Distributions with rational transforms

Joint work with Mogens Bladt

New Frontiers in Applied Probability

honouring Søren Asmussen

Sandbjerg Castle 2/8 2011

Outline

- Moment recursions
- Kulkarni's multivariate phase–type distributions (MPH^*)
- Multivariate definition and main theorem (MVME)
- Some constructions
- Classification of multivariate gamma distributions
- Distributions on the reals: uni- and multivariate
- The killing of a conjecture
- Further work

Laplace transform from moments

- An m–dimensional ME distribution is uniquely determined from its first $2m-1$ moments.
- Solve for f_i and g_i in the first $2m-1$ moment equations.
- g_i obtained from certain Hankel matrices.

$$\mu_i = \frac{M_i}{i!} \qquad \mu_i = \sum_{j=0}^{i} (-1)^j f_{m-j} \psi_{i-j} \quad i = 0, 1, \ldots \quad ,$$

where $\psi_0 = \frac{1}{g_m}$ and $\psi_i = \sum_{j=0}^{i-1} (-1)^j \frac{\psi_{i-1-j} g_{m-1-j}}{g_m}$

- Moments of higher order are given recursively by

$$\mu_{m+j} = \sum_{i=0}^{m-1} g_i (-1)^{m+i} \mu_{i+j} \text{ for } j \geq 0.$$

Kulkarni's multivariate phase–type distributions – MPH*

- n different reward rates for each state of T given by R

$$X_j = \sum_{i=1}^{m} \sum_{k=1}^{N_k} R_{ij} Z_{ik} \quad .$$

  - ◇ Here N_k is the number of visits to state k before absorption and Z_{ik} are the k'th sojourn in state j

- Partial differential equations for joint survival function.
- Joint Laplace-Stieltjes transform

$$H(\boldsymbol{s}) = \boldsymbol{\gamma}\left((-T)^{-1} \Delta\left(R\boldsymbol{s}\right) + I \right)^{-1} \boldsymbol{e}.$$

- Includes previous work by Assaf defining the class MPH.

Joint transform and moments

Theorem 1 *The cross–moments* $\mathbb{E}\left(\prod_{i=1}^{n} Y_i^{r_i}\right)$*, where* $\boldsymbol{Y}$ *follows an MME* distribution with representation* $(\boldsymbol{\gamma}, T, R)$*, and where* $r_i \in \mathbb{N}$*, are given by*

$$\boldsymbol{\gamma} \sum_{\ell=1}^{r!} \prod_{i=1}^{r} (-T)^{-1} \Delta(\boldsymbol{r}_{\sigma_\ell(i)}) \boldsymbol{e}.$$

Here $r = \sum_{i=1}^{n} r_i$*,* $\boldsymbol{r}_j$ *is the* j*th column of* R *and* σ_ℓ *is one of the* $r!$ *possible ordered permutations of the derivatives, with* $\sigma_\ell(i)$ *being the value among* $1 \ldots n$ *at the* i*'th position of that permutation.*

General definition of multivariate matrix exponential distributions

Definition 1 *A non–negative random vector* $\boldsymbol{X} = (X_1, ..., X_n)$ *of dimension* n *is said to have multivariate matrix–exponential distribution (MVME) if the joint Laplace transform* $L(\boldsymbol{s}) = \mathbb{E}\left[\exp(- < \boldsymbol{X}, \boldsymbol{s} >)\right]$ *is a multi–dimensional rational function, that is, a fraction between two multi–dimensional polynomials. Here* $< \cdot, \cdot >$ *denotes the inner product in* $\mathbb{R}^n$ *with* $\boldsymbol{s} = (s_1, \ldots, s_n)'$.

Our main theorem characterizes the class of MVME.

Theorem 2 *A vector* $\boldsymbol{X} = (X_1, \ldots, X_n)$ *follows a multivariate matrix–exponential distribution if and only if* $< \boldsymbol{X}, \boldsymbol{a} >= \sum_{i=1}^{n} a_i X_i$ *has a univariate matrix–exponential distribution for all non–negative vectors* $\boldsymbol{a} \neq \mathbf{0}$.

Outline of proof

- Only if part: Suppose $\mathbb{E}\left(e^{-<\boldsymbol{X},\boldsymbol{s}>}\right)$ is rational in $\boldsymbol{s}$. Then consider $\mathbb{E}\left(e^{-s<\boldsymbol{X},\boldsymbol{a}>}\right) = \mathbb{E}\left(e^{-<\boldsymbol{X},s\boldsymbol{a}>}\right)$ that is obviously rational in s.
- If part: Suppose $< \boldsymbol{X}, \boldsymbol{a} >$ has ME representation $(\boldsymbol{\beta}(\boldsymbol{a}), D(\boldsymbol{a}), \boldsymbol{d}(\boldsymbol{a}))$ for all $\boldsymbol{a} > \boldsymbol{0}$.
  - ◇ The dimension of D is bounded by some integer m.
  - ◇ Using the moment relations we express the coefficients $f_i(\boldsymbol{a})$ and $g_i(\boldsymbol{a})$ of the Laplace transform in terms of certain determinants of the moments.
  - ◇ The jth moment is a sum of jth order monomials in the components of $\boldsymbol{a}$.
  - ◇ We conclude that f_i and g_i are rational in $\boldsymbol{a}$.

The transform is of a particular simple form

Lemma 1 *If* $\langle \boldsymbol{X}, \boldsymbol{a} \rangle$ *is MVME distributed then we may write its Laplace transform for* $\langle \boldsymbol{X}, \boldsymbol{a} \rangle$ *as*

$$\frac{\tilde{f}_1(\boldsymbol{a})s^{m-1} + \tilde{f}_2(\boldsymbol{a})s^{m-2} + ... + \tilde{f}_{m-1}(\boldsymbol{a})s + 1}{\tilde{g}_0(\boldsymbol{a})s^m + \tilde{g}_1(\boldsymbol{a})s^{m-1} + ... + \tilde{g}_{m-1}(\boldsymbol{a})s + 1},$$

where the terms $\tilde{f}_i(\boldsymbol{a})$ *and* $\tilde{g}_i(\boldsymbol{a})$ *are sums of* n*–dimensional monomials in* $\boldsymbol{a}$ *of degree* $m - i$ *and* m *is the common order except a set of measure zero.*

Farlie Gumbel Morgenstern construction

Consider

$$F(x_1, x_2) = F_1(x_1)F_2(x_2)\left(1 + \rho\left(1 - F_1(x_1)\right)\left(1 - F_2(x_2)\right)\right) \quad ,$$

where F_i are univariate cumulative distribution functions. This expression can be rewritten as

$$\begin{aligned} F(x_1, x_2) \quad &= \quad \frac{1+\rho}{4} F_{1,M}(x_1)F_{2,M}(x_2) + \frac{1-\rho}{4} F_{1,M}(x_1)F_{2,m}(x_2) \\ &+ \quad \frac{1-\rho}{4} F_{1,m}(x_1)F_{2,M}(x_2) + \frac{1+\rho}{4} F_{1,m}(x_1)F_{2,m}(x_2) \quad , \end{aligned}$$

where $F_{i,m}(x) = 1 - (1 - F_i(x))^2$ and $F_{i,M}(x) = F_i^2(x)$ i.e. the distribution of minimum and maximum respectively of two F_i distributed independent random variables.

Theorem 3 *The bivariate Farlie-Gumbel-Morgenstern distribution formed from two matrix-exponential distributions is in MME*. An MME* representation is*

$$\left(\boldsymbol{\gamma}_1 \otimes \boldsymbol{\gamma}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}\right)$$

$$\begin{bmatrix} S_1 \oplus S_1 & \frac{1}{2}\left(\boldsymbol{s}_1 \oplus \boldsymbol{s}_1\right) & \frac{1-\rho}{4}\left(\boldsymbol{s}_1 \oplus \boldsymbol{s}_1\right)\boldsymbol{e}\tilde{\boldsymbol{\gamma}}_{2,M} & \frac{1+\rho}{4}\left(\boldsymbol{s}_1 \oplus \boldsymbol{s}_1\right)\boldsymbol{e}\tilde{\boldsymbol{\gamma}}_{2,m} \\ 0 & S_1 & \frac{1+\rho}{2}\boldsymbol{s}_1\tilde{\boldsymbol{\gamma}}_{2,M} & \frac{1-\rho}{2}\boldsymbol{s}_1\tilde{\boldsymbol{\gamma}}_{2,m} \\ 0 & 0 & \Delta_{2,M}^{-1} S_2' \Delta_{2,M} & \Delta_{2,M}^{-1}(\boldsymbol{s}_2 \oplus \boldsymbol{s}_2')\Delta_{2,m} \\ 0 & 0 & 0 & \tilde{S}_{2,m} \end{bmatrix}$$

with

$$\boldsymbol{\pi}_2 = \mu_2^{-1}\boldsymbol{\alpha}_2\left(-S_2\right)^{-1}, \quad \tilde{\boldsymbol{\alpha}}_2 = \mu_2^{-1}\boldsymbol{\pi}_2 \circ \boldsymbol{s}_2,$$

$$\boldsymbol{\pi}_{2,m} = \mu_{2,m}^{-1}\left(\boldsymbol{\alpha}_2 \otimes \boldsymbol{\alpha}_2\right)\left(-S_2 \oplus S_2\right)^{-1}, \;\; \boldsymbol{\pi}_{2,M} = \left(\frac{\mu_{2,m}}{\mu_{2,M}}\boldsymbol{\pi}_{2,m}, 1 - \frac{\mu_{2,m}}{\mu_{2,M}}\boldsymbol{\pi}_2\right),$$

$$\tilde{\boldsymbol{\alpha}}_{2,m} = \left(\mu_{2,m}\right)^{-1}\boldsymbol{\pi}_2^{(m)} \circ (\boldsymbol{s}_2 \oplus \boldsymbol{s}_2), \qquad \tilde{\boldsymbol{\alpha}}_{2,M} = \left(\mu_{2,M}\right)^{-1}\left(\mathbf{0}, \boldsymbol{\pi}_{2,M} \circ \boldsymbol{s}_2\right)$$

$\frac{1}{\mu} f(\boldsymbol{x})$

- Suppose $f(x)$ is (univariate) ME
- Then $f(\boldsymbol{x})$ is (proportional to) an MME* density
- For $n = 2$ we get

$$\left(\left(\frac{\boldsymbol{\alpha}(-C)^{-1}}{\mu}, \boldsymbol{0} \right), \begin{bmatrix} C & -C \\ 0 & C \end{bmatrix}, \begin{bmatrix} \boldsymbol{e} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e} \end{bmatrix} \right)$$

- Not always the most interesting representation
- Joint distribution of age and residual life time in equilibrium renewal process. Closely related to size–biased distributions
- The result can be generalized to apply for the nth order moment distributions, but we have no probabilistic interpretation at this point.

Bi and multivariate exponentials and gammas

- A multitude of various definitions
- Most of these have rational joint Laplace transform for integer shape parameter
- Many of these are in MPH and most are in MPH^*
- The MME^* provides a framework for categorization

Moran and Downton's Bivariate Exponential

The MME* representation of this distribution is

$\boldsymbol{\gamma}(\boldsymbol{a}) = (\alpha_1, \alpha_2)$

$$T = \begin{bmatrix} -\lambda_1 & \lambda_1(1-p_1) \\ \lambda_2(1-p_2) & -\lambda_2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

$$f(x_1, x_2) = \lambda_1 \lambda_2 p_2 e^{-(\lambda_1 x_1 + \lambda_2 x_2)} \sum_{i=1}^{\infty} \frac{(\lambda_1(1-p_1)x_1\lambda_2(1-p_2)x_2)^{i-1}}{((i-1)!)^2} .$$

with (slightly more general) Laplace transform

$$\frac{(\alpha_1 s_2 \lambda_1 p_1 \lambda_2 + \alpha_2 s_1 \lambda_1 \lambda_2 p_2) + \lambda_1\lambda_2(1-(1-p_1)(1-p_2))}{s_1 s_2 + (s_2\lambda_1 + s_1\lambda_2) + \lambda_1\lambda_2(1-(1-p_1)(1-p_2))} .$$

Cheriyan-Ramabhadran's Bivariate Gamma

With MME* representation $\boldsymbol{\gamma} = (1, 0, \ldots, 0)$, the matrix T is an $(m_0 + m_1 + m_2) \times (m_0 + m_1 + m_2)$ matrix of Erlang structure

$$T = \begin{bmatrix} -\lambda & \lambda & \ldots & 0 \\ 0 & -\lambda & \ldots & 0 \\ \vdots & \vdots & \vdots\vdots & \vdots \\ 0 & 0 & \ldots & -\lambda \end{bmatrix}, \qquad R = \begin{bmatrix} \boldsymbol{e}_{m_0} & \boldsymbol{e}_{m_0} \\ \boldsymbol{e}_{m_1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{e}_{m_2} \end{bmatrix} .$$

The density is given by $f(x_1, x_2) =$

$$\frac{e^{-x_1-x_2}}{(m_0-1)!(m_1-1)!(m_2-1)!} \int_0^{\min(x_1,x_2)} x^{m_0-1}(x_1-x)^{m_1-1}(x_2-x)^{m_2-1}e^x \mathrm{d}x$$

Dussauchoy-Berland's bivariate gamma

$\boldsymbol{\gamma} = (1, 0, 0, 0)$ and

$$T = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & \lambda_1 \left(1 - \frac{\lambda_2}{\rho\lambda_1}\right)^2 & 2\rho\lambda_2 \left(1 - \frac{\lambda_2}{\rho\lambda_1}\right) \\ 0 & 0 & -\lambda_2 & \lambda_2 \\ 0 & 0 & 0 & -\lambda_2 \end{bmatrix}, \quad R = \begin{bmatrix} \rho & 1 \\ \rho & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

- $X_1 - \rho X_2$ and X_2 are independent with LST

$$\left(\frac{\lambda_1 + \rho s_1}{\lambda_1 + \rho s_1 + s_2}\right)^{l_1} \left(\frac{\lambda_2}{\lambda_2 + s_1}\right)^{l_2},$$

in MME* for positive integer values of l_1 and l_2. An MME* representation, (even in MPH) for $l_1 = l_2 = 2$ and $\rho\lambda_1 \geq \lambda_2$ is

Bivariate exponentials with arbitray correlations

- Can be seen as a generalization of Farlie–Gumbel–Morgenstern distributions.
- Mixtures of combinations of order statistics
- A distribution can be seen as the average the distribution of its order statistics
- Eksplicit form of generator

$$\begin{bmatrix} -2\lambda & \lambda & p_{11}\lambda & p_{12}\lambda \\ 0 & -\lambda & p_{21}\lambda & p_{22}\lambda \\ 0 & 0 & -\mu & \mu \\ 0 & 0 & 0 & -2\mu \end{bmatrix}$$

Joint density of the bivariate exponential

Theorem 4 *The joint density for* $\mathbf{Y}^{(n)} = \left(Y_1^{(n)}, Y_2^{(n)}\right)$ *is given by*

$$f(y_1, y_2) = \sum_{\ell=1}^{n} \sum_{k=1}^{n} c_{\ell k} \ell \lambda e^{-\ell \lambda y_1} k \mu e^{-k \mu y_2},$$

with

$$\begin{aligned} c_{\ell k} &= \frac{(-1)^{\ell+k-(n+1)}}{n} \begin{pmatrix} n \\ \ell \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix} \\ &\quad \cdot \sum_{i=n+1-\ell}^{n} \sum_{j=1}^{k} p_{ij} (-1)^{-i-j} \begin{pmatrix} \ell - 1 \\ n - i \end{pmatrix} \begin{pmatrix} k - 1 \\ k - j \end{pmatrix} . \end{aligned}$$

Krishnamoorthy and Parthasarathy's multivariate exponential

- $H(\boldsymbol{s}) = |I + P\Delta(\boldsymbol{s})|^{-1}$. For $n = 3$ we have with

$$P = \begin{bmatrix} 1 & \rho & \tau \\ \rho & 1 & \eta \\ \tau & \eta & 1 \end{bmatrix}, \qquad H(s) = \frac{1}{s^3 g_0^* + s^2 g_1^* + s g_2^* + 1},$$

where

$$\begin{aligned} g_0^* &= a_1 a_2 a_3 (1 + 2\rho\tau\eta - \rho^2 - \tau^2 - \eta^2) \\ g_1^* &= (a_1 a_2 (1 - \rho^2) + a_1 a_3 (1 - \tau^2) + a_2 a_3 (1 - \eta^2)) \\ g_2^* &= (a_1 + a_2 + a_3) \end{aligned}$$

- Only in MME*(3) when $\tau = \rho\eta$, $\rho = \tau\eta$, or $\eta = \rho\tau$

Rational moment generating functions (distributions on the reals)

- The characterization result generalizes directly giving rise to the class of BMVME distributions
- Ahn and Ramaswami - bilateral phase-type distributions - an MPH* construction with general rewards but just one variable.
  - ◇ Explicit representation of the two sided distribution
- Asmussen - like Ahn and Ramaswami but with a state dependent diffusion term.
  - ◇ Explicit representation of the two sided distribution - i.e. also the diffusion can be written on the MPH* form.

Generalization of Asmussens result on a univariate diffusion

Let $\boldsymbol{Y} = (Y_1, \ldots, Y_\ell) \sim \mathsf{MME}^*(\boldsymbol{\alpha}, T, R)$, where T is of dimension m. Now consider a multidimensional vector $\boldsymbol{X} = (X_1, \ldots, X_k)$ such that

$$X_j = \sum_{i=1}^{\ell} B_{ij}, \quad j = 1, \ldots, k$$

where $\mathbf{B}_i = (B_{i1}, \ldots, B_{ik}) \sim \mathsf{N}_k(Y_i \mathbf{r}(i), Y_i \Sigma(i))$, with $\mathbf{r}(i) = (r_1(i), \ldots, r_k(i))$ and $\Sigma(i)$ is a covariance matrix, $i = 1, \ldots, \ell$. Then $\boldsymbol{X}$ has a rational (multi-dimensional) moment-generating function, i.e. $\boldsymbol{X}$ belongs to the class of Bilateral Multivariate Matrix-Exponential distributions (BMVME).

Two independent Brownian motions observed at the same (exponential) time

- With both diffusion parameters being $\sqrt{2}$ and the exponential parameter being one, the moment generating function is

$$\frac{1}{1 - s_1^2 - s_2^2}$$

- Which cannot be expressed in the MPH* form.

Further work

- Estimation
- Numerical evaluation
- Statistical estimation, fitting, tests?
- When is an MME* representation a distribution?
- Understanding the general case better
- Extension of $f(\boldsymbol{x})$ results.
- Further analytical results - extensions?
- Applications in Computer Science, Transportation Science, possibly Hydrology, and other fields