Characterization of tails through hazard rate and convolution closure properties

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Let denote as f the density function and F the corresponding distribution. In what follows, we will need the hazard rate function,

$$h(x) := \frac{f(x)}{\overline{F}(x)},$$

where $\overline{F}(x) = 1 - F(x)$ denotes the right tail of any distribution F. For every function g we use the following notation for the upper and the lower limit,

$$g^{\star}(u) := \limsup_{x\to\infty} \frac{g(ux)}{g(x)} \ \ and \ \ g_{\star}(u) := \liminf_{x\to\infty} \frac{g(ux)}{g(x)} \, .$$

Let also introduce the upper and the lower limit,

$$M_1 = \liminf_{x\to\infty} x\, h(x) \ \ and \ \ M_2 = \limsup_{x\to\infty} x\, h(x) \, .$$

We write $m(x) \sim g(x)$ as $x \to \infty$ for the limit relation

$$\lim_{x\to\infty} \frac{m(x)}{g(x)} = 1.$$

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Whenever we consider a sequence F_i, $i = 1, 2, \ldots,$ of such distributions, we will use the corresponding symbols h_i, f_i, M_1^i and M_2^i.

Consider also the convolution formula for the distributions

$$\overline{F_1 * F_2}(x) = \overline{F}_2(x) + \int_0^x \overline{F}_1(x-y) dF_2(y)\,.$$

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- F is said to belong to the class $\mathcal{D}$ of distribution function with dominatedly varying tails if:
  1. $\overline{F}_{\star}(u) > 0$ for all (or, equivalently, for some) $u > 1$,
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Note: The class $\mathcal{ER}$ extend out of heavy-tails, on the contrary the rest of the classes are well known heavy tails.

Matuszewska Indices

Recall that for a positve function g on $(0,\infty)$ the:
Upper Matuszewska index γ_g is defined as the infimum of those values α for which there exists a constant C such that for each $U > 1$, as $x \to \infty$,

$$\frac{g(ux)}{g(x)} \leq C(1 + o(1))\, u^{\alpha} \quad \text{uniformly in } u \in [1, U],$$

Lower Matuszewska index δ_g is defined as the supremum of those values β for which, for some $D > 0$ and all $U > 1$, as $x \to \infty$,

$$\frac{g(ux)}{g(x)} \geq D(1 + o(1))\, u^{\beta} \quad \text{uniformly in } u \in [1, U].$$

There exist a connection between Matuszewska indices and classes of distributions $\mathcal{D}$ and $\mathcal{ER}$. More specific,

Proposition (Cline-Samorodnisky, 1994)

For any distribution F on $(0, \infty)$ it holds:

- $F \in \mathcal{D}$ *if and only if* $\gamma_F < \infty$,
- $F \in \mathcal{ER}$ *if and only if* $\delta_F > 0$.

For our work it is important to introduce the Matuszewska indices for a density function. In what follows we always assume that F has a positive Lebesque density f.

Matuszewska Indices for Densities

For a positive Lebesque density f the following relations hold:

$$\gamma_f = \inf\left\{-\frac{\log f_\star(u)}{\log u} : u > 1\right\} = -\lim_{u\to\infty} \frac{\log f_\star(u)}{\log u},$$

where $f_\star(u) = \liminf_{x\to\infty} f(ux)/f(x)$, and

$$\delta_f = \sup\left\{-\frac{\log f^\star(u)}{\log u} : u > 1\right\} = -\lim_{u\to\infty} \frac{\log f^\star(u)}{\log u},$$

where $f^\star(u) = \limsup_{x\to\infty} f(ux)/f(x)$.

Potter Type Inequalities

Using Matuszewska Indices we can establish inequalities for f. For example

- If $\gamma_f < \infty$, then for every $\gamma > \gamma_f$ there exist constants $C'(\gamma)\,,\ x_0' = x_0'(\gamma)$ such that

$$\frac{f(y)}{f(x)} \geq C'(\gamma)\left(\frac{y}{x}\right)^{-\gamma}\,, \quad y \geq x \geq x_0'. \tag{2.1}$$

- If $\delta_f > -\infty$ then for every $\delta < \delta_f$ there exist constants $C(\delta)\,,\ x_0 = x_0(\delta)$ such that

$$\frac{f(y)}{f(x)} \leq C(\delta)\left(\frac{y}{x}\right)^{-\delta}\,, \quad y \geq x \geq x_0. \tag{2.2}$$

We will say that a density has bounded increase if $\delta_f > -\infty$

The assumption of a density that has bounded increase is commonly fulfilled. Most of the densities of interest in statistics and probability theory satisfy this condition, e.g. the Gamma and the Weibull. The first part of the presentation will be about the classes $\mathcal{ER}$ and $\mathcal{D}$ and the connection with the limits M_1 and M_2. Similar results where presented by Konstantinides-Tang-Tsitsiashvili (2002) and Klüppelberg (1988) under the assumptions of a monotone density (i.e. eventually non-increasing). More specific,

1. $F \in \mathcal{ER}$ if and only if $M_1 > 0$,
2. $F \in \mathcal{D}$ if and only if $F \in \mathcal{D} \cap \mathcal{L}$ if and only if $M_2 < \infty$.

As a result by the assumption of a bounded increase density we avoid to verify the monotonicity property and restricts the calculation to that of δ_f through $f^\star(u)$.

Theorem

Assume F is a distribution supported on $(0, \infty)$ with positive Lebesgue density f such that f has bounded increase. Then $F \in \mathcal{ER}$ if and only if $M_1 > 0$.

Proposition

If f has bounded increase with $\delta_f > 1$ then $F \in \mathcal{ER}$ and for any $\delta \in (1, \delta_f)$ there are positive constants x_0 , $C(\delta)$, defined in (2.2), such that for all $x \geq x_0$:

$$xh(x) \geq \frac{(\delta - 1)}{C(\delta)} .$$

Theorem

Assume that $F_1, F_2 \in \mathcal{ER}$ with positive Lebesgue densities on $(0,\infty)$ and that the following conditions hold:

1. *The density f_1 has bounded increase with $\delta_{f_1} > 0$,*
2. $\delta_{\overline{F}_1} < \delta_{\overline{F}_2}$ *and* $\liminf_{x\to\infty} x^{\delta}\, \overline{F}_1(x) > 0$ *for some* $\delta \in \left[\delta_{\overline{F}_1}, \delta_{\overline{F}_2}\right)$.

Then $F_1 * F_2 \in \mathcal{ER}$.

We verify that the two conditions of the previous Theorem do not contradict considering two Pareto distribution,
$\overline{F}_i(x) = x^{-\alpha_i}$, $i = 1, 2$.

Theorem

Assume that F is supported on $(0, \infty)$ with a positive Lebesgue density f which has bounded increase. Then the following statements are equivalent:

$$1.\ F \in \mathcal{D} \qquad 2.\ F \in \mathcal{D} \cap \mathcal{L} \quad \text{and} \quad 3.\ M_2 < \infty\,.$$

So for the class of distributions with bounded increase Lebesque density it holds $\mathcal{D} = \mathcal{D} \cap \mathcal{L}$. In general this is not true. This made us wonder if this assumption can lead to similar results?

Proposition

Assume that F has a positive Lebesgue density on $(0,\infty)$ and $\gamma_f<\infty$. Then $F\in\mathcal{D}\cap\mathcal{L}$, and for any $\gamma>\gamma_f$ there are positive constants x_0', $C'(\gamma)$, defined in (2.1), such that for all $x\geq x_0'$ and $\lambda>1$:

$$xh(x)\leq C'(\gamma)\,V(\lambda,\gamma)\,,$$

where

$$V(\lambda,\gamma)=\begin{cases}\dfrac{(\lambda^{-\gamma+1}-1)}{(-\gamma+1)}, & \text{if } \gamma\neq 1,\\ \log\lambda, & \text{if } \gamma=1.\end{cases}$$

Theorem (Pitman, 1980)

Suppose F is absolutely continuous with density f and hazard rate $h(x)$ eventually decreasing to 0. Then $F \in \mathcal{S}$ if and only if

$$\lim_{x\to\infty} \int_0^x \exp\{yh(x)\} f(y) dy = 1\,.$$

This theorem is a complete answer for $\mathcal{S}-$membership for absolutely continuous distributions. However is not straightforward the verification of the eventually monotone hazard rate (such that $h(y) \leq h(x)$ for all $y \geq x \geq x_0$). In the next result we prove the previous statement assuming that the hazard rate h has positive decrease, which might be checked easier.

Definition

We say that h has positive decrease if $\delta_h > 0$ or equivalently $h^{\star}(u) < 1$ for some $u > 1$.

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Theorem

Let F be a distribution on $(0, \infty)$ with positive Lebesgue density f. Assume that the hazard rate h has positive decrease. Then $F \in \mathcal{S}$ if and only if

$$\lim_{x\to\infty} \int_0^x \exp\{\kappa y h(x)\} f(y) dy = 1 \quad \textit{for every } \kappa > 0.$$

Definition (Konstantinides-Tang-Tsitsiashvili (2002))

F is said to belong to the class $\mathcal{A}$ if $F \in \mathcal{S} \cap \mathcal{ER}$.

We can say that $\mathcal{A}$ is the 'heavy tailed' part of $\mathcal{ER}$. Interesting was the study of a certain subclass, which will denote as $\mathcal{D} \cap \mathcal{A} = \mathcal{D} \cap \mathcal{L} \cap \mathcal{ER}$. This class contains well known classes as regular varying and extended regular varying. The first result concerns closure under convolution and max-sum equivalence.

Proposition

Assume $F_i \in \mathcal{D} \cap \mathcal{A}$, $i = 1, 2$. *Then* $F_1 * F_2 \in \mathcal{D} \cap \mathcal{A}$ *and*

$$\overline{F_1 * F_2}(x) \sim \overline{F}_1(x) + \overline{F}_2(x)\,, \quad x \to \infty\,. \tag{6.1}$$

Corollary

Assume F is a distribution supported on $(0, \infty)$ with positive Lebesgue density f such that f has bounded increase. Then $F \in \mathcal{D} \cap \mathcal{A}$ if and only if one of the following statements holds

1. $0 < M_1 \leq M_2 < \infty$,
2. $0 < \overline{F}_{\star}(u) \leq \overline{F}^{\star}(u) < 1$.

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Assume F is a distribution supported on $(0, \infty)$ with positive Lebesgue density f such that f has bounded increase. Then $F \in \mathcal{D} \cap \mathcal{A}$ if and only if one of the following statements holds

1. $0 < M_1 \leq M_2 < \infty$,
2. $0 < \overline{F}_{\star}(u) \leq \overline{F}^{\star}(u) < 1$.

Corollary

Let F be a distribution on $(0, \infty)$ with positive Lebesgue density f. Assume that the hazard rate h has positive decrease. Then $F \in \mathcal{A}$ if and only if $M_1 > 0$ and

$$\lim_{x \to \infty} \int_0^x \exp\{\kappa y h(x)\} f(y) dy = 1 \quad \text{for every } \kappa > 0 ,$$

For the following results will need to introduce the following to classes of distributions.

- A distribution function F belongs to the class $\mathcal{C}$, if

$$\lim_{u\uparrow 1} \overline{F}^{\star}(u) = 1, \quad or \quad \lim_{u\downarrow 1} \overline{F}_{\star}(u) = 1\,.$$

  Such a distribution function F is said to have a consistently varying tail.

- A distribution function F belongs to the class $\mathcal{R}_{-\infty}$, if

$$\lim_{x\to\infty} \frac{\overline{F}\,(ux)}{\overline{F}\,(x)} = \begin{cases} 0\,, & \text{for some } u > 1\,, \\ \infty\,, & \text{for some } 0 < u < 1\,, \end{cases}$$

  Such a distribution function F is said to have a rapidly varying tail.

Some know results are $\mathcal{R}_{-\infty} \subseteq \mathcal{ER}$ and $\mathcal{C} \subsetneq \mathcal{D} \cap \mathcal{L}$.

Theorem

For the class of distributions with bounded increase Lebesque density it holds:

$$\mathcal{S} = (\mathcal{S} \cap \mathcal{R}_{-\infty}) \cup (\mathcal{D} \cap \mathcal{L}) \tag{7.1}$$

and

$$\mathcal{D} \cap \mathcal{S} = \mathcal{C} = (\mathcal{D} \cap \mathcal{A}) \cup \mathcal{R}_0 \tag{7.2}$$

Equation (7.1) was an expected result. For example Tang and Tsitsiashvili (2003) mentioned this relation using the notation $\approx$.

Thank you for your attention!

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