Ruin Probabilities in a Diffusion Environment

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1 Cox Models

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- $G(y)$: distribution function of Y_i, $G(0) = 0$
- $\mu_n = \mathbb{E}[Y_i^n]$, $\mu = \mu_1$, $h(r) = \mathbb{E}[\mathrm{e}^{rY} - 1]$.

Diffusion Intensities

Let $\{Z_t\}$ be a diffusion process following the stochastic differential equation

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Let $\Lambda(t) = \int_0^t \ell(Z_s)\,\mathrm{d}s$ for some function ℓ. We define

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Thus, given $\{Z_t\}$, the claim number process $\{N_t\}$ is conditionally an inhomogeneous Poisson process with rate $\{\ell(Z_t)\}$.

The Martingale

The process $M = \{g(Z_t)\mathrm{e}^{-r(X_t-x)-\theta(r)t}\}$ is a martingale if

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We norm g, such that $\lim_{t\to\infty} \mathbb{E}[g(Z_t)] = 1$.

The Change of Measure

Consider the measure

$$\mathbb{Q}[A] = \frac{\mathbb{E}[g(Z_T)\mathrm{e}^{-r(X_T - x) - \theta(r)T}; A]}{\mathbb{E}[g(Z_0)]} .$$

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$$\tilde{\mathfrak{A}} f = \frac{ga + b^2 g'}{g} f' + \tfrac{1}{2} b^2 f'' \ .$$

The Change of Measure

Typically, the function $\theta(r)$ will be convex. Since

$$\mathbb{E}_{\mathbb{Q}}[X_t] = \mathbb{E}_{\mathbb{P}}[X_t g(Z_t) \mathrm{e}^{-rX_t} \mathrm{e}^{-\theta(r)t}]$$

we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}r} \mathbb{E}_{\mathbb{P}}[g(Z_t) \mathrm{e}^{-rX_t} \mathrm{e}^{-\theta(r)t}] \\ &= \mathbb{E}_{\mathbb{P}}\Bigl[\Bigl(\frac{\mathrm{d}}{\mathrm{d}r} g(Z_t)\Bigr) \mathrm{e}^{-rX_t} \mathrm{e}^{-\theta(r)t}\Bigr] - \mathbb{E}_{\mathbb{Q}}[X_t] - t\theta'(r)\,. \end{aligned}$$

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Typically, dividing by t and letting $t \to \infty$, the first term will vanish. Thus $t^{-1}\mathbb{E}_{\mathbb{Q}}[X_t]$ will converge to $-\theta'(r)$. Hence the safety loading condition will not be fulfilled for $r \ge r_0$, where r_0 is the solution to $\theta'(r) = 0$. That means that $\mathbb{Q}[T_u < \infty] = 1$ if and only if $r \ge r_0$.

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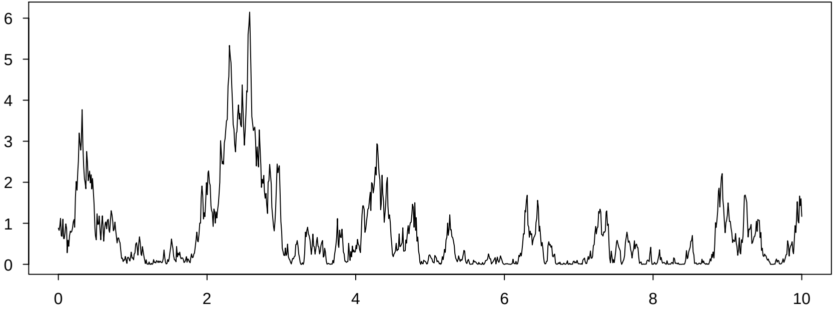
We consider the intensity $\lambda_t = Z_t^2$.

Randomly Generated Intensity



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We try $g(z) = \kappa e^{kz^2}$ for some $k < \frac{a}{b^2}$. The restriction is in order to ensure that $\mathbb{E}[g(Z_0)] < \infty$. From $\mathbb{E}[g(Z_0)] = 1$ we find $\kappa = \sqrt{1 - b^2 k/a}$.

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The equation reduces to

$$\frac{1}{2}b^2(4z^2k^2 + 2k) - 2az^2k - (\theta(r) + cr) + z^2 h(r) = 0 \; .$$

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Thus we find

$$k = \frac{a}{2b^2} - \sqrt{\frac{a^2}{4b^4} - \frac{h(r)}{2b^2}}, \qquad \theta(r) = \frac{a - \sqrt{a^2 - 2b^2h(r)}}{2} - cr\ ,$$

and

$$\kappa = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{b^2h(r)}{2a^2}}}\ .$$

The Martingale

M is a martingale. The generator of the diffusion after the change of measure becomes

$$\widetilde{\mathfrak{A}}f(z) = -(a - 2kb^2)zf'(z) + \tfrac{1}{2}b^2 f''(z)\,.$$

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Hence under $\mathbb{Q}$ the process Z is an Ornstein–Uhlenbeck process with the same diffusion coefficient b and drift $-\sqrt{a^2 - 2b^2 h(r)}\, z$. Z will turn back to its mean more slowly than under $\mathbb{P}$ if $r > 0$. The (stationary under $\mathbb{Q}$) drift of the process X under $\mathbb{Q}$ is then

$$c - \frac{b^2}{2\sqrt{a^2 - 2b^2 h(r)}}\, \tilde{h}(r) \frac{h'(r)}{\tilde{h}(r)} = -\theta'(r)\,.$$

Cramér–Lundberg Inequalities

Let R be the (non-trivial) solution to $\theta(r) = 0$. Then

$$\psi(u) = \mathbb{E}_{\mathbb{Q}}\Bigl[\frac{1}{g(Z_{T_u})}\, \mathrm{e}^{R(u+X_{T_u})}\Bigr]\, \mathrm{e}^{-Ru} < \kappa^{-1}\mathrm{e}^{-Ru}\,.$$

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In the same way we obtain the two finite-time Lundberg inequalities

$$\begin{aligned}\psi(u,yu) &< \kappa^{-1}\mathrm{e}^{-R(0,y)u}\,, \quad (y < y_0)\,,\\ \psi(u) - \psi(u,yu) &< \kappa^{-1}\mathrm{e}^{-R(y,\infty)u}\,, \quad (y > y_0)\,,\end{aligned}$$

where $y_0 = 1/\theta'(R)$, $R(0,y) = \sup_{r\ge 0} r - y\theta(r)$ and $R(y,\infty) = \sup_{r\ge 0} r - y\theta(r)$.

Cramér–Lundberg Approximation

Using a renewal approach we get

$$\lim_{u\to\infty} \psi(u)\, \mathrm{e}^{Ru} = C\, \mathbb{E}_{\mathbb{P}}[g(Z_0)]$$

for some constant C.

Subexponential Distributions

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A distribution function $F(y)$ is in the class $\mathcal{S}^*$, if $\mu_F < \infty$ and

$$\lim_{x\to\infty} \int_0^x \frac{(1 - F(x-y))(1 - F(y))}{1 - F(x)} \,\mathrm{d}y = 2\mu_F \,.$$

For example are the Pareto, the Weibull and the Lognormal distributions in $\mathcal{S}^*$.

$F \in \mathcal{S}^*$ implies $F(y)$ and $F^{\mathrm{s}}(y) = \mu_F^{-1} \int_0^y 1 - F(z) \,\mathrm{d}z$ are in $\mathcal{S}$.

Exponential Moments of the Intensity

Let $\varepsilon > 0$, $S = \inf\{t > 0 : Z_t = \varepsilon\}$, $T_1 = \inf\{t > S : Z_t = 0\}$.

Lemma

Let either $Z_0 = 0$ or Z_0 be normally distributed with mean zero and variance $b^2/(2a)$. There exists a $\gamma > 0$ such that $\exp\{\gamma \int_0^{S_1} (1 + Z_t^2) \, \mathrm{d}t\}$ is integrable.

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Proof.

Construct an appropriate martingale. □

The Tails

Lemma

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This shows that $-X(S_1)$ and $cS_1 - X(S_1)$ have the same distribution tail.

The Asymptotic Behaviour

Theorem

Suppose that both $G(x)$ *and* $G^{\mathrm{s}}(x)$ *are subexponential, and that* $Z_0 = 0$ *or that* Z_0 *is normally distributed with mean zero and variance* $b^2/(2a)$. *Then*

$$\lim_{u\to\infty} \frac{\psi(u)}{1 - G^{\mathrm{s}}(u)} = \frac{\mu b^2/(2a)}{c - \mu b^2/(2a)} = \frac{\mu b^2}{2ac - \mu b^2} .$$

Extensions

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- Claim size distribution depends on λ_t: $Y_i \sim F_{\lambda(t)}$.

References

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Thank you for your attention