(\gamma - \gamma_{ref})} \frac{D_{ref}}{\gamma_{ref}} \right\} \frac{1}{\gamma^2 + \omega^2}$$
$$+ \alpha_i^2 \frac{\gamma^2}{(\gamma + \gamma_{ref})(\gamma - \gamma_{ref})} D_{ref} \frac{1}{\gamma_{ref}^2 + \omega^2}$$



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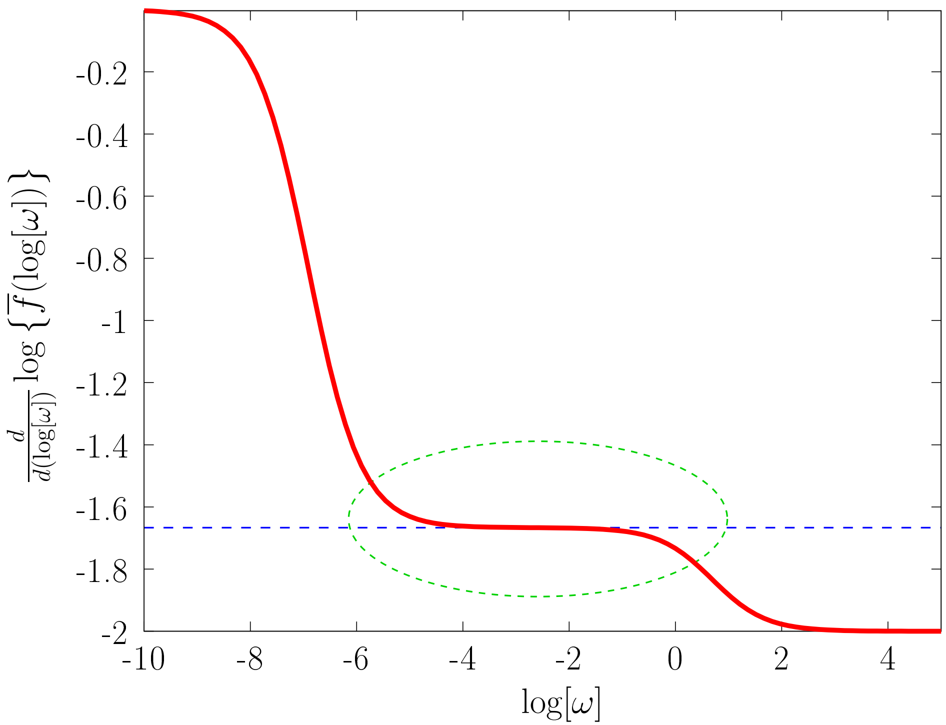
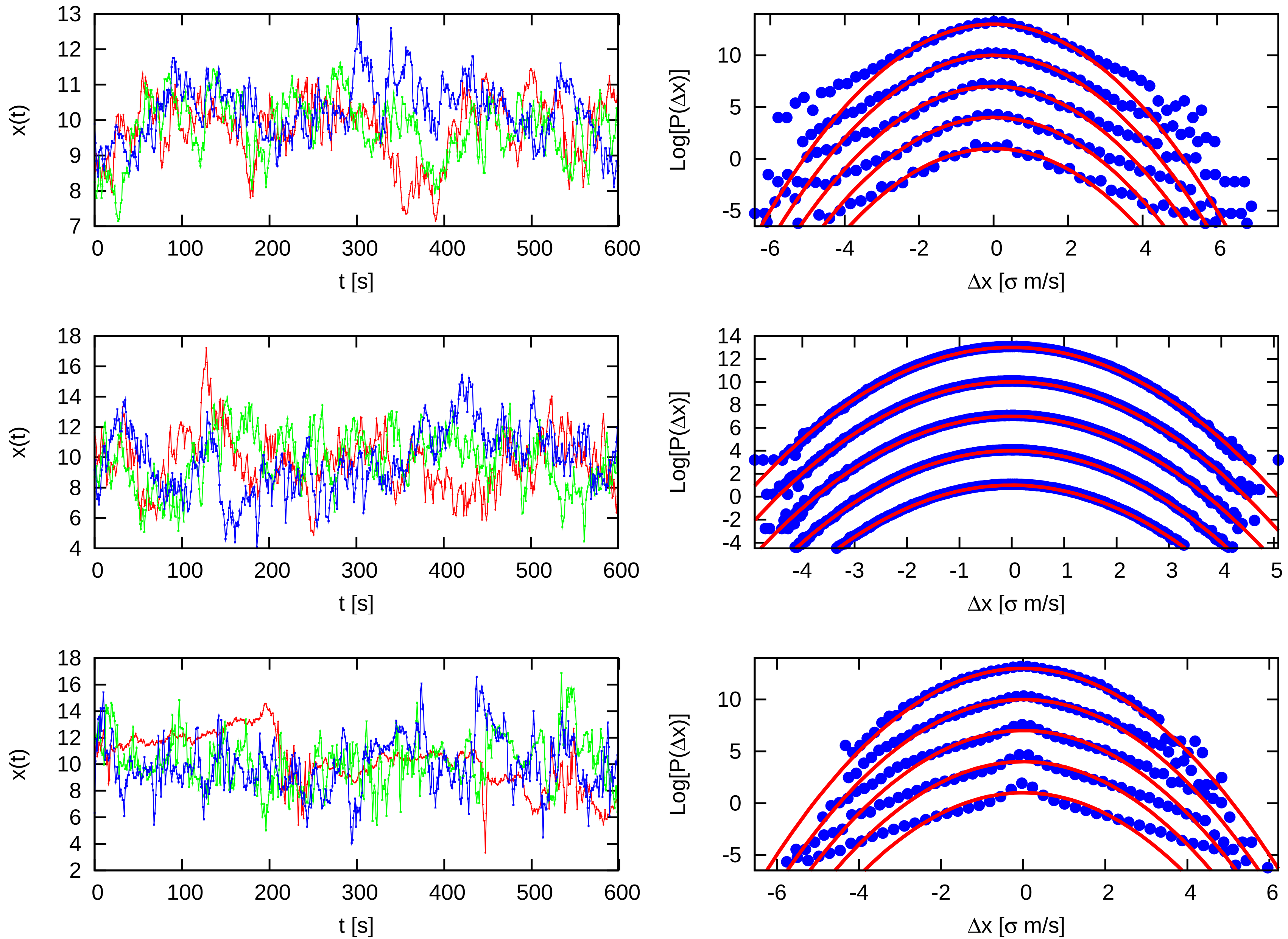
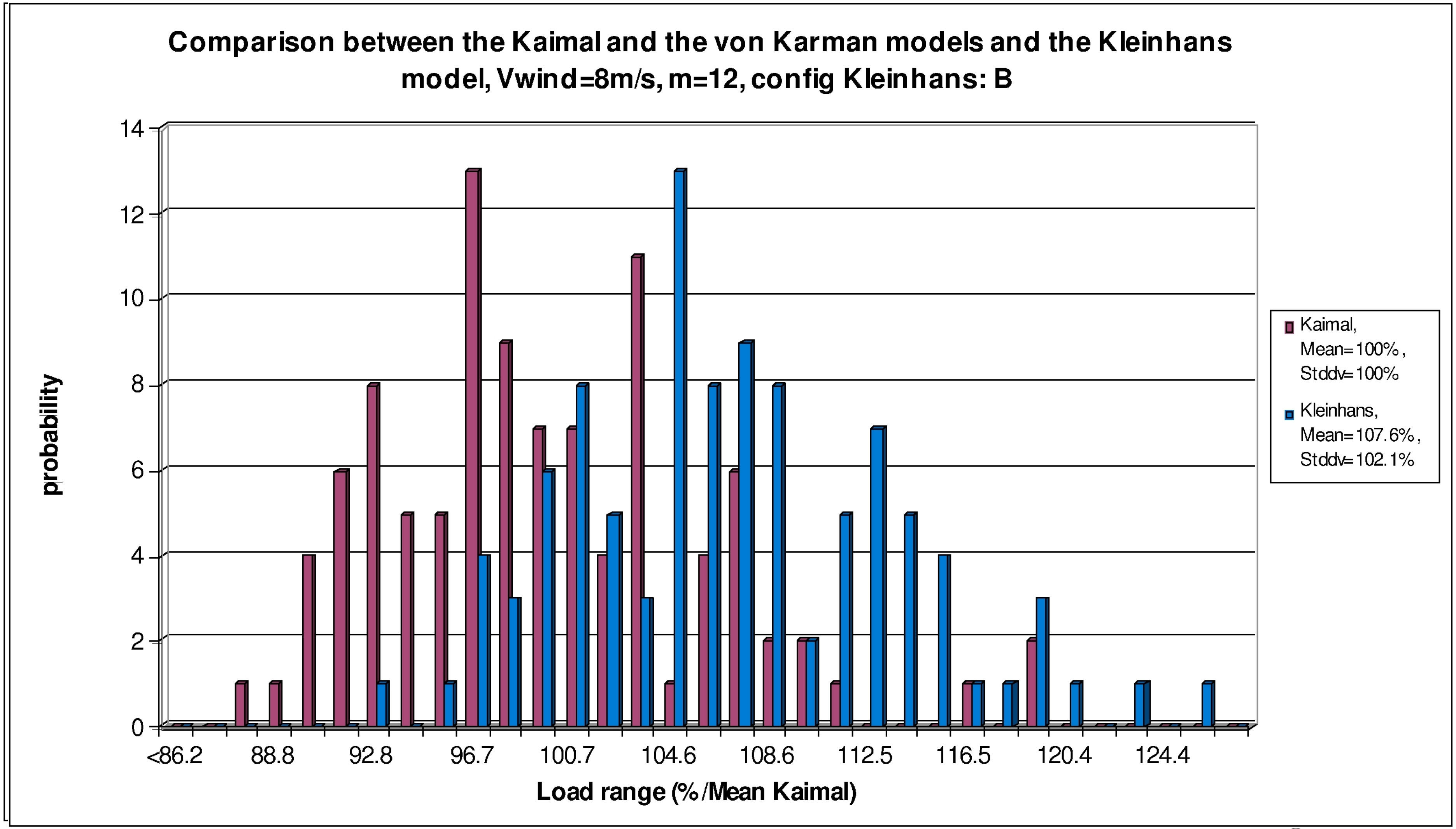
$$\begin{aligned} \overline{f}(\omega) &= \left\{ D + \alpha_i^2 \frac{\gamma^3}{(\gamma + \gamma_{ref})(\gamma - \gamma_{ref})} \frac{D_{ref}}{\gamma_{ref}} \right\} \frac{1}{\gamma^2 + \omega^2} \\ &\quad + \alpha_i^2 \frac{\gamma^2}{(\gamma + \gamma_{ref})(\gamma - \gamma_{ref})} D_{ref} \frac{1}{\gamma_{ref}^2 + \omega^2} \end{aligned}$$



Comparisson: Measurement $\leftrightarrow$ Model

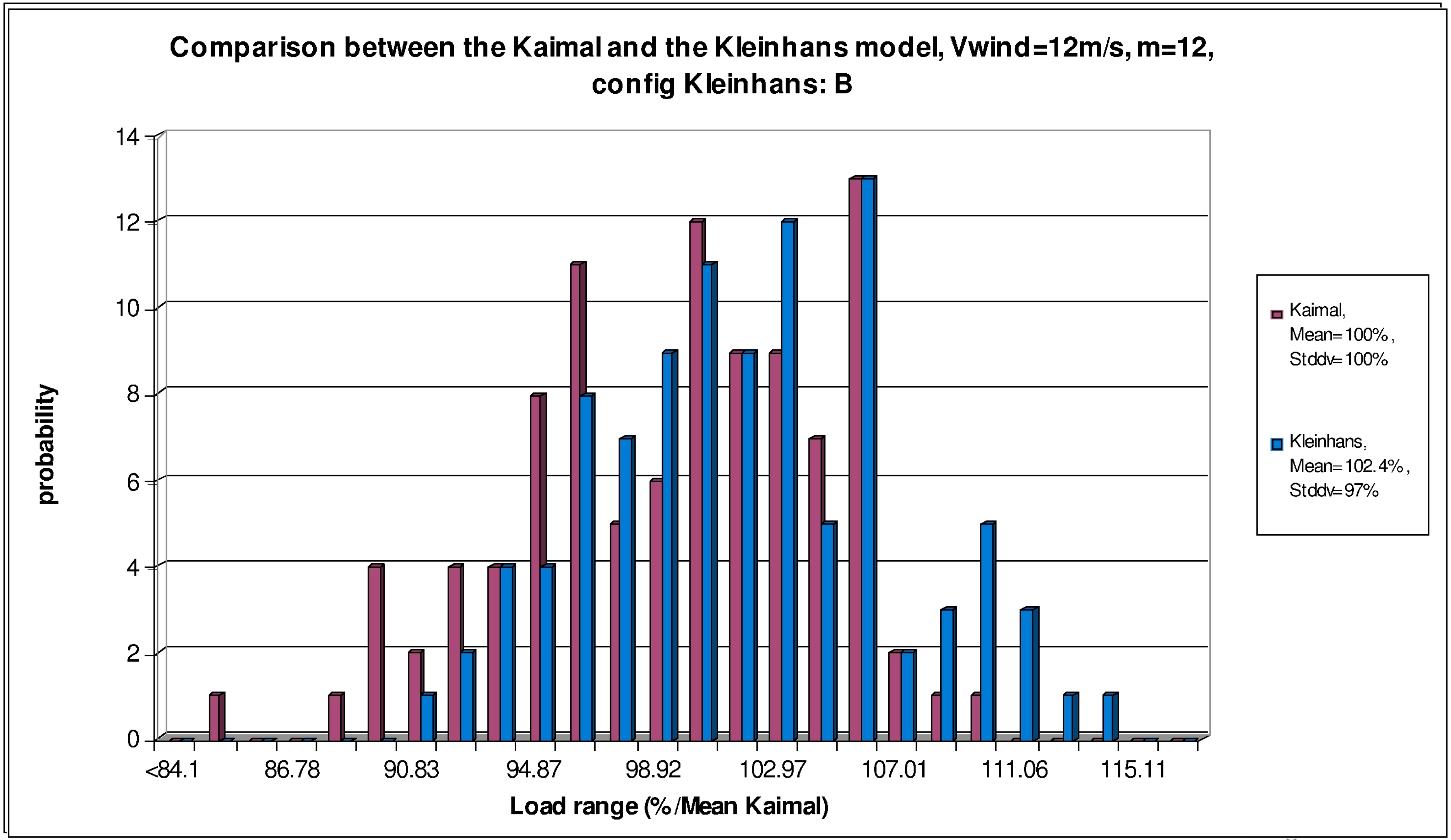


Results: Loads on Wind Turbine Blade



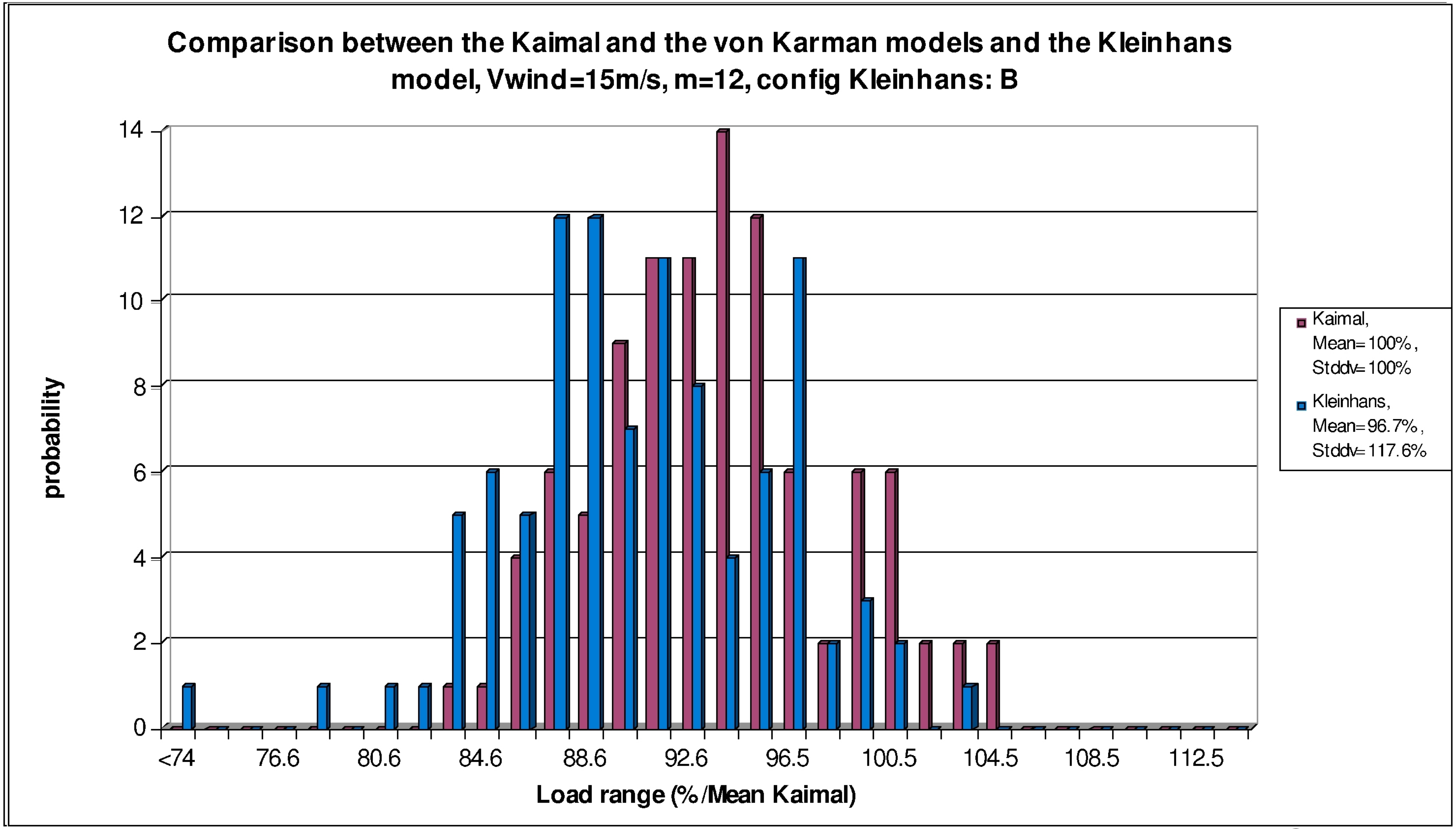
[H. Gontier, A.P. Schaffarczyk, Fachhochschule Kiel, Germany]

Results: Loads on Wind Turbine Blade



[H. Gontier, A.P. Schaffarczyk, Fachhochschule Kiel, Germany]

Results: Loads on Wind Turbine Blade



[H. Gontier, A.P. Schaffarczyk, Fachhochschule Kiel, Germany]

Conclusion

Conclusion

CTRWs

- Powerfull tool for numerical simulation of fractional processes
- Simulation of continuous sample paths

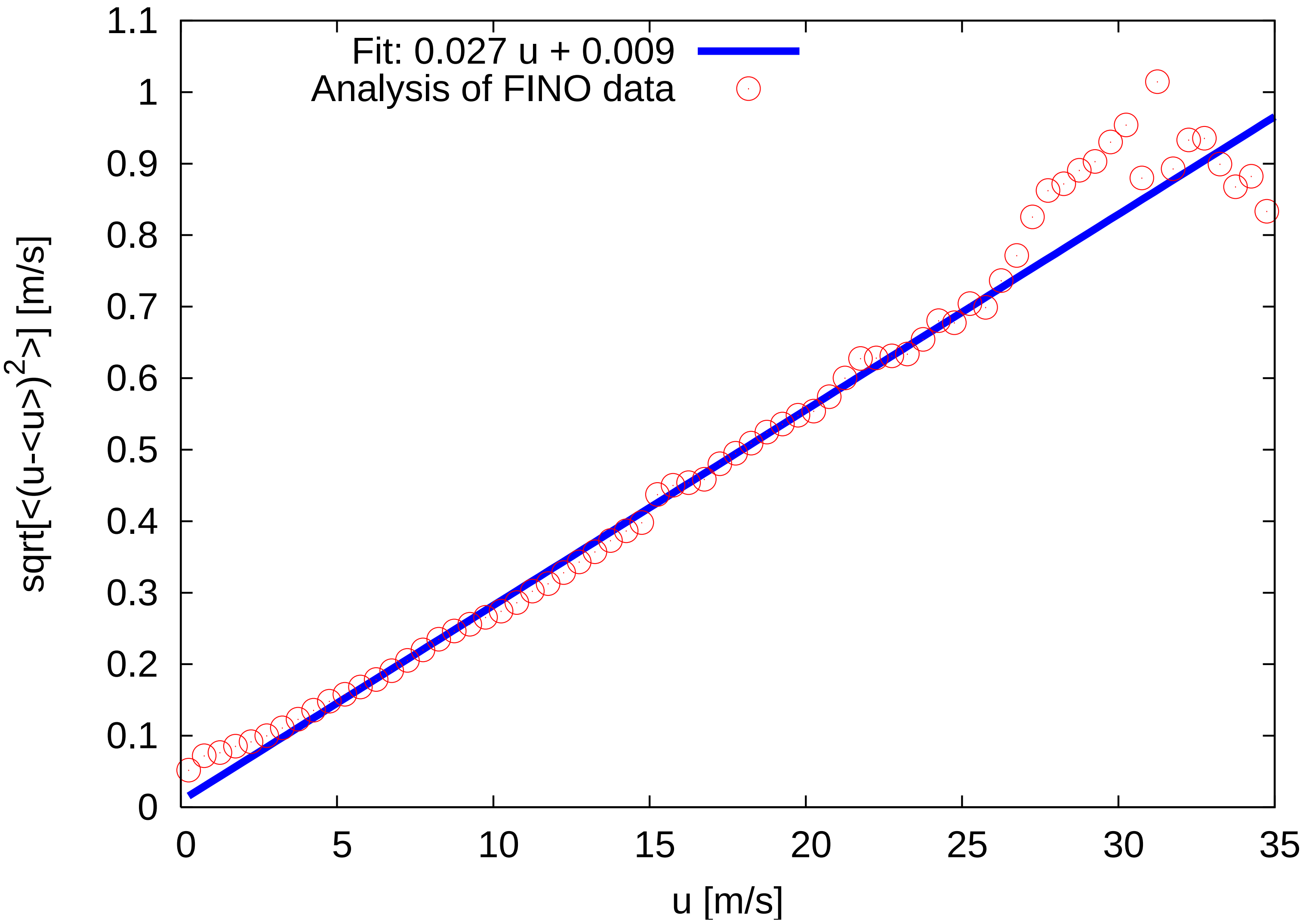
  $\Rightarrow$ Accurate reproduction of fractional dynamics at finite time
- Multivariate joint statistics?

Wind field model

- Modells currently applied: Poor reproduction of intermittent statistics
- Stochastic approach based on CTRW

  $\Rightarrow$ Intermittency can be controlled
- Infinite waiting times unphysical: Truncation of power laws ($\lightning$)
- Multiplicative character of atmospheric turbulence?

Conclusion



Fit: 0.027 u + 0.009
Analysis of FINO data
sqrt[<(u-<u>)^2>] [m/s]
u [m/s]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
0
5
10
15
20
25
30
35

Thank you

for your attention!