Moment distributions of phase–type

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Outline

- In this talk we consider moment distributions which underlying distribution is of either phase–type or matrix–exponential.
- We show that moment distributions of any order are again phase–type or matrix–exponential.
- Moment distributions have applications in various fields like demography and engineering.
- We especially focus on demographic applications relating to the Lorenz curve and Gini index in which case explicit formulas may be obtained.

Moment distributions

- Let f be the density a distribution on $[0, \infty)$. Let $\mu_n = \int_0^\infty x^n f(x)dx$ be its n'th moment.
- Then
$$g_n(x) = \frac{x^n f(x)}{\mu_n}$$
is a density and is called the n'th moment distribution of f.
- We shall now assume that f is either of phase–type or matrix–exponential.
- We shall start with the first order moment distribution.

First order moment distribution

- Let f be a density on $[0, \infty)$ and let F denotes its corresponding distribution function.
- Consider a stationary renewal process with inter–arrival distribution F.
- Hence the renewal process is delayed with initial arrival distribution given by the density

$$f_e(x) = \frac{1 - F(x)}{\mu_1} = \frac{\bar{F}(x)}{\mu_1}.$$

- Let F_e denote the corresponding distribution function.
- Let A_t be the age of the process at time t (time from previous arrival) and R_t the residual life–time (time until next arrival).

First order moment distributions

- Then

$$\mathbb{P}(A_t > x, R_t > y) = \bar{F}_e(x+y).$$

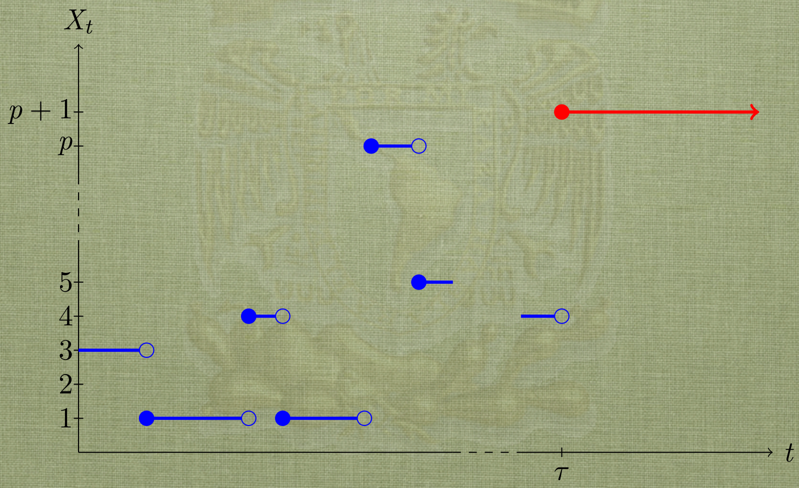
- Differentiating twice w.r.t. x and y,

$$f_{(A_t,R_t)}(x,y) = \frac{f(x+y)}{\mu_1}.$$

- From this formula we read that A_t and R_t have the same marginal distribution.
- The spread $S_t = A_t + R_t$ has density

$$f_{S_t}(x) = \int_0^x f_{(A_t,R_t)}(x-t,t)\,dt = \frac{xf(x)}{\mu_1}.$$

Phase–type distributions



Phase–type distributions

- A Phase–type distribution is the time until absorption in a Markov jump process with finitely many states, one of which is absorbing and the rest being transient.
- We write $\tau \sim \mathrm{PH}(\pi, T)$ (note that $t = -Te$, where e is the column vector of ones.
- Phase-type distributions is a flexible tool in applied probability. Allows for many closed form solutions to complex problems.
- They are dense in the class of distributions on the positive reals.
- They are, however, light tailed.

An example of use

A general and easy to prove result states that

$$P^s = e^{\Lambda s} = \begin{pmatrix} e^{Ts} & e - e^{Ts}e \\ 0 & 0 \end{pmatrix}.$$

Let $\tau \sim \mathrm{PH}(\pi, T)$. Let f denote density of τ. Then

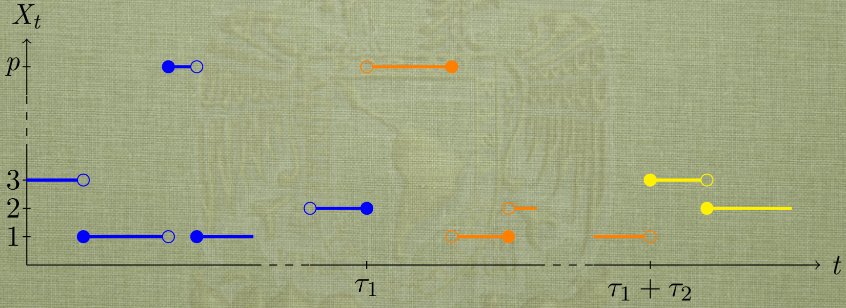
$$\begin{aligned} f(x)dx &= \mathbb{P}(\tau \in (x, x+dx]) \\ &= \sum_{i,j=1}^{p} \pi_i p_{ij}^x t_j dx \\ &= \sum_{i,j=1} \pi_i \left(e^{Tx}\right)_{ij} t_j dx \\ &= \pi e^{Tx} t dx. \end{aligned}$$

Hence

$$f(x) = \pi e^{Tx} t$$

Renewal theory

Consider a renewal process with inter–arrival times $T_1, T_2, \ldots$ being i.i.d. $\sim \mathrm{PH}(\pi, T)$.



Renewal density: $u(x)$ =probability of an arrival in $[x, x + dx)$.

The concatated process is a Markov process $\{J_t\}_{t \geq 0}$ with intensity matrix $R = T + t\pi$:

$$r_{ij}dx = t_{ij}dx + t_i dx \pi_j.$$

Renewal theory

Transition probabilities of $\{J_t\}_{t\geq 0}$:

$$P^s = \exp\left((T + t\pi)s\right).$$

Hence

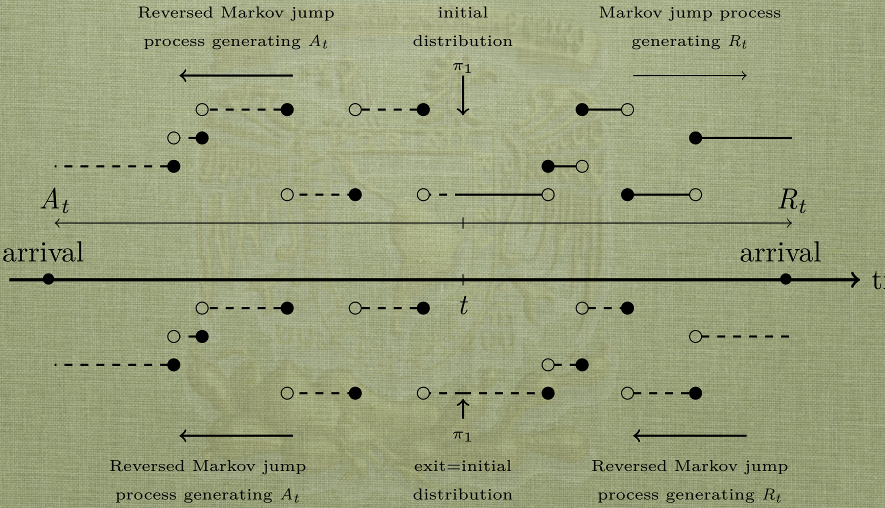
$$\begin{aligned} u(x)dx &= \sum_{i,j=1}^{p} \pi_i p_{ij}^x t_j dx \\ &= \sum_{i,j=1}^{p} \pi_i \left(e^{(T+t\pi)x}\right)_{ij} t_j dx \\ &= \pi e^{(T+t\pi)x} t dx. \end{aligned}$$

Hence

$$u(x) = \pi e^{(T+t\pi)x} t.$$

Stationary renewal process

- A stationary renewal process with phase–type inter–arrival times $T_2, T_3, ...$ i.i.d. $\sim \mathrm{PH}(\alpha, T)$ is a delayed renewal process with $T_1 \sim \mathrm{PH}(\pi_1, T)$, where $\pi_1 = \frac{\alpha T^{-1}}{\alpha T^{-1} e}$.
- π_1 is the stationary distribution of the Markov jump process with intensity matrix $T + t\alpha$.
- π_1 is also the stationary distribution for the time reversed process.
- Time reversing a PH distribution essentially works as for Markov jump processes.


Reversed Markov jump process generating A_t
initial distribution
π_1
Markov jump process generating R_t
A_t
R_t
arrival
arrival
t
π_1
Reversed Markov jump process generating A_t
exit=initial distribution
Reversed Markov jump process generating R_t

First moment distribution of phase–type

- Let f be the density of a $\mathrm{PH}(\pi, S)$.
- Let $\pi_1 = \pi S^{-1}/\pi S^{-1} e$. Then

$$A_t, R_t \sim \mathrm{PH}(\pi_1, S) \text{ or } \; A_t, R_t \sim \mathrm{PH}(\pi_1, \hat{S})$$

  where $\hat{S} = \Delta(m_1)^{-1} S' \Delta(m_1)$ and $m_1 = -\alpha S^{-1}$.
- We time reverse A_t or R_t. If $R_t \sim \mathrm{PH}(\pi_1, S)$, then we time reverse A_t with the choice of representation $\mathrm{PH}(\pi_1, \hat{S})$. If we time reverse $R_t \sim \mathrm{PH}(\pi_1, S)$, then we use $A_t \sim \mathrm{PH}(\pi_1, \hat{S})$.
- The exit distribution the A_t–process is π_1, the same as the initial distribution of R_t. Hence we may generate the initial distribution of the R_t process by realizing the time–reversed of A_t and then realize the process of R_t. The total time it takes for both processes to exit is just the spread S_t.

First moment distribution of phase–type

- If we reverse R_t we get a representation is for the first moment distribution $(\hat{\alpha}_1, \hat{S}_1)$, where

$$\hat{\alpha}_1 = (s'\Delta(m_2), 0)$$

$$\hat{S}_1 = \begin{pmatrix} \Delta^{-1}(m_2)S'\Delta(m_2) & \rho_1^{-1}\Delta^{-1}(m_2)\Delta(m_1) \\ 0 & \Delta^{-1}(m_1)S'\Delta(m_1) \end{pmatrix},$$

with $\rho_i = \alpha(-S^{-i})e$ and $m_i = \rho_{i-1}^{-1}\alpha(-S)^{-i}$.

- If we reverse A_t we get a representation (α_1, S_1) with

$$\alpha_1 = \left(\rho_1^{-1}\alpha\Delta(r), 0\right)$$

$$S_1 = \begin{pmatrix} \Delta^{-1}(r)S\Delta(r) & \Delta^{-1}(r) \\ 0 & S \end{pmatrix},$$

where $r = (-S)^{-1}e$.

Moment distributions based on matrix–exponentials

- Let f be the density of a matrix–exponential distribution with representation (α, S, s) with $s = -Se$ (the latter only being notationally convenient).
- Then its n'th moment distribution is again matrix–exponential with representation (α_n, S_n, s_n), where

$$\alpha_n = \left(\frac{\alpha S^{-n}}{\alpha S^{-n} e}, 0, ..., 0 \right) \quad S_n = \begin{pmatrix} S & -S & 0 & ... & 0 \\ 0 & S & -S & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & S \end{pmatrix}, \quad s_n = \begin{pmatrix} 0 \\ 0 \\ .. \\ s \end{pmatrix}$$

Moment distribution based on matrix–exponentials

- This easily follows from

$$\alpha_n e^{S_n x} s_n = \frac{\alpha S^{-n}}{\alpha S^{-n} e} \frac{(-1)^n}{n!} x^n S^n e^{Sx} s = \frac{x^n \alpha e^{Sx} s}{(-1)^n n! \alpha S^{-n} e} = \frac{x^n \alpha e^{Sx} s}{\mu_n}.$$

- To obtain the corresponding distribution function F_n we integrate partially and obtain

$$F_n(x) = 1 - \frac{\alpha S^{-n}}{\alpha S^{-n} e} \sum_{i=0}^{n} \frac{(-xS)^i}{i!} e^{Sx} e$$

- In particular for $n = 1$ we get

$$F_1(x) = 1 - \frac{\alpha S^{-n}}{\alpha S^{-n} e} \left(e^{Sx} e + x e^{Sx} s \right)$$

Higher order moment distributions of phase–type

- In principle we now conclude that moment distributions of any order are again phase–type if the original distribution is.
- This follows trivially from the n'th order moment distributions is the first order moment distribution of the $n-1$'th order moment distribution!
- Hence, in principle, there is an algorithm for generating a PH representation.
- The order, however, will blow up unnecessarily.
- The following result provides a lower order representation of the n'th moment distribution, but we lack a probabilistic proof :-(

Higher order moment distributions of phase–type

Consider a phase–type distribution with representation (α, S). Then the n'th order moment distribution is again of phase–type with representation (α_n, S_n), where

$$\alpha_n = \left(\frac{\rho_{n+1}}{\rho_n}\pi_{n+1} \bullet s, 0, \dots, 0\right)$$

$$S_n = \begin{bmatrix} C_{n+1} & D_{n+1} & 0 & \dots & 0 & 0 \\ 0 & C_n & D_n & \dots & 0 & 0 \\ 0 & 0 & C_{n-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & C_2 & D_2 \\ 0 & 0 & 0 & \dots & 0 & C_1 \end{bmatrix}$$

and $\rho_i = \mu_i / i! = \alpha(-S)^{-i} e$ are the reduced moments,

$$\pi_i = \rho_i^{-1}\alpha(-S)^{-i}, \;\; C_i = \Delta(\pi_i)^{-1} S' \Delta(\pi_i), \;\; D_i = \frac{\rho_{i-1}}{\rho_i}\Delta(\pi_i)^{-1}\Delta(\pi_{i-1}).$$

Lorenz curve and Gini index

- If F is a distribution function and F_1 the corresponding first moment distribution, then the parametric curve

$$t \to (F(t), F_1(t))$$

is called the Lorenz curve or concentration curve.

- By definition,

$$\frac{dF_1(x)}{dF(x)} = \frac{x}{\mu_1} > 0$$

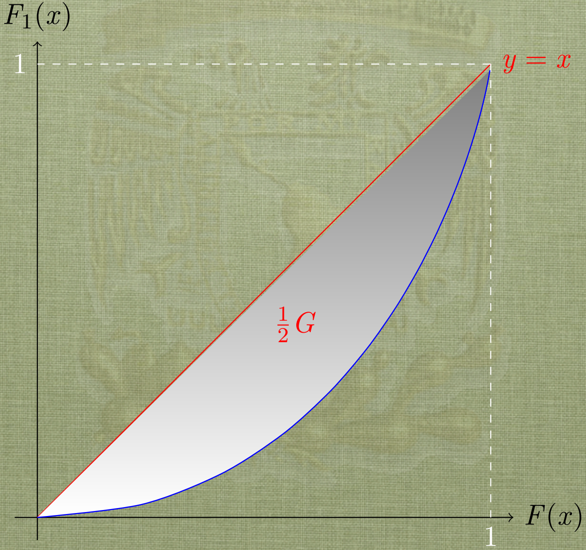
and

$$\frac{d^2F_1(x)}{dF^2(x)} = \frac{dx}{\mu_1 dF(x)} = \frac{1}{\mu_1}\frac{1}{f(x)} > 0.$$

- Hence the Lorenz curve is convex. For the ME (and PH) we get

$$t \to \left(1 - \alpha e^{St} e, 1 - \frac{\alpha S^{-1}}{\alpha S^{-1} e}\left(e^{St} e + t e^{St} s\right)\right).$$

Gini index



Gini index

- The Gini index is defined as twice the area enclosed by the Lorenz curve and the line $y = x$.
- The Lorenz curve starts in $(0, 0)$ and ends in $(1, 1)$. Since the curve is convex it "lies under" $y = x$.
- The area under the $y = x$ for $x = 0$ to $x = 1$ is $1/2$.
- The area A under the Lorenz curve is

$$\begin{aligned} A &= \int_0^\infty F'(t) F_1(t)\, dt \\ &= \int_0^\infty \alpha e^{St} s \left(1 - \alpha_1 e^{S_1 t} e\right) dt \\ &= 1 - \int_0^\infty \alpha e^{St} s \alpha_1 e^{S_1 t} e dt \\ &= 1 + (\alpha \otimes \alpha_1)(S \oplus S_1)^{-1}(s \otimes e) \end{aligned}$$

Some examples

- The Gini index G hence amounts to

$$G = 2(\frac{1}{2} - A) = 2(\alpha \otimes \alpha_1)\left(-\left(S \oplus S_1\right)\right)^{-1}(s \otimes e) - 1.$$

- Consider three examples:

$$f(x) = 4xe^{-2x}, g(x) = 9e^{-10x} + \frac{1}{91}e^{-10x/91}, h(x) = \frac{2}{3}e^{-x}\left(1 + \cos(x)\right).$$

- Representations for the Erlang and Hyper–exponential distributions are taken to be

$$\left((1,0), \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}\right), \qquad \left(\left(\frac{9}{10}, \frac{1}{10}\right), \begin{pmatrix} -10 & 0 \\ 0 & -\frac{10}{91} \end{pmatrix}\right)$$

while a representation for the ME distribution is

$$\left((0,0,1), \begin{pmatrix} -1 & 0 & 0 \\ -\frac{2}{3} & -1 & 1 \\ \frac{2}{3} & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{4}{3} \end{pmatrix}\right).$$

Graphs for the Erlang distribution

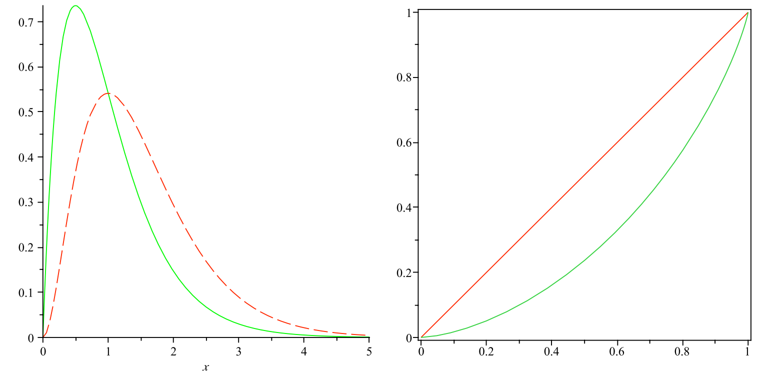


Figura: Left: Densities f and f_1. Right: Corresponding Lorenz curve. The Gini indiex is 0.3750.

Graphs for the hyper–exponential distribution

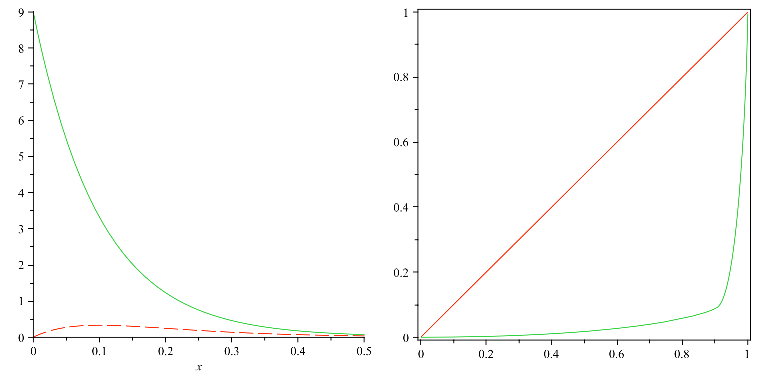


Figura: Left: Densities g and g_1. Right: Corresponding Lorenz curve. The Gini indiex is 0.8962.

Graphs for the ME distribution

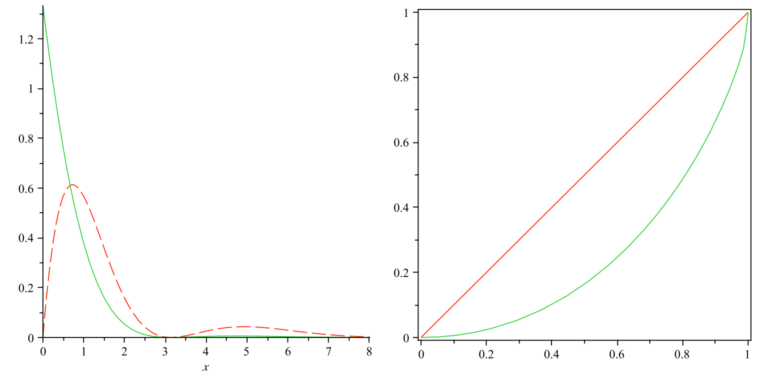


Figura: Left: Densities h and h_1. Right: Corresponding Lorenz curve. The Gini indiex is 0.4917.

Conclusion

- Explicit formulas for moment representations of any order, both ME and PH.
- Closure property.
- Explicit formulas for Gini index, important e.g. in economics
- Open problem of how to estimate grouped data in general.

Hard work in applied probability...