Approximations of the Laplace Transform of a Lognormal Random Variable

Leonardo Rojas Nandayapa

Joint work with

Søren Asmussen & Jens Ledet Jensen

The University of Queensland
School of Mathematics and Physics

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Outline

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Sums of Lognormal random variables

Due to the popularity of the Lognormal random variables sums of lognormal appear in a wide variety of applications

1. **Finance** Stock prices are modeled as lognormals. Sums of lognormals arise in portfolio and option pricing.
2. **Insurance** Individual claims are also modeled lognormal: Total claim amount is a sum of lognormals.
3. **Engineering.** Sums of lognormals arise in a large amount of applications. Most prominent in telecommunications.
4. **Biology, Geology,. . .**

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Sums of Lognormal random variables

Since the distribution of the sum of lognormals is not available a large number of numerical and approximative methods have been developed.

1. **Approximating distributions.** A popular approach is using another lognormal distribution. More recently Pearson Type IV, left skew normal, log-shifted gamma, power lognormal distributions have been used.
2. **Transforms Inversion.**
3. **Bounds.**
4. **Monte Carlo methods.**

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Sums of Lognormal random variables

However, most of these methods have drawbacks:

1. Inaccuracies in certain regions. Lower regions and upper tail.
2. Poor approximations for large/low number of summands. Same for extreme parameters.
3. Difficulties arising from non-identically distributed.
4. Complicated methods.

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Laplace Transform

We denote the Laplace transform of a density f

$$\mathcal{L}_f(\theta) = \int_0^\infty \mathrm{e}^{-\theta x} f(x) dx = \mathbb{E}\left[\mathrm{e}^{-\theta X}\right].$$

where the domain of convergence of the transform is

$$\Theta = \{\theta \in \mathbb{R} : \theta \geq 0\}.$$

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Some applications of Laplace transforms

- **Cumulative Distribution Functions:** It follows that the Laplace transform of its cdf F is

$$\mathcal{L}_F(\theta) = \frac{\mathcal{L}_f(\theta)}{\theta}, \qquad \theta > 0.$$

Thus we can compute probabilities by using any of the numerical inversion methods available in the literature.

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Common applications of Laplace transforms

Example (Bromwich inversion integral)

If F is supported over $[0, \infty]$ with no atoms then

$$F(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{e}^{x\theta} \mathcal{L}_F(\theta) d\theta, \qquad \gamma > 0.$$

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Common applications of Laplace transforms

- **Sums of Independent Random Variables:** Let $X_1, \ldots, X_n$ be independent random variables with pdf's $f_1, \ldots, f_n$ and let F be the cdf of $S_n := X_1 + \cdots + X_n$. Then

$$\mathcal{L}_F(\theta) = \frac{\prod_{i=1}^{n} \mathcal{L}_{f_i}(\theta)}{\theta^n}, \qquad \theta > 0.$$

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Common applications of Laplace transforms

- **Exponential families generated by a random variable**
  Let X be a random variable with distribution F. The family of distributions defined by

$$dF_\theta(x) = \frac{\mathrm{e}^{-\theta x} dF(x)}{\mathcal{L}_f(\theta)}, \qquad \theta \in \Theta.$$

  is known as the *exponential family of distributions generated by X*.

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Common applications of Laplace transforms

Example

In some applications (saddlepoint approximation and rare-event simulation for example) it is often required to find the solution θ to the equation

$$\mathbb{E}_{\theta}[X] = y, \qquad y \text{ fixed.}$$

Here $\mathbb{E}_{\theta}$ is the expectation w.r.t. F_{θ}.

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Laplace transform of a Lognormal

No closed form of the Laplace transform of a Lognormal random variable is known

$$\mathcal{L}_f(\theta) = \int_0^\infty \frac{1}{x\sqrt{2\pi}\sigma} \exp\left\{ -\theta x - \frac{(\log x - \mu)^2}{2\sigma^2} \right\} dx$$

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Intuitive approach

We consider for $k = 0, 1, 2, \dots$

$$\mathbb{E}\left[X^k \mathrm{e}^{-\theta X}\right] = \int_0^\infty \frac{x^{k-1}}{\sigma\sqrt{2\pi}} \exp\left\{ -\theta x - \frac{(\log x - \mu)^2}{2\sigma^2} \right\} dx$$

$$= \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\theta \mathrm{e}^y + ky - \frac{(y - \mu)^2}{2\sigma^2} \right\} dy.$$

The change of variable $y = \log x$ was used here.

Intuitive approach

The Laplace method suggest to replace the expression

$$-\theta e^{y} + ky - \frac{(y-\mu)^2}{2\sigma^2} \tag{1}$$

by a Taylor approximation of second order around the value ρ_k that maximizes this expression. That is

$$-\theta e^{\rho_k}\left[1 + (y-\rho_k) + \frac{(y-\rho_k)^2}{2}\right] + ky - \frac{(y-\mu)^2}{2\sigma^2}. \tag{2}$$

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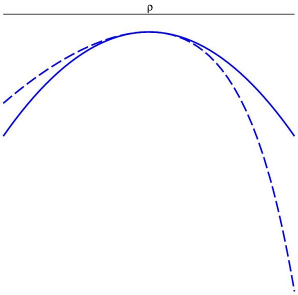
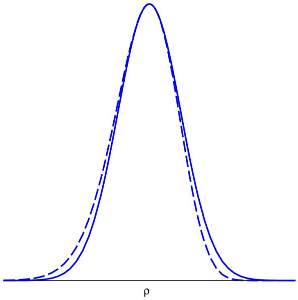
Intuitive approach

The figures illustrate the idea





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Intuitive approach

Moreover, the method works because the resulting integral can be explicitly obtained.

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left\{-\theta e^{\rho_k}\left[1+(y-\rho_k)+\frac{(y-\rho_k)^2}{2}\right]+ky-\frac{(y-\mu)^2}{2\sigma^2}\right\}dy.$$

(Notice that the expression in the brackets is simply a second order polynomial).

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Intuitive approach

The novelty in the lognormal case is the explicit calculation of the value

$$\rho_k \;=\; -\mathrm{LW}(\theta\sigma^2 \mathrm{e}^{k\sigma^2+\mu}) + k\sigma^2 + \mu,$$

where the function $\mathrm{LW} : [-\mathrm{e}^{-1}, \infty) \to \mathbb{R}$, known as the LambertW, is the inverse of

$$f(W) = W\mathrm{e}^{W}.$$

Intuitive approach

Moreover, the property

$$\mathrm{LW}(x)\,\mathrm{e}^{\mathrm{LW}(x)} = x, \qquad x \in \mathbb{C}.$$

is useful to prove that $\mathbb{E}\left[X^k \mathrm{e}^{-\theta X}\right]$ can be approximated with

$$\frac{1}{\sqrt{\mathrm{LW}(\theta\sigma^2 \mathrm{e}^{k\sigma^2+\mu})+1}} \times \exp\left\{-\frac{\mathrm{LW}^2(\theta\sigma^2\mathrm{e}^{k\sigma^2+\mu}) + 2\,\mathrm{LW}(\theta\sigma^2\mathrm{e}^{k\sigma^2+\mu}) - 2k\sigma^2\mu - k^2\sigma^4}{2\sigma^2}\right\}$$

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Laplace transform

In particular, with $k = 0$

$$\mathcal{L}_f(\theta) \approx \frac{1}{\sqrt{\mathrm{LW}(\theta\sigma^2 \mathrm{e}^{\mu}) + 1}} \exp\left\{ -\frac{\mathrm{LW}^2(\theta\sigma^2\mathrm{e}^{\mu}) + 2\,\mathrm{LW}(\theta\sigma^2\mathrm{e}^{\mu})}{2\sigma^2} \right\}$$

We will use the notation $\widetilde{\mathcal{L}_f}(\theta)$ for this approximation.

Outline

1. Introduction
   - Sums of Lognormal Random Variables
   - Laplace transforms in probability
2. Approximations of the Laplace transform
   - The Laplace method
   - The exponential family generated by a Lognormal
3. Applications
   - Cdf of a sum of lognormal via inversion
   - Tail probabilities and rare-event simulation

Exponential Family

Then we can approximate the solution of

$$\mathbb{E}_{\theta}[X] = y.$$

where $\mathbb{E}_{\theta}[X]$ is the expectation w.r.t.

$$dF_{\theta}(x) = \frac{\mathrm{e}^{-\theta x} dF(x)}{\mathcal{L}_f(\theta)}.$$

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Exponential family

Following an analogous procedure we arrive at

$$F_\theta(x) = \int_0^x \frac{e^{-\theta y}}{\mathcal{L}_f(\theta)} dF(y)$$
$$\approx \frac{\widetilde{\mathcal{L}_f}(\theta)}{\mathcal{L}_f(\theta)} \int_0^x \frac{1}{y\sqrt{2\pi}\sigma} \exp\left\{ -\frac{\left(\log y - \mu_\theta\right)^2}{2\sigma_\theta^2} \right\} dy.$$

where

$$\mu_\theta := \mu - \mathrm{LW}(\theta\sigma^2 e^\mu), \qquad \sigma_\theta^2 := \frac{\sigma^2}{1 + \mathrm{LW}(\theta\sigma^2 e^\mu)}.$$

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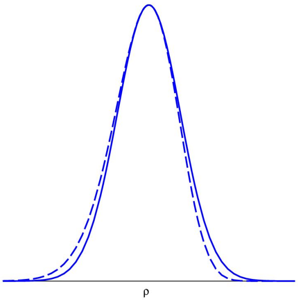
Exponential Family

That is

$$F_\theta(x) \approx G(x), \qquad G \sim \mathrm{LN}(\mu_\theta, \sigma_\theta^2)$$

The exponential family can be approximated with a lognormal distribution.



Exponential family

This result enable us to approximate $\mathbb{E}_\theta[X]$ with the expected value of a lognormal

$$\mathbb{E}_\theta[X] \approx e^{\mu_\theta + \sigma_\theta^2/2}.$$

Moreover, the solution of $\mathbb{E}_\theta[X] = y$ for θ is given by

$$\theta = \frac{\gamma e^\gamma}{\sigma^2 e^\mu}, \qquad \gamma := \frac{-1 + \mu - \log y + \sqrt{(1 - \mu - \log y)^2 + 2\sigma^2}}{2}.$$

Final notes

In the paper version we obtain an expression of the remainder, i.e.

$$\mathcal{L}_f(\theta) = \widetilde{\mathcal{L}_f(\theta)}(1 + \mathcal{R}(\theta)),$$

where $\mathcal{R}(\theta)$ is a series. Using higher order terms we can sharp the results above.

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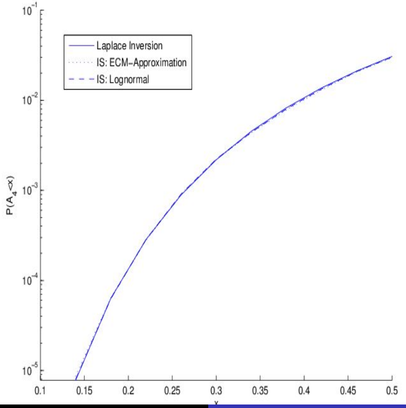
Sums of Lognormals

The most obvious application is to approximate the cdf of a sum of independent lognormals. Figure below shows a numerical comparison with simulation results



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Left tail

Commonly, approximations available are inaccurate in lower regions of the cdf.

$$\mathbb{P}(X_1 + \cdots + X_n < ny), \qquad y \to 0.$$

When the X_i's are i.i.d. an importance sampling algorithm with exponential change of measure can be implemented. In fact, we prove that if θ is such that $\mathbb{E}_\theta[X] = y$ then this algorithm is strongly efficient as $y \to 0$.

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Left tail

Moreover, if $y < \mathbb{E}[X]$ and we want to estimate

$$\mathbb{P}(X_1 + \cdots + X_n < ny), \qquad n \to \infty.$$

The same importance sampling algorithm is efficient as $n \to \infty$.

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Future Work

- A numerical analysis of available methods in the literature.
- A Monte Carlo method for the sum of lognormals which is efficient
  - Across the whole support of the distribution
  - In the case when $n \to \infty$.
- A valid saddlepoint approximation for the lower region.
- Extend the results to the non-independent case.