On Perfect Sampling and Conditional Large Deviations

Jose Blanchet (joint work with J. Dong, K. Sigman, A. Wallwater)

Columbia University

[07/2011] July, 2011

Outline

1. Model and Problem Statement
2. Exact Simulation and Regeneration
3. Fast Exact Simulation and Rare Events

What is This Talk About?

- From a Monte Carlo standpoint:

Exact Simulation and Rare-Event Simulation

What is This Talk About?

- From a Monte Carlo standpoint:

Exact Simulation and Rare-Event Simulation

- Two areas fundamentally influenced by Søren's work!

What is This Talk About?

- This talk relates to:

What is This Talk About?

- This talk relates to:
- Asmussen, Glynn and Thorisson '92 (A-G-T): Stationary Detection in the Initial Transient Problem

What is This Talk About?

- This talk relates to:
- Asmussen, Glynn and Thorisson '92 (A-G-T): Stationary Detection in the Initial Transient Problem
- Asmussen, Binswanger and Højggard '00 (A-B-H): Rare-event Simulation for Heavy-tailed Distributions

What is the main message?

Main message 1:

Conditional distribution given rare events $->$ key for fast exact simulation

What is the main message?

Main message 2:
Some interesting examples...

Sampling $\max\{S_k : k \geq 0\}$ for a random walk S_k *without* bias

What is the main message?

Other interesting examples

- Exact sampling of perpetuities (joint with K. Sigman)

What is the main message?

Other interesting examples

- Exact sampling of perpetuities (joint with K. Sigman)
- Exact sampling of steady-state infinite server queues (joint with J. Dong)

What is the main message?

Other interesting examples

- Exact sampling of perpetuities (joint with K. Sigman)
- Exact sampling of steady-state infinite server queues (joint with J. Dong)
- Exact sampling of steady-state multi-server queues (joint with A. Wallwater)

Outline

Exact Simulation and Regeneration

- Let $\{W_k : k \geq 0\}$ be non-delayed (positive recurrent) regenerative process at times $T_0 = 0 < T_1 < T_2 < ...$

Exact Simulation and Regeneration

- Let $\{W_k : k \geq 0\}$ be non-delayed (positive recurrent) regenerative process at times $T_0 = 0 < T_1 < T_2 < \ldots$
- Known fact

$$P(W_\infty \in \cdot) = \frac{E\left(\sum_{k=0}^{T_1-1} I(W_k \in \cdot)\right)}{ET_1}$$

$$= \sum_{k=0}^{\infty} \frac{P(W_k \in \cdot | T_1 > k) P(T_1 > k)}{ET_1}$$

Exact Simulation and Regeneration

- A-G-T note implications for simulation:

$$P\left(W_{\infty}\in\cdot\right)=\sum_{k=0}^{\infty}P\left(W_{k}\in\cdot\,|\,T_{1}>k\right)\times\left(\frac{P\left(T_{1}>k\right)}{ET_{1}}\right)$$

Exact Simulation and Regeneration

- A-G-T note implications for simulation:

$$P\left(W_{\infty}\in\cdot\right)=\sum_{k=0}^{\infty}P\left(W_{k}\in\cdot\middle|T_{1}>k\right)\times\left(\frac{P\left(T_{1}>k\right)}{ET_{1}}\right)$$

- **Step 1:** Simulate τ_{eq} such that $P\left(\tau_{eq}=k\right)=P\left(T_{1}>k\right)/ET_{1}$.

Exact Simulation and Regeneration

- A-G-T note implications for simulation:

$$P\left(W_{\infty}\in\cdot\right)=\sum_{k=0}^{\infty}P\left(W_{k}\in\cdot|\,T_{1}>k\right)\times\left(\frac{P\left(T_{1}>k\right)}{ET_{1}}\right)$$

- **Step 1:** Simulate τ_{eq} such that $P\left(\tau_{eq}=k\right)=P\left(T_{1}>k\right)/ET_{1}$.
- **Step 2:** Given $\tau_{eq}=k$ simulate W_k given that $T_1>k$ and OUTPUT W_k

Connection to Bernoulli Factory Problem

- **Important remark:** Execution of **Step 1** motivates the "*Bernoulli factory problem*"

Connection to Bernoulli Factory Problem

- **Important remark:** Execution of **Step 1** motivates the "*Bernoulli factory problem*"
- Say $E[(1+T_1)^p] \leq m$ for $p > 1$ then

$$P(\tau_{eq} = k) = \frac{P(T_1 > k)}{ET_1} \leq \frac{m}{ET_1} \times \frac{1}{(1+k)^p} < \frac{2m}{ET_1} \times \frac{1}{(1+k)^p}.$$

Connection to Bernoulli Factory Problem

- **Important remark:** Execution of **Step 1** motivates the "*Bernoulli factory problem*"
- Say $E[(1+T_1)^p] \leq m$ for $p > 1$ then

$$P(\tau_{eq} = k) = \frac{P(T_1 > k)}{ET_1} \leq \frac{m}{ET_1} \times \frac{1}{(1+k)^p} < \frac{2m}{ET_1} \times \frac{1}{(1+k)^p}.$$

- Acceptance / rejection: Propose from Z such that $P(Z = k) \propto 1/(1+k)^p$ & **accept with probability**

$$\frac{(1+Z)^p}{2m} \times P(T_1 > Z | Z)$$

Connection to Bernoulli Factory Problem

- **Important remark:** Execution of **Step 1** motivates the "*Bernoulli factory problem*"
- Say $E\left[(1+T_1)^p\right] \leq m$ for $p > 1$ then

$$P\left(\tau_{eq} = k\right) = \frac{P\left(T_1 > k\right)}{ET_1} \leq \frac{m}{ET_1} \times \frac{1}{(1+k)^p} < \frac{2m}{ET_1} \times \frac{1}{(1+k)^p}.$$

- Acceptance / rejection: Propose from Z such that $P\left(Z = k\right) \propto 1/(1+k)^p$ & **accept with probability**

$$\frac{(1+Z)^p}{2m} \times P\left(T_1 > Z | Z\right) \leq 1/2$$

- Keane-O'Brien (1994), Nacu-Peres (2005), Latuszynski, Kosmidis, Papaspiliopoulos and Roberts, (2009)

Simple Bounds Give Exact Simulation Algorithms

Well, we know methods to bound $E\left(1+T_1\right)^p$, that's all we need.
So, exact simulation is "easy"?

Exact Simulation and Regeneration

- **Unfortunately... algorithm SLOW (regardless of factory)**

Exact Simulation and Regeneration

- **Unfortunately... algorithm SLOW (regardless of factory)**
- Given $\tau_{eq} = k$ need $1/P(T_1 > k)$ cycles of W to see $T_1 > k$.

Exact Simulation and Regeneration

- **Unfortunately... algorithm SLOW (regardless of factory)**
- Given $\tau_{eq} = k$ need $1/P(T_1 > k)$ cycles of W to see $T_1 > k$.
- Total expected number of cycles

$$\sum_{k=0}^{\infty} \frac{1}{P(T_1 > k)} \times P(\tau_{eq} = k) = \sum_{k=0}^{\infty} \frac{1}{ET_1} = \infty.$$

Exact Simulation and Regeneration

- **Unfortunately... algorithm SLOW (regardless of factory)**
- Given $\tau_{eq} = k$ need $1/P(T_1 > k)$ cycles of W to see $T_1 > k$.
- Total expected number of cycles

$$\sum_{k=0}^{\infty} \frac{1}{P(T_1 > k)} \times P(\tau_{eq} = k) = \sum_{k=0}^{\infty} \frac{1}{ET_1} = \infty.$$

- This takes me to **Message 1: Cond. Dist. Given Rare Events Key for Fast Exact Simulation** ... *Why?*

Outline

- *KEY OBSERVATION*: Want to sample from

$$P(W_n \in \cdot \,|\, T_1 > n)$$

with complexity $C^*(n)$ so that

$$EC^*(\tau_{eq}) = \sum_{n=0}^{\infty} C^*(n) \frac{P(T_1 > n)}{ET_1} < \infty.$$

- *KEY OBSERVATION*: Want to sample from

$$P(W_n \in \cdot | T_1 > n)$$

with complexity $C^*(n)$ so that

$$EC^*(\tau_{eq}) = \sum_{n=0}^{\infty} C^*(n) \frac{P(T_1 > n)}{ET_1} < \infty.$$

- Need method that is good enough for large n...

Exact Simulation and Conditioning on Rare Events

- *KEY OBSERVATION*: Want to sample from

$$P(W_n \in \cdot \,|\, T_1 > n)$$

with complexity $C^*(n)$ so that

$$EC^*(\tau_{eq}) = \sum_{n=0}^{\infty} C^*(n) \frac{P(T_1 > n)}{ET_1} < \infty.$$

- Need method that is good enough for large n...
- Connections to rare-event simulation...

Steady-state Simulation for the Single-Server Queue

- This takes me to **Message 2: interesting examples**...

Steady-state Simulation for the Single-Server Queue

- This takes me to **Message 2: interesting examples**...
- *Waiting time sequence in a single-server queue*

$$W_{k+1} = (W_k + X_{k+1})^+ ,$$

where X_k's are i.i.d.

Steady-state Simulation for the Single-Server Queue

- This takes me to **Message 2: interesting examples**...
- *Waiting time sequence in a single-server queue*

$$W_{k+1} = (W_k + X_{k+1})^+ ,$$

  where X_k's are i.i.d.
- Assume $EX_k < 0$.

Steady-state Simulation for the Single-Server Queue

- This takes me to **Message 2: interesting examples**...
- *Waiting time sequence in a single-server queue*

$$W_{k+1} = (W_k + X_{k+1})^+ ,$$

where X_k's are i.i.d.

- Assume $EX_k < 0$.
- $W_0 = 0$, regeneration times $T_0 = 0 < T_1 < \ldots$ are subsequent visits to zero.

Steady-state Simulation for the Single-Server Queue

- This takes me to **Message 2: interesting examples**...
- *Waiting time sequence in a single-server queue*

$$W_{k+1} = (W_k + X_{k+1})^+ ,$$

  where X_k's are i.i.d.
- Assume $EX_k < 0$.
- $W_0 = 0$, regeneration times $T_0 = 0 < T_1 < \ldots$ are subsequent visits to zero.
- $T_1 = \min\{k \geq 1 : W_k = 0\}$.

Simulation Conditional on Long Busy Periods

Goal: Want to simulate W_n given $T_1 > n$ with *good complexity* for large n...

The Associated Random Walk

- Define the associated random walk $S_n = X_1 + ... + X_n$...

The Associated Random Walk

- Define the associated random walk $S_n = X_1 + ... + X_n$...
- Note that on $T_1 > n$ we have that $W_k = S_k > 0$ for $0 \leq k \leq n$.

The Associated Random Walk

- Define the associated random walk $S_n = X_1 + ... + X_n$...
- Note that on $T_1 > n$ we have that $W_k = S_k > 0$ for $0 \leq k \leq n$.
- **Equivalent Goal: Simulate S_n given that $S_1 > 0, ..., S_n > 0$ with good complexity for large k**

Light-tailed Random Walks

- Let $\psi(\theta) = \log E \exp(\theta X_n)$, suppose there is $\theta_0 > 0$ with $\psi'(\theta_0) = 0$.

Light-tailed Random Walks

- Let $\psi(\theta) = \log E \exp(\theta X_n)$, suppose there is $\theta_0 > 0$ with $\psi'(\theta_0) = 0$.
- *Since* $\psi'(0) = EX_k < 0$ *convexity yields* $\psi(\theta_0) < 0$

Light-tailed Random Walks

- Let $\psi(\theta) = \log E \exp(\theta X_n)$, suppose there is $\theta_0 > 0$ with $\psi'(\theta_0) = 0$.
- *Since* $\psi'(0) = EX_k < 0$ *convexity yields* $\psi(\theta_0) < 0$
- $P_{\theta_0}(\cdot)$ is *exponential tilting* using θ_0:

$$P_{\theta_0}(S_n \in A) = E\{\exp(\theta_0 S_n - n\psi(\theta_0)) I(S_n \in A)\}.$$

Light-tails Random Walks

- Note

$$
\begin{aligned}
& P\left(W_n \in dx \middle| T_1 > n\right) \\
& = \frac{P\left(S_n \in dx\right) I\left(S_k > 0 : k \leq n\right)}{P\left(T_1 > n\right)} \\
& = \frac{\exp\left(n\psi\left(\theta_0\right)\right)}{P\left(T_1 > n\right)} \times P_{\theta_0}\left(S_n \in dx\right) \exp\left(-\theta_0 S_n\right) I\left(S_k > 0 : k \leq n\right) \\
& \leq \frac{\exp\left(n\psi\left(\theta_0\right)\right)}{P\left(T_1 > n\right)} \times P_{\theta_0}\left(S_n \in dx\right) \exp\left(-\theta_0 S_n\right) I\left(S_n > 0\right) \\
& \leq \frac{\exp\left(n\psi\left(\theta_0\right)\right)}{P\left(T_1 > n\right)} \times P_{\theta_0}\left(S_n \in dx\right)
\end{aligned}
$$

Efficient Acceptance / Rejection for Light-tails

- We know that

$$P(S_n > 0) = \exp(n\psi(\theta_0) + o(n)),$$

and also (Palmowski and Rolski (2006)) that

$$P(T_1 > n) = \exp(n\psi(\theta_0) + o(n)).$$

Efficient Acceptance / Rejection for Light-tails

- We know that

$$P(S_n > 0) = \exp(n\psi(\theta_0) + o(n)),$$

and also (Palmowski and Rolski (2006)) that

$$P(T_1 > n) = \exp(n\psi(\theta_0) + o(n)).$$

- We get $C^*(n) = \exp(o(n))$,

$$EC^*(\tau_{eq}) = \sum_{n=0}^{\infty} \exp(n\psi(\theta_0) + o(n)) < \infty.$$

Efficient Acceptance / Rejection for Light-tails

- We know that

$$P\left(S_n > 0\right) = \exp\left(n\psi\left(\theta_0\right) + o\left(n\right)\right),$$

and also (Palmowski and Rolski (2006)) that

$$P\left(T_1 > n\right) = \exp\left(n\psi\left(\theta_0\right) + o\left(n\right)\right).$$

- We get $C^*\left(n\right) = \exp\left(o\left(n\right)\right)$,

$$EC^*\left(\tau_{eq}\right) = \sum_{n=0}^{\infty} \exp\left(n\psi\left(\theta_0\right) + o\left(n\right)\right) < \infty.$$

- **Conclusion:** With light tails we can simulate steady-state waiting time of $GI/GI/1$ queue (equivalently $\max\{S_k : k \geq 0\}$) with good complexity.

Heavy-tails and Conditional Large Deviations Behavior

- Suppose that X_n's have power-law tail (reg. varying) with index $\alpha > 3$

Heavy-tails and Conditional Large Deviations Behavior

- Suppose that X_n's have power-law tail (reg. varying) with index $\alpha > 3$
- $P(X_n > x) = x^{-\alpha} K (1 + o(1))$ as $x \to \infty$ for some $K > 0$.

Heavy-tails and Conditional Large Deviations Behavior

- Suppose that X_n's have power-law tail (reg. varying) with index $\alpha > 3$
- $P(X_n > x) = x^{-\alpha} K (1 + o(1))$ as $x \to \infty$ for some $K > 0$.
- How does the event $T_1 > n$ occur for large n?

Heavy-tails and Conditional Large Deviations Behavior

- Suppose that X_n's have power-law tail (reg. varying) with index $\alpha > 3$
- $P(X_n > x) = x^{-\alpha} K (1 + o(1))$ as $x \to \infty$ for some $K > 0$.
- How does the event $T_1 > n$ occur for large n?
- Zwart '01: One large service requirement large early in busy period <– important feature!

Heavy-tails and Conditional Large Deviations Behavior

- Suppose that X_n's have power-law tail (reg. varying) with index $\alpha > 3$
- $P(X_n > x) = x^{-\alpha} K (1 + o(1))$ as $x \to \infty$ for some $K > 0$.
- How does the event $T_1 > n$ occur for large n?
- Zwart '01: One large service requirement large early in busy period <– important feature!
- Proposal distribution to incorporate important feature... related to ABH '00

Discussing Assumptions in Heavy-tailed Setting

- Note

$$P\left(W_n \in dx \middle| T_1 > n\right) \leq \frac{P\left(S_n > 0\right)}{P\left(T_1 > n\right)} P\left(S_n \in dx \middle| S_n > 0\right).$$

Discussing Assumptions in Heavy-tailed Setting

- Note

$$P\left(W_n \in dx \middle| T_1 > n\right) \leq \frac{P\left(S_n > 0\right)}{P\left(T_1 > n\right)} P\left(S_n \in dx \middle| S_n > 0\right).$$

- If one samples S_n given $S_n > 0$ with complexity $O\left(n\right)$ then

$$C^*\left(n\right) = n \times \frac{P\left(S_n > 0\right)}{P\left(T_1 > n\right)} = \frac{n}{P\left(T_1 > n \middle| S_n > 0\right)} = O\left(n^2\right)$$

Discussing Assumptions in Heavy-tailed Setting

- Note

$$P\left(W_n \in dx \middle| T_1 > n\right) \leq \frac{P\left(S_n > 0\right)}{P\left(T_1 > n\right)} P\left(S_n \in dx \middle| S_n > 0\right).$$

- If one samples S_n given $S_n > 0$ with complexity $O\left(n\right)$ then

$$C^*\left(n\right) = n \times \frac{P\left(S_n > 0\right)}{P\left(T_1 > n\right)} = \frac{n}{P\left(T_1 > n \middle| S_n > 0\right)} = O\left(n^2\right)$$

- Since $P\left(\tau_{eq} = n\right) = \Theta(n^{-\alpha})$, $EC^*\left(\tau_{eq}\right) < \infty$ implies $\alpha > 3$.

Discussing Assumptions in Heavy-tailed Setting

- Note

$$P\left(W_n \in dx \middle| T_1 > n\right) \leq \frac{P\left(S_n > 0\right)}{P\left(T_1 > n\right)} P\left(S_n \in dx \middle| S_n > 0\right).$$

- If one samples S_n given $S_n > 0$ with complexity $O\left(n\right)$ then

$$C^*\left(n\right) = n \times \frac{P\left(S_n > 0\right)}{P\left(T_1 > n\right)} = \frac{n}{P\left(T_1 > n \middle| S_n > 0\right)} = O\left(n^2\right)$$

- Since $P\left(\tau_{eq} = n\right) = \Theta(n^{-\alpha})$, $EC^*\left(\tau_{eq}\right) < \infty$ implies $\alpha > 3$.
- Can do $\alpha > 2$ by replacing event $S_n > 0$ for another (better chosen) event.

Formulation of a Sufficient Goal

Goal: Sample $S_1, ..., S_n$ given $S_n > 0$ with complexity $O\left(n^{1+\delta}\right)$ for $\delta < \alpha - 3$.

In other words: Average Number of Proposals $= O\left(n^{\delta}\right)$

Carrying Over the Goal: Conditional Sampling Given Large Deviations

- Note

$$\frac{P\left(S_n \in \cdot, S_n > 0\right)}{P\left(S_n > 0\right)}$$

$$= w_n \frac{P\left(S_n \in \cdot, S_n > 0 | \exists k \leq n : X_k - EX_k \leq n^{1-\delta}\right)}{P\left(S_n > 0\right)}$$

$$+ \left(1 - w_n\right) \frac{P\left(S_n \in \cdot, S_n > 0 | \forall k \leq n : X_k - EX_k \leq n^{1-\delta}\right)}{P\left(S_n > 0\right)},$$

with

$$w_n = P\left(\exists k \leq n : X_k - EX_k > n^{1-\delta}\right).$$

Back to a Calculation Involving Light Tails

- Note that

$$E\left(X_k | X_k \leq n^{1-\delta}\right) = EX_k + O\left(n^{-(\alpha-1)\times(1-\delta)}\right)$$

Back to a Calculation Involving Light Tails

- Note that

$$E\left(X_k | X_k \leq n^{1-\delta}\right) = EX_k + O\left(n^{-(\alpha-1)\times(1-\delta)}\right)$$

- Therefore

$$\phi_n := E\left(\exp\left(\frac{S_n - nEX_k}{n^{1-\delta}}\right) | \forall k \leq n : X_k - EX_k \leq n^{1-\delta}\right)$$
$$= 1 + O\left(\frac{1}{n^{(1-2\delta)}}\right)$$

Back to a Calculation Involving Light Tails

- Apply tilting parameter $1/n^{1-\delta}$ to $X_k - EX_k \mid X_k - EX_k \leq n^{1-\delta}$ & **call that measure** $P_n(\cdot)$**:**

$$
\begin{aligned}
& \frac{P\left(S_n \in \cdot, S_n > 0 | \forall k \leq n : X_k - EX_k \leq n^{1-\delta}\right)}{P\left(S_n > 0\right)} \\
& = \frac{\phi_n}{P\left(S_n > 0\right)} E_n\left(\exp\left(-\frac{S_n - nEX_k}{n^{1-\delta}}\right) I\left(S_n \in \cdot, S_n > 0\right)\right) \\
& \leq \phi_n \frac{\exp\left(n^{\delta} EX_k\right)}{P\left(S_n > 0\right)} = o\left(1\right)
\end{aligned}
$$

The Proposal Distribution for Conditional Heavy-tailed Sampling

- **Summary:** Proposal distribution (change-of-measure) $P'(\cdot)$,

$$
\begin{aligned}
& P'(S_1, ..., S_n \in \cdot) \\
& = \frac{w_n}{w_n + \exp(n^{\delta} EX_k)} P\left(S_n \in \cdot | \exists k \leq n : X_k - EX_k > n^{1-\delta}\right) \\
& + \frac{\exp\left(n^{\delta} EX_k\right)}{w_n + \exp(n^{\delta} EX_k)} P_n(S_n \in \cdot),
\end{aligned}
$$

recall $w_n = P\left(\exists k \leq n : X_k - EX_k > n^{1-\delta}\right)$.

The Proposal Distribution for Conditional Heavy-tailed Sampling

- **Summary:** Proposal distribution (change-of-measure) $P'(\cdot)$,

$$
\begin{aligned}
& P'(S_1, \ldots, S_n \in \cdot) \\
& = \frac{w_n}{w_n + \exp\left(n^{\delta} EX_k\right)} P\left(S_n \in \cdot | \exists k \leq n : X_k - EX_k > n^{1-\delta}\right) \\
& + \frac{\exp\left(n^{\delta} EX_k\right)}{w_n + \exp\left(n^{\delta} EX_k\right)} P_n\left(S_n \in \cdot\right),
\end{aligned}
$$

recall $w_n = P\left(\exists k \leq n : X_k - EX_k > n^{1-\delta}\right)$.

- Likelihood ratio

$$
\frac{dP}{dP'} \times \frac{I\left(S_n > 0\right)}{P\left(S_n > 0\right)} \leq 2\phi_n \frac{w_n + \exp\left(n^{\delta} EX_k\right)}{P\left(S_n > 0\right)} = O\left(n^{\delta}\right)
$$

Conclusions

- Message 1: "Fast" exact simulation strongly related to conditional large deviations simulation

Conclusions

- Message 1: "Fast" exact simulation strongly related to conditional large deviations simulation
- Message 2: Interesting examples such as G/G/1 queue both light and heavy-tailed (reg. varying with finite variance) & others