DUALITY OF STOCHASTIC FLUID FLOWS

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QBD Analogy & Implications **QBD:** (X_n, J_n)

State Space: $\{(i,j) : i = 0, 1, \ldots, \quad 1 \leq j \leq m\}$

Partitioned Generator

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & \ldots \\ A_2 & A_1 & A_0 & 0 & \ldots \\ 0 & A_2 & A_1 & A_0 & \ldots \\ 0 & 0 & A_2 & A_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- QBD Properties:
  - skip-free in level
  - rates depend only on phase and not on level
- MMFF is similar
  - environmental CTMC J_t; rate of fluid is r_k when $J_t = k$
  - MMFF has continuous paths; skip-free in the continuum
  - fluid rates depend only on phase and not on level

New Algorithms for MMFF

- VR: MMFF can be analyzed by similar arguments as used in the MGM for QBD
- Ahn & Ramaswami: Powerful algorithms for transient & steady state analysis of MMFF
- Much further work along MGM theory by Ramaswami, Ahn, Latouche, Taylor, Bean, & colleagues

The above approach has many parallels to the matrix-geometric approach to QBD.

Dual QBDs

Set $A = A_0 + A_1 + A_2$; $\pi = \pi A$; $\Delta = diag(\pi)$; $\tilde{A}_i = \Delta^{-1} A_i' \Delta$

Consider the QBDs

$$
\begin{bmatrix}
\cdots & & & & & \\
 & A_2 & A_1 & A_0 & & \\
 & & A_2 & A_1 & A_0 & \\
 & & & A_2 & A_1 & \\
 & & & & & \cdots
\end{bmatrix}
\quad
\begin{bmatrix}
\cdots & & & & & \\
 & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & & \\
 & & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & \\
 & & & \tilde{A}_0 & \tilde{A}_1 & \\
 & & & & & \cdots
\end{bmatrix}
$$

Note the reversal of indices on the A-matrices. We have a level & phase reversal.

Asmussen & Ramaswami: Each QBD above is the time reversal of the other.

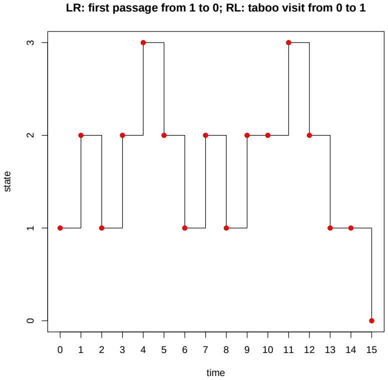
Duality of QBDs

$$\begin{bmatrix} \cdots & & & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & A_2 & A_1 & \\ & & & & & \cdots \end{bmatrix} \quad \begin{bmatrix} \cdots & & & & & \\ & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & & \\ & & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & \\ & & & \tilde{A}_0 & \tilde{A}_1 & \\ & & & & & \cdots \end{bmatrix}$$

Let $G = A_2 + A_1 G + A_0 G^2$ and $\tilde{R} = \tilde{A}_2 + \tilde{R}\tilde{A}_1 + \tilde{R}^2\tilde{A}_0$.
Ramaswami: $G = \Delta^{-1}\tilde{R}'\Delta$
A result connecting downward passage time in a QBD to the upward renewal function in its dual.

Duality Interpretation

Asmussen & Ramaswami: Interpretation in terms of time reversal.



Note visits to 1 occur more than once; hence markov renewals when you consider the reversed path.

Holds more generally for $M/G/1$-type and $GI/M/1$-types; useful in many ways.

IS THERE A DUALITY FOR MMFFs? HOW DOES IT LOOK?

Dual MMFFs

Primal MMFF

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

Rate of fluid change in state k is r_k; $\quad r_k > 0$ in set 1 (Ω_1) and $r_k < 0$ in set 2 (Ω_2).

Let $\pi = (\pi_1, \pi_2)$ be such that $\pi S = 0$, and

$\Delta = diag(\pi) = diag(\Delta_1, \Delta_2)$

Let $\tilde{S}_{ij} = \Delta_i^{-1} S'_{ji} \Delta_j,\ j = 1, 2.$ – time reversal.

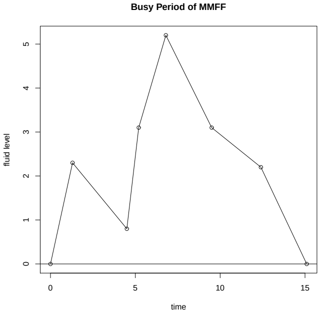
Dual MMFF

$$\tilde{S} = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{bmatrix}$$

with fluid rates $-r_k$ in state k. NOTE: the sets of up and down states are switched in addition to reversal of phase sequence.

Dual MMFF is the time reversal of the Primal.

Busy Period of MMFF



Let $\Psi(s)$ = busy period transform matrix of the Primal MMFF.
$\Psi_{ij}(s)$ is the transform with $J(0) = i$ and $J(\tau) = j$.
$\Psi(s)$ is $m \times n$, where $m = \#$ up-states and $n = \#$ down states.

Duality Conjecture

Let $\tilde{\Psi}(s)$ of order $n \times m$ be the BP-transform of the dual MMFF.

Conjecture based on QBD & the picture:

$$\Psi(s) = \Delta_1^{-1} \tilde{\Psi}(s)' \Delta_2.$$

Is this conjecture true?

Duality Result

Let $\tilde{\Psi}(s)$ of order $n \times m$ be the BP-transform of the dual MMFF. We have:

$$\Psi(s) = C_1^{-1}\Delta_1^{-1}\tilde{\Psi}(s)'\Delta_2 C_2,$$

where C_1 and C_2 are diagonal matrices of $c_k = |r_k|$, the absolute fluid rates in sets Ω_1 and Ω_2.

Explanations:

1. In dynamical systems, one needs to consider "dynamical states" which include the velocities and not just "kinematic" states (position only) !! Seems to be known in quantum physics and time reversal considerations therein.

2. The matrix $\Psi(s)$ is actually the transform of a Markov renewal density and involves a space variable based argument (density at 0 in space), whereas time reversal involves a time based argument. When the phase is k, in a dt interval of time, fluid changes by $\pm c_k\, dt$ and c_k appear as Jacobians in changing between the space and time variables.

Proofs of Duality Results

- Derive a nonlinear equation for $\Psi(s)$ of Algebraic Ricatti type using level crossing arguments.
- Establish minimality of solution for $s \geq 0$.
- Make a transform in the equation
- Identify the resulting equation as characterizing $\tilde{\Psi}(s)$.

These details are routine though a bit arduous. See the paper by Ahn & Ramaswami.

Some Interesting Open Questions

- How do duality results for queues "with speeds" look like ? My guess is the speeds will crop up in the results.
- How do duality results look like for the fluid models with Brownian noise? Here we need to consider duality results for hitting times, local times, and the like. Can we deduce these from the fluid approximation to the Brownian models as considered recently by Ramaswami using the corresponding results for fluids without the Brownian?

Our Best Wishes
to Soren
for many many more decades
of Productive Research
and Good Health!