Threshold Strategies for Risk Processes and their Relation to Queueing Theory

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This talk

I. Setting / Literature

II. Risk and queues / Dualities

III. Ruin probability

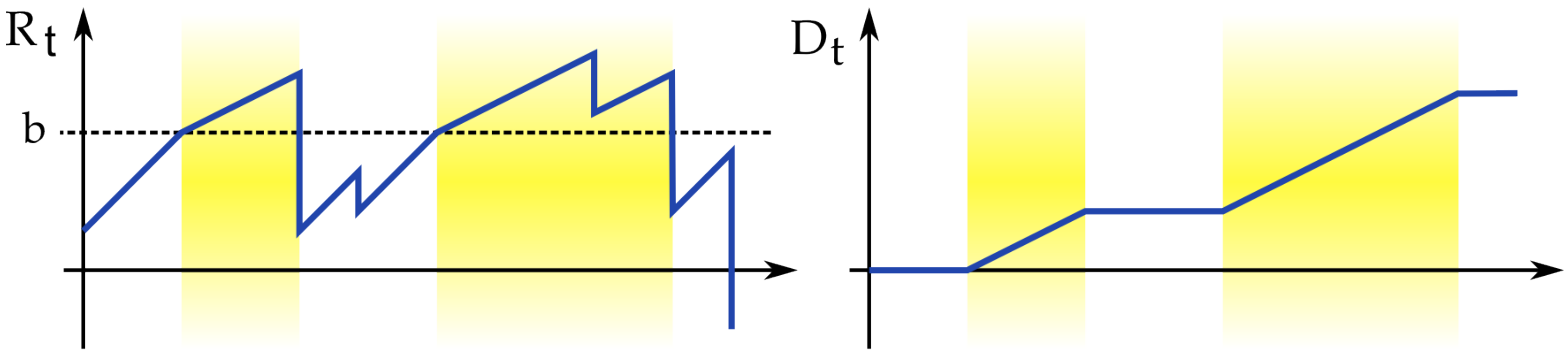
IV. More general ideas

(V. If time allows: Ruin time)

I. Setting / Literature

Risk process with threshold dividend strategy

Setting



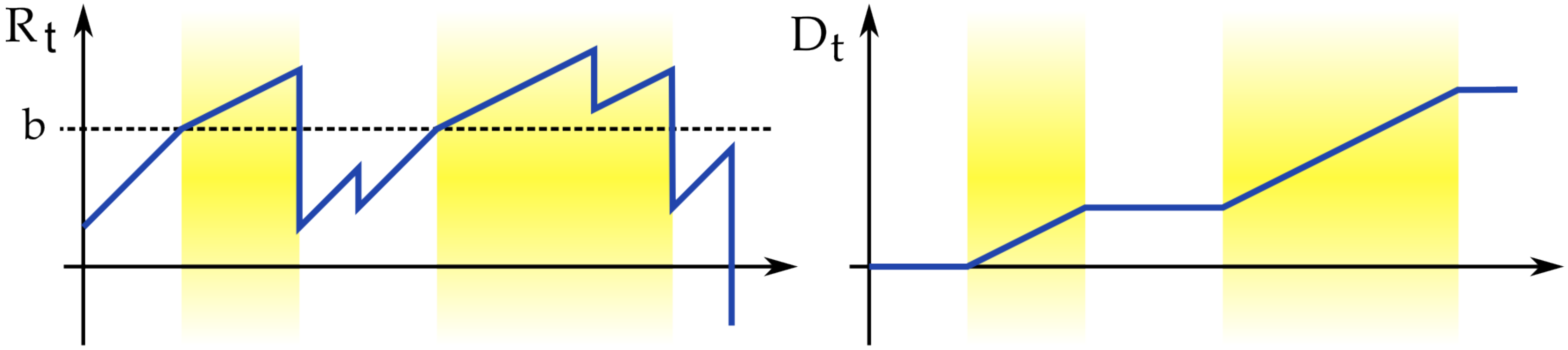
We consider a simple insurance risk model (Cramèr-Lundberg regime), where R_t denotes the surplus of an insurance company at time t.

Some of the income is re-distributed as dividends: whenever R_t is larger than some threshold b, a fraction of γ is paid out as dividends.

$\Rightarrow$ **Threshold strategy (refracting barrier)**

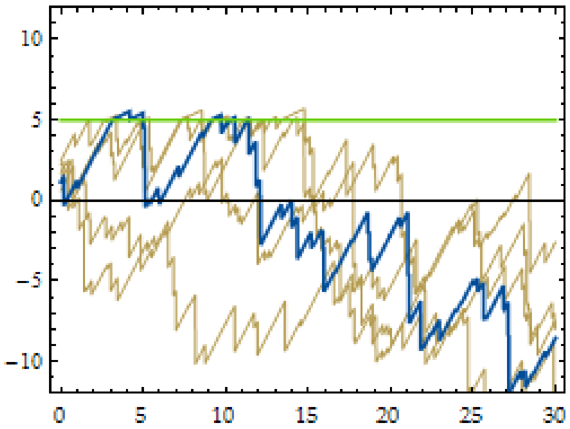
Risk process: $dR_t = r(R_t)\,dt + dS_t$, with

- aggregated claims $S_t = \sum_{i=1}^{N_t} X_i$,
- i.i.d. claims $(X_i)_{i=1,2,\dots}$, distribution G, $\mathbb{E}[X_1] = 1/\lambda$,
- Poisson claim number process N_t with rate μ,
- "plowback rate" $r(x) = 1 - \gamma\mathbf{1}(x \geq b)$,
- dividend process $D_t = \gamma \int_0^t \mathbf{1}(R_s > b)\,ds$,



We let $\rho = \mu/\lambda$ and $\psi(x) = \mathbb{P}_x(\tau < \infty)$.

Three scenarios

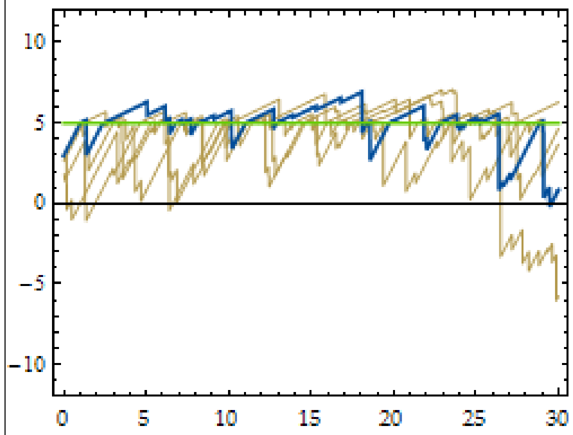


(I)

$\rho > 1$,

$R_t \to -\infty$

$\psi(x) = 1$

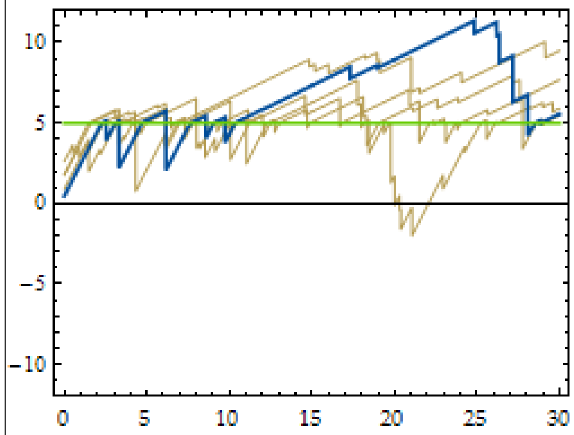


(II)

$1 - \gamma < \rho < 1$,

R_t pos. recurrent

$\psi(x) = 1$



(III)

$\rho < 1 - \gamma$,

$R_t \to \infty$

$\psi(x) < 1$

We omit the cases $\rho = 1$ and $\rho = 1 - \gamma$.

Literature

Ruin theory + dividends

- de Finetti (1957)
- ⋮
- Gerber (1969)
- Bühlmann (1970)
- ⋮
- Albrecher, Kainhofer (2002)
- Lin,Willmot, Drekic (2003)
- Li, Garrido (2004)
- Zhou (2005)
- Albrecher, Hartinger, Tichy (2005)
- Gerber, Shiu (2006)
- Lin, Pavlova (2006)
- Avanzi (2009) **review**

Ruin theory

- Lundberg (1903)
- ⋮
- Gerber (1969)
- Bühlmann (1970)
- Dickson (1993)
- ⋮
- Gerber, Shiu (1997, 1998)
- Rolski, Schmidli, Schmidt, Teugels (1999)
- **Asmussen** (2000)
  Albrecher, **Asmussen** (2010)
- Mikosch (2004)

Queues & Risk

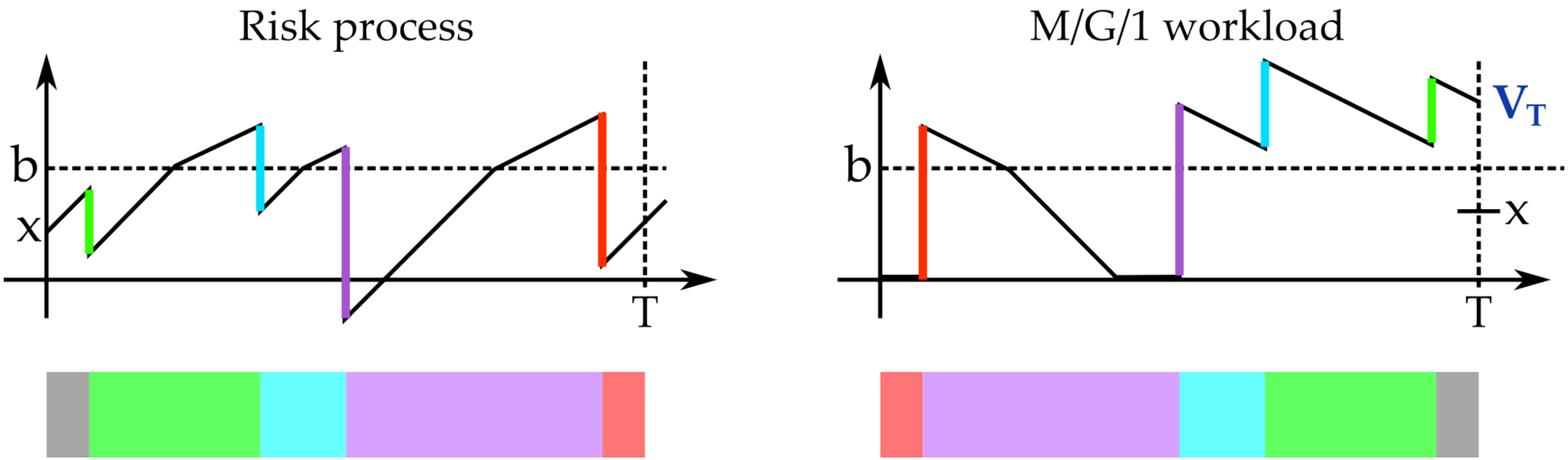
- Prabhu (1961)
- Seal (1972)
- Siegmund (1976)
- Harrison & Resnick (1978)
- **Asmussen** (1987/2003)
- **Asmussen**, Petersen (1988)
- **Asmussen**, Sigman (1998)
- Sigman, Ryan (2000)

Queueing theory

II. Risk and queues

Dualities

Duality I

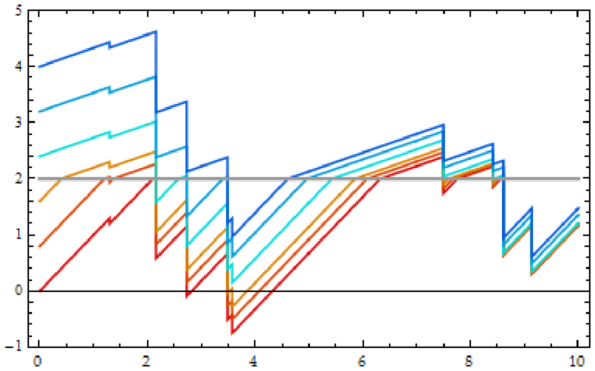


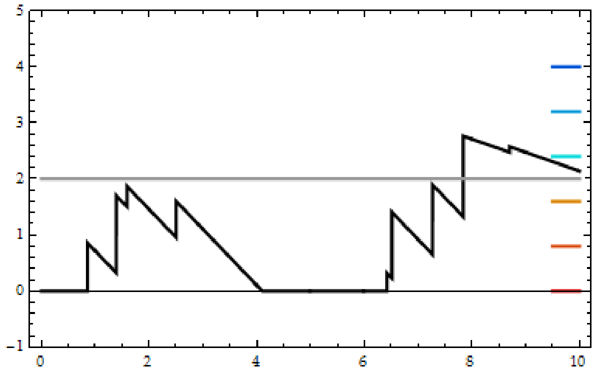
Costruct dual process V_t as follows:

- fix a time T,
- use the same jump sizes and inter-jump times, but in reversed order and reversed direction,
- set $\frac{dV_t}{dt} = 0$ for $V_t \leq 0$ and $\frac{dV_t}{dt} = -r(V_t)$ else.

V_t is the workload process of a M/G/1 with service time distribution G, arrival rate μ and server speed $1 - \gamma \mathbf{1}(V_t \geq b)$.

Duality I





Then surprisingly (Asmussen & Petersen (1988))

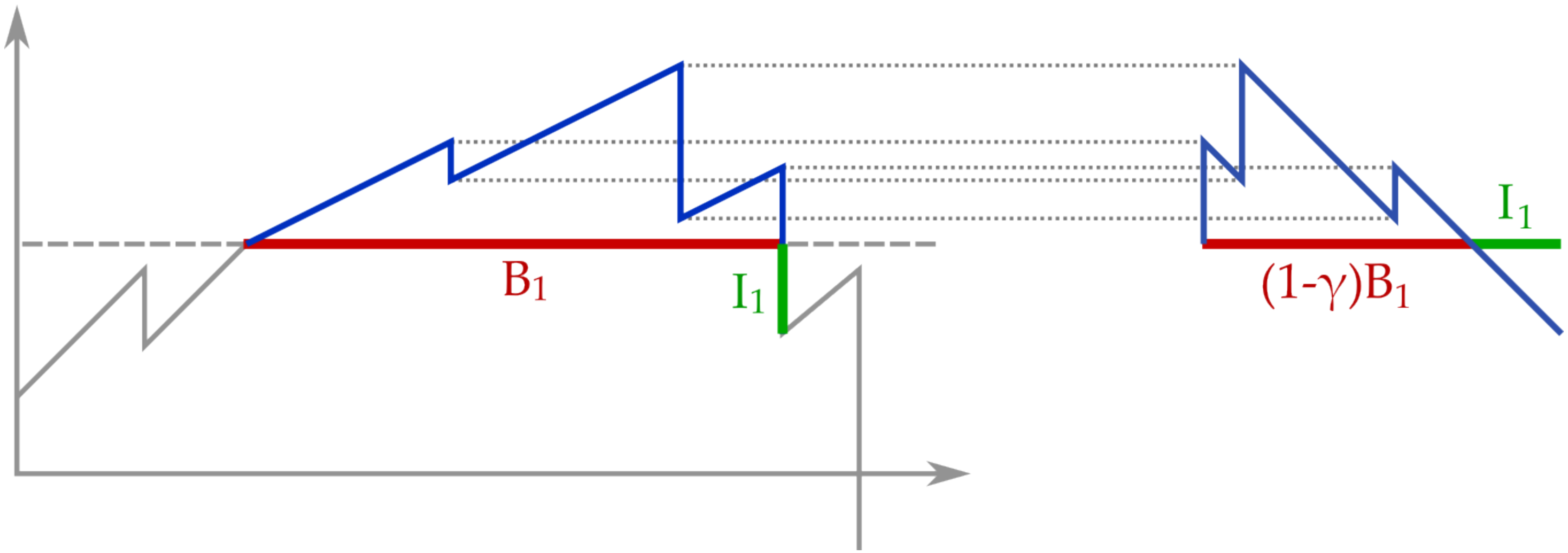
$$\mathbb{P}_x(\tau \leq T) = \mathbb{P}(V_T > x | V_0 = 0)$$

and, if $\rho < 1 - \gamma$,

$$\overline{\psi}(x) = \mathbb{P}_x(\tau = \infty) = F(x),$$

where F is the stationary distribution of the dual queue.

There is another (more obvious) duality:



Risk process
Arrival rate μ
Claim size mean $\frac{1}{\lambda}$

$\Leftrightarrow$

Workload G/M/1
Service mean $\frac{1-\gamma}{\mu}$
Arrival rate λ

III. Ruin probability

Case $\rho < 1 - \gamma$ (upward drift)

The survival probability is positive:

$$\overline{\psi}(x) = \mathbb{P}_x(\tau = \infty) > 0$$

It follows from duality I that

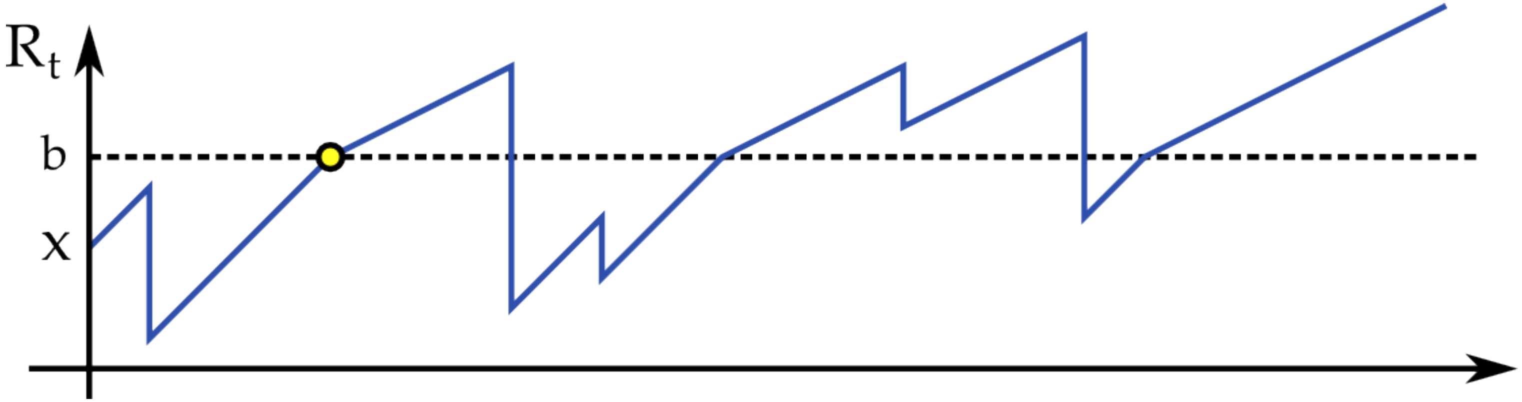
$$\overline{\psi}(x) = F(x),$$

where F is the stationary distribution of an M/G/1 queue with server speed $1 - \gamma \mathbf{1}(V_t > b)$, service distribution G and arrival rate μ.

The Laplace transform of F has been derived by Gaver & Miller (1962) (context: storage processes).

Aim: express $\overline{\psi}$ in terms of $F^{(\gamma)}$ and $F^{(0)}$ (stationary distribution of the standard queue), where $F^{(\gamma)}$ denotes the stationary distribution of an M/G/1 queue with server speed $1 - \gamma$.

Ruin probability



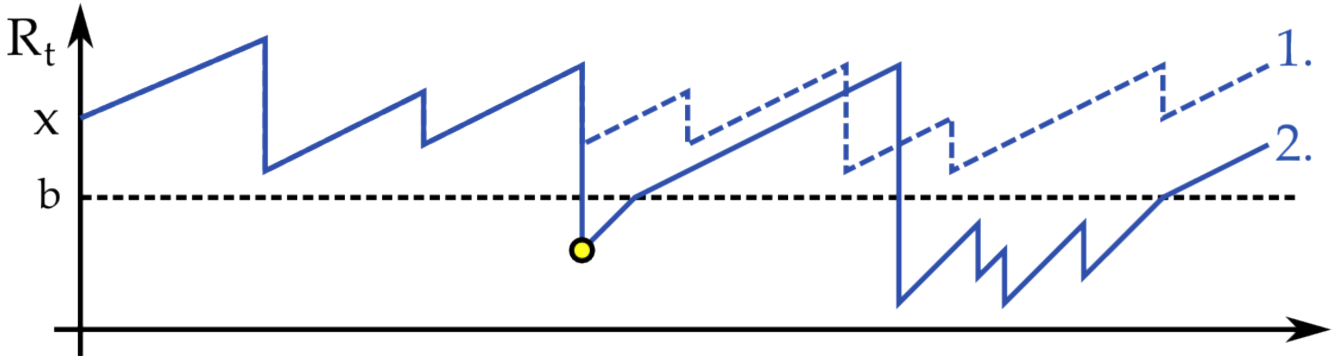
(A) For $x < b$:

$$\overline{\psi}(x) = \theta(x,b)\overline{\psi}(b),$$

where $\theta(x,b) = \mathbb{P}_x(R_t = b \text{ for some } t < \tau)$.
The same formula holds in the $\gamma = 0$ case, too, hence

$$\theta(x,b) = \frac{F^{(0)}(x)}{F^{(0)}(b)}.$$

by duality I.



(B) For $x \geq b$ we have $\tau = \infty$ if either

1. R_t never reaches b, the probability being

$$\lim_{y \to \infty} \theta(x-b, y) = F^{(\gamma)}(x-b).$$

2. $R_s < b$ for some $s > 0$, but never jumps below 0, the probability being

$$\overline{F}^{(\gamma)}(x-b) \int_0^b \theta(b-u, b)\, dH_{x-b}(u) \psi(b),$$

where $H_{x-b}(u)$ is the p.d.f. of the excess.

We obtain:

$$\overline{\psi}(x) = \begin{cases} \theta(x,b)\overline{\psi}(b) & ; x < b \\ F^{(\gamma)}(x-b) & \\ \quad + \overline{F}^{(\gamma)}(x-b)\overline{\psi}(b) \int_0^b \theta(b-u,b)\, H_{x-b}(du) & ; x \geq b \end{cases}$$

Still to determine:

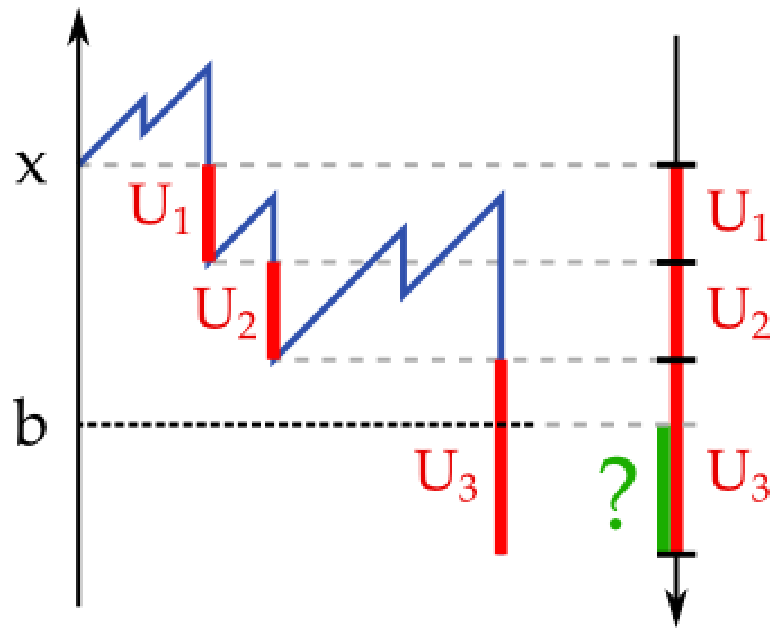
i) excess distribution $H_{x-b}(u)$ and

ii) survival probability $\overline{\psi}(b)$.

i) c.f. Figure.

Excesses $U_1, U_2, \ldots$ form a renewal process.

$H_{x-b}(u)$ is the distribution of the forward recurrence time at time $x-b$ of a renewal process with inter-occurrence time distribution $H_0(u)$.

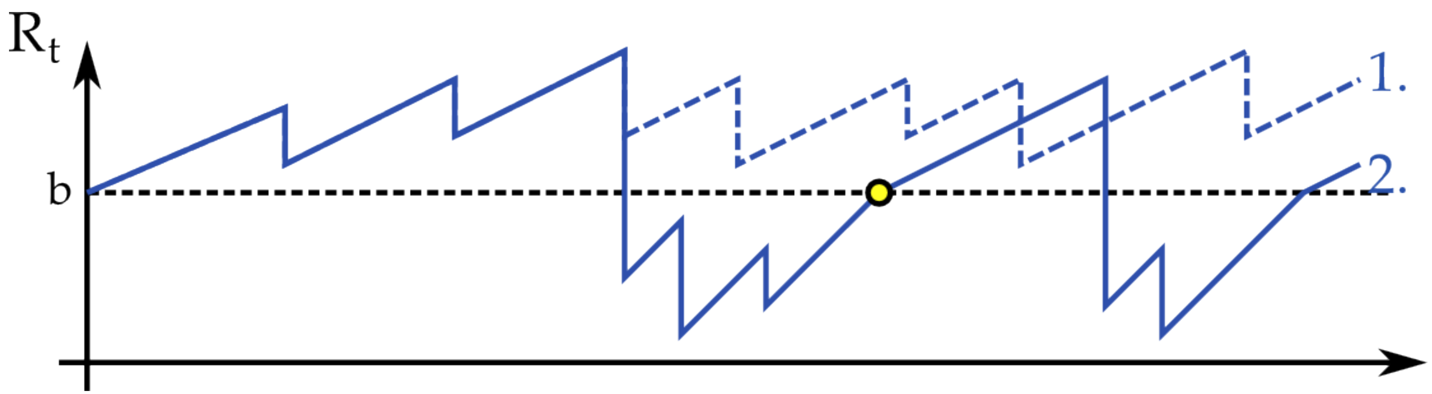


The distribution $H_0(u)$ equals the distribution of the idle period of a transient G/M/1 queue (or deficit at ruin):

$$H_0(u) = \lambda \int_0^u (1 - G(t))\, dt.$$

(e.g. Prabhu (1997) - apparently well known in risk theory: Bowers, Gerber, Hickmann, Jones & Nesbitt (1987))

ii) $\overline{\psi}(b)$



Two possibilites

1. Process never reaches b again, the probability being

$$1 - \rho_\gamma = 1 - \rho/(1-\gamma).$$

(e.g. by duality I: steady state idle probability)

2. Process jumps below b, but $\tau = \infty$, the probability being

$$\rho_\gamma \int_0^b \theta(b-u, b)\, dH_0(u) \overline{\psi}(b).$$

It follows that

$$\overline{\psi}(b) = 1 - \rho_\gamma + \rho_\gamma \int_0^b \theta(b-u, b)\, dH_0(u) \overline{\psi}(b).$$

After rearranging we obtain

$$\overline{\psi}(b) = \frac{1-\rho_\gamma}{1 - \rho_\gamma \int_0^b \theta(b-u,b)\, dH_0(u)},$$

$$\overline{\psi}(b) = \frac{1-\rho_\gamma}{1 - \frac{\rho_\gamma}{F^{(0)}(b)} \int_0^b F^{(0)}(b-u)\, dH_0(u)}.$$

The term

$$\int_0^b F^{(0)}(b-u)\, dH_0(u)$$

looks familiar...

Ruin probability

Cohen (1982): For the G/G/1 queue with

- V_∞=steady-state workload,
- W_∞=steady-state waiting time,
- W_r=residual waiting time,

$$V_\infty | (V_\infty > 0) \overset{d}{\sim} W_\infty + W_r.$$

With Poisson arrivals and PASTA we obtain

$$F^{(0)}(y) = 1 - \rho + \rho \int_0^y F^{(0)}(y-u)\, dH_0(u).$$

(follows also by integration of the well known integro-differential equation for F).

Inserting into equation for $\overline{\psi}(b)$ yields

$$\overline{\psi}(b) = F^{(0)}(b)\frac{1-\rho-\gamma}{1-\rho-\gamma F^{(0)}(b)}. \tag{1}$$

Theorem: *The survival probability is given by*

$$\overline{\psi}(x) = \begin{cases} \theta(x,b)\overline{\psi}(b) & ; x < b \\ F^{(\gamma)}(x-b) & \\ \quad + \overline{F}^{(\gamma)}(x-b)\overline{\psi}(b)\int_0^b \theta(b-u,b)\, H_{x-b}(du) & ; x \geq b \end{cases}$$

with $\overline{\psi}(b)$ *given in* (1) *and* $H_{x-b}(u)$ *the distribution of the forward recurrence time at time* $x-b$ *of a renewal process with renewal distribution* $H_0(u)$.

(c.f. Lin & Pavlova (2006)).

IV. Some general ideas

Case $\mathbb{P}_x(\tau < \infty) = 1$.

Gerber-Shiu discounted penalty function

For the dividend-free risk process ($\gamma = 0$) the functional

$$\Phi_{\alpha,w}(x) = \mathbb{E}_x[e^{-\alpha\tau} w(R_{\tau-}, |R_\tau|)]$$

was introduced by Gerber & Shiu (1998).
On can show that

$$\Phi'_{\alpha,w}(x) = (\mu+\alpha)\Phi_{\alpha,w}(x) - \mu \int_x^\infty w(x, y-x)\, dG(y) - \mu \int_0^x \varphi_\alpha(x-y)\, dG(y).$$

Gerber-Shiu-ism: Extensive literature about solutions.
Approaches available for the usual suspects:

- exponential G,
- Erlang G,
- phase-type G.

Generalization

More generally one could investigate the two functionals

$$\Psi_{v,\beta}(x) = \mathbb{E}_x[\int_0^\tau e^{-\int_0^t v(R_s)\,ds}\beta(R_t)\,dt].$$

and

$$\Phi_{v,w}(x) = \mathbb{E}_x[e^{-\int_0^\tau v(R_s)\,ds}w(R_{\tau-},|R_\tau|)],$$

(β, v, w non-negative and bounded) for a risk process

$$dR_t = r(R_t)\,dt + dS_t$$

with general ("plowback"-)rate $r : \mathbb{R} \to (0,1]$.

Why is

$$\Psi_{v,\beta}(x) = \mathbb{E}_x[\int_0^\tau e^{-\int_0^t v(R_s)\,ds}\beta(R_t)\,dt]$$

useful?

- Expected present value of the discounted dividends:

$$\Psi_{\delta,1-r}(x) = \mathbb{E}_x[\int_0^\tau e^{-\delta t}(1 - r(R_t))\,dt].$$

- Expected value of the total dividends (undiscounted):

$$\Psi_{0,1-r}(x) = \mathbb{E}_x[\int_0^\tau (1 - r(R_t))\,dt].$$

Why is

$$\Phi_{v,w}(x) = \mathbb{E}_x[e^{-\int_0^\tau v(R_s)\,ds} w(R_{\tau-}, |R_\tau|)]$$

useful?

- Gerber-Shiu-functional if $v(x) = \alpha$:

$$\Phi_{\alpha,w}(x) = \mathbb{E}_x[e^{-\alpha\tau} w(R_{\tau-}, |R_\tau|)]$$

- Laplace-transform of the total dividends:

$$\Phi_{\alpha(1-r),1}(x) = \mathbb{E}_x[e^{-\alpha \int_0^\tau (1-r(R_s))\,ds}].$$

- Laplace-transform of the ruin time (Gerber & Shiu (2006))

$$\Phi_{\alpha,1}(x) = \mathbb{E}_x[e^{-\alpha\tau}]$$

Theorem: *Consider the integro-differential equation*

$$\begin{aligned} r(x)S'(x) \\ = (\mu + v(x))S(x) - \mu \int_0^x S(x-u)dG(u) - h(x). \qquad (2) \end{aligned}$$

- *Then* $\Psi_{v,\beta}(x)$ *is a solution of* (2) *with* $h(x) = \beta(x)$ *and*
- $\Phi_{v,w}(x)$ *is a solution of* (2) *with* $h(x) = \mu \int_x^\infty w(x, u-x)\, dG(u)$.

Classic proof: condition on the number of jumps during $[0, \Delta t]$ and let $\Delta t \to 0$.

Here: Approach via PDMPs (c.f. DASSIOS & EMBRECHTS (1989),EMBRECHTS & SCHMIDLI (1994))

The first functional

"Proof" for $\Psi_{v,\beta}(x) = \mathbb{E}_x[\int_0^\tau e^{-\int_0^t v(R_s)\,ds}\beta(R_t)\,dt]$:
Rewrite the equation

$$r(x)S'(x) = (\mu + v(x))S(x) - \mu \int_0^x S(x-u)dG(u) - \beta(x)$$

into $\mathscr{G}S(x) = v(x)S(x) - \beta(x)$, with generator $\mathscr{G}$ of the killed Markov process R_t:

$$\mathscr{G}f(x) = r(x)f'(x) + \mu \int_0^x (f(x-y) - f(x))\,dG(y) \\ - \mu f(x)(1 - G(x)).$$

Use the martingale

$$e^{-\int_0^t v(R_s)\,ds}S(R_t) - \int_0^t e^{-\int_0^s v(R_u)\,du}\big(\mathscr{G}S(R_s) - v(R_s)S(R_s)\big)\,ds.$$

+ optional stopping with $S(\Delta) = 0$ (Δ the cemetery state).

The second functional

"Proof" for $\Phi_{v,w}(x) = \mathbb{E}_x[e^{-\int_0^\tau v(R_s)\,ds} w(R_{\tau-}, |R_\tau|)]$:
Rewrite

$$r(x)S'(x) = (\mu + v(x))S(x) - \mu \int_0^x S(x-u)dG(u) - \mu \int_x^\infty w(x, u-x)\,dG(u).$$

into eigenfunction equation $\mathscr{G}S(x) = v(x)S(x)$, where

$$\mathscr{G}f(x) = r(x)f'(x) + \mu \int_0^x (f(x-u) - f(x))dG(u) + \mu \int_x^\infty (w(x, u-x) - f(x))\,dG(u).$$

(! uncountable number of outer states)
Finally use the same martingale as before:

$$e^{-\int_0^t v(R_s)\,ds} S(R_t).$$

Drawback: solutions of

$$r(x)S'(x) = (\mu + v(x))S(x) - \mu \int_0^x S(x-u)dG(u) - h(x)$$

are difficult to find.

- Let $\theta(x) = \exp\left(\int_0^x \frac{\mu+f(z)}{r(z)}dz\right)$. Then

$$S(x) = \theta(x)\Big(S(0) - \int_0^x \frac{S^*(w)}{\theta(w)}, dw\Big)$$

  and S^* solves the Volterra integral equation

$$S^*(x) = b(x) - \int_0^x K(w,x)S^*(w)\,dw$$

  with certain functions K and b.

- If G is absolutely continuous having a density g with

$$g(x-u) = \sum_{k=0}^{m} A_k(x) B_k(y),$$

(e.g. exponential, Erlang, hyper-exponential distributions) then one can rewrite

$$r(x)S'(x) = (\mu + v(x))S(x) - \mu \int_0^x S(x-u)dG(u) - h(x)$$

into a system of first order linear differential equations.

Observation: Two functionals

$$\Psi_{v,\beta}(x) = \mathbb{E}_x[\int_0^\tau e^{-\int_0^t v(R_s)\,ds}\beta(R_t)\,dt].$$

$$\Phi_{v,w}(x) = \mathbb{E}_x[e^{-\int_0^\tau v(R_s)\,ds}w(R_{\tau-},|R_\tau|)],$$

solve the same equation

$$r(x)S'(x) = (\mu + v(x))S(x) - \mu\int_0^x S(x-u)dG(u) - h(x).$$

with $h(x) = \beta(x)$ and $h(x) = \mu\int_x^\infty w(x,u-x)\,dG(u)$.

It is tempting to equate $\beta(x)$ and $\mu\int_x^\infty w(x,u-x)\,dG(u)$:

$$\Phi_{v,w}(x) = \mu\mathbb{E}_x[\int_0^\tau e^{-\int_0^t v(R_s)\,ds}\int_{R_t}^\infty w(R_s,u-R_s)\,dG(u)\,dt].$$

V. The ruin time

Back to the threshold paper

Laplace transform of the ruin time:

$$\varphi_\alpha(x) = \mathbb{E}_x[e^{-\alpha\tau}\mathbf{1}(\tau < \infty)]$$

We have seen that (see Gerber & Shiu (2006))

$$\begin{aligned}(1-\gamma\mathbf{1}(x \geq b))\varphi'_\alpha(x) \\ = (\mu+\alpha)\varphi_\alpha(x) - \mu\overline{G}(x) - \mu\int_0^x \varphi_\alpha(x-y)\,dG(y) \quad (3)\end{aligned}$$

Connection to Queueing theory: We can write

$$\varphi_\alpha(x) = \mathbb{P}_0(V_T > x)$$

with $T \overset{d}{\sim} \exp(\alpha)$ and V_t the workload process of $M/G/1$. Then (3) follows from a result in Gaver & Miller (1962) .

Equation (3):

$$(1-\gamma\mathbf{1}(x \geq b))\varphi'_\alpha(x) = (\mu+\alpha)\varphi_\alpha(x) - \mu\overline{G}(x) - \mu\int_0^x \varphi_\alpha(x-y)\,dG(y).$$

In general no hope for explicit solutions (several results in Lin & Pavlova (2006).)

Our suggestion: define (double-)transforms

$$\Psi_\alpha^-(s) = \int_0^\infty e^{-sx}\varphi_\alpha^-(x)\,dx,$$
$$\Psi_\alpha^+(s) = \int_b^\infty e^{-sx}\varphi_\alpha(x)\,dx,$$

where $\varphi_\alpha^-(x)$ is a solution on $[0,\infty)$ of (3) with $\gamma = 0$.

Then

$$\Psi_\alpha^-(s) = \frac{\varphi_\alpha^-(0) - \mu\frac{1-G^*(s)}{s}}{s - \mu(1 - G^*(s)) - \alpha}$$

Inversion of $\Psi_\alpha^-(s)$ yields $\varphi_\alpha(x)$ for $x < b$.
Moreover,

$$\Psi_\alpha^+(s) = \frac{(1-\gamma)\varphi_\alpha(b) - \mu\int_b^\infty \overline{G}(x)e^{-sx}\,dx - \mu W(s,x)}{s(1-\gamma-\alpha-\mu(1-G^*(s)))},$$

with

$$W(s,x) = \int_0^b \Big(G^*(s)\varphi_\alpha(x) - \int_0^x \varphi_\alpha(x-u)\,dG(u)\Big)e^{-sx}\,dx.$$

Thank you!