On growth collapse processes with stationary structure and their shot-noise counterparts

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The Hebrew University of Jerusalem

joint work with

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Technical University Eindhoven

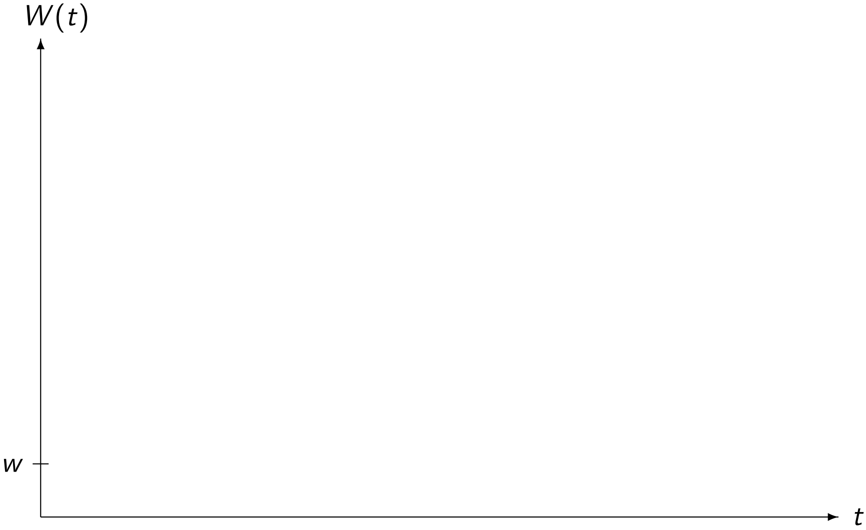
and

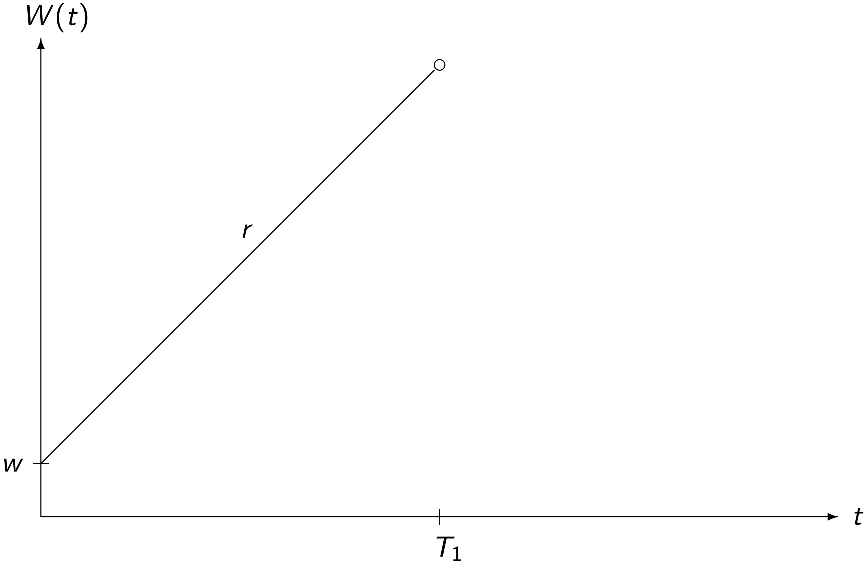
David Perry
Haifa University

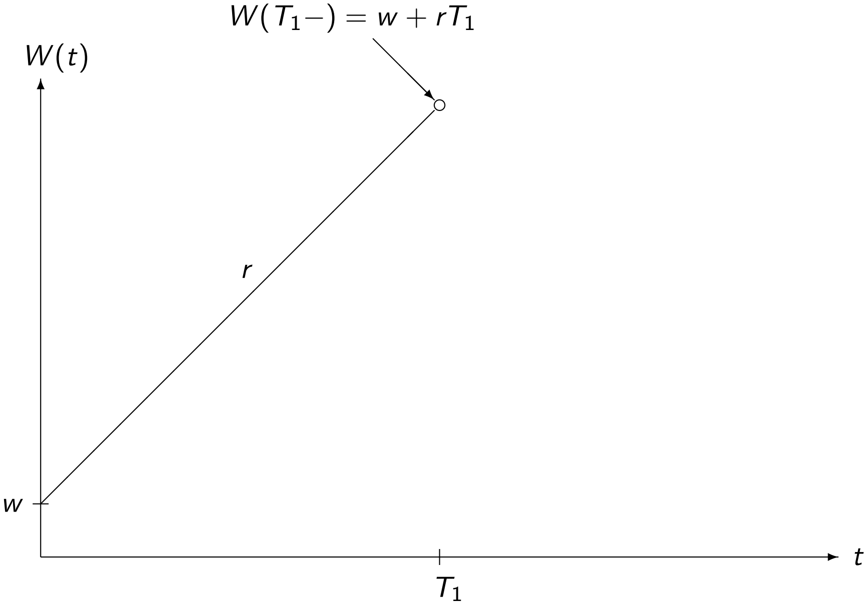
Edinburgh

Danish Garden

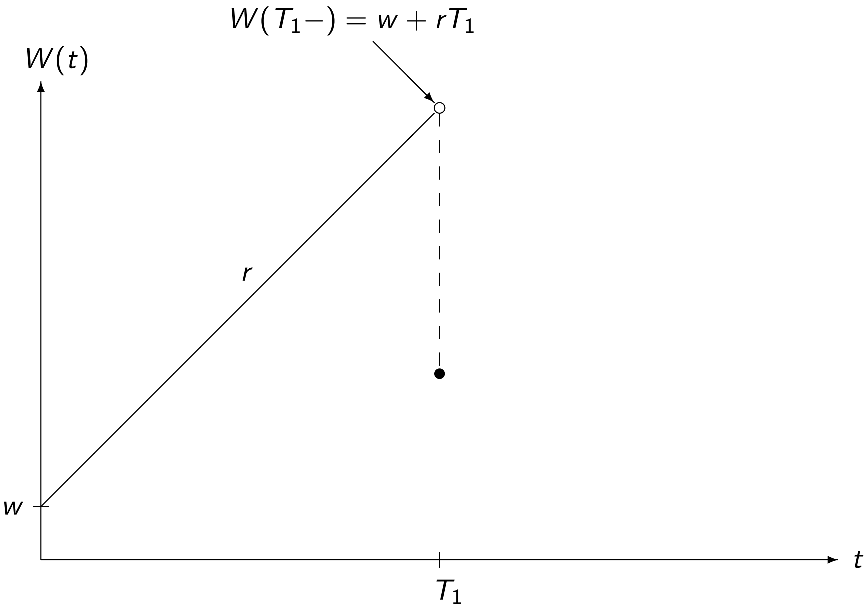
Conference in Honour of Søren Asmussen




W(t)
r
w
t
T1



$W(T_1-) = w + rT_1$
$W(t)$
r
w
t
T_1



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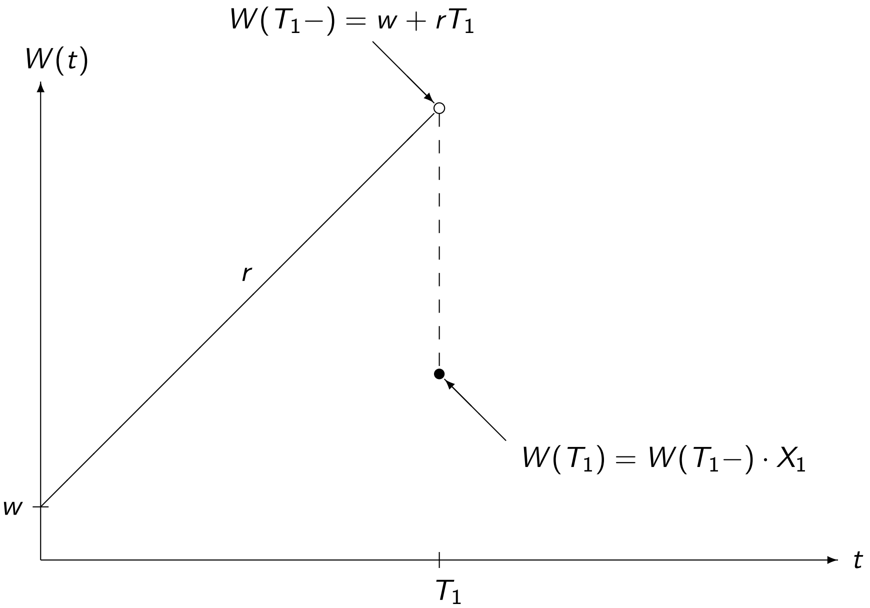
$W(T_1-) = w + rT_1$
$W(t)$
r
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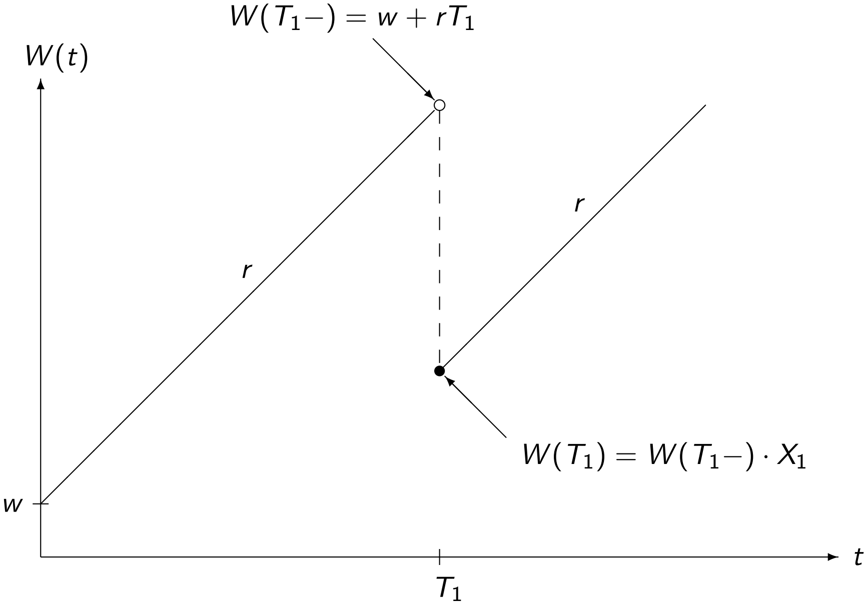
w
t
T_1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- With $N(t) = \sup\{n | T_n \leq t\}$

$$W(t) = W_{N(t)} + r(t - T_{N(t)})$$

- **K** - 2009, *J. Appl. Probab.*, **46**, 363-371.

- **K** - 2009, *J. Appl. Probab.*, **46**, 363-371.
- **K** and Löpker - 2010, *Prob. Eng. Inf. Sci.*, **24**, 99-107.

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- **K** and Yor - 2010, *Ann. Appl. Prob.*, **20**, 367-381.

Assumptions:

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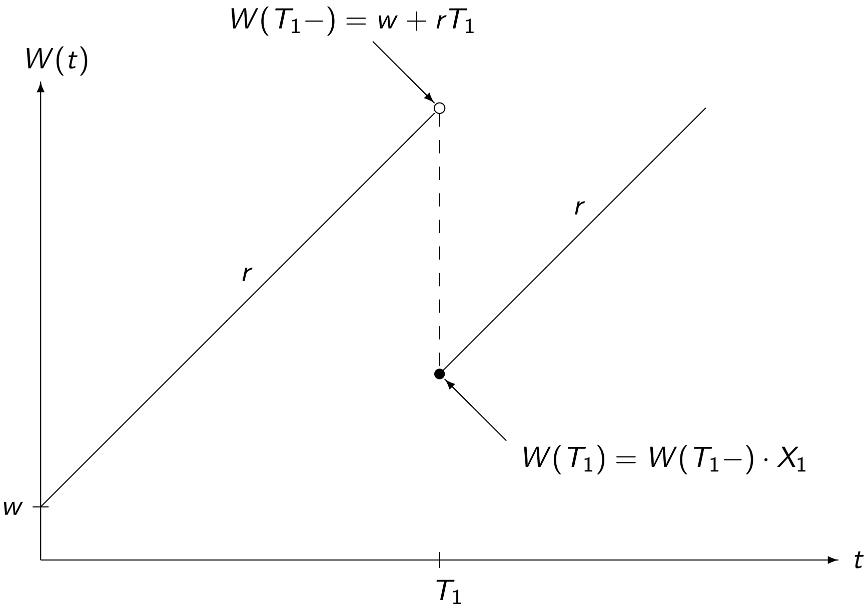

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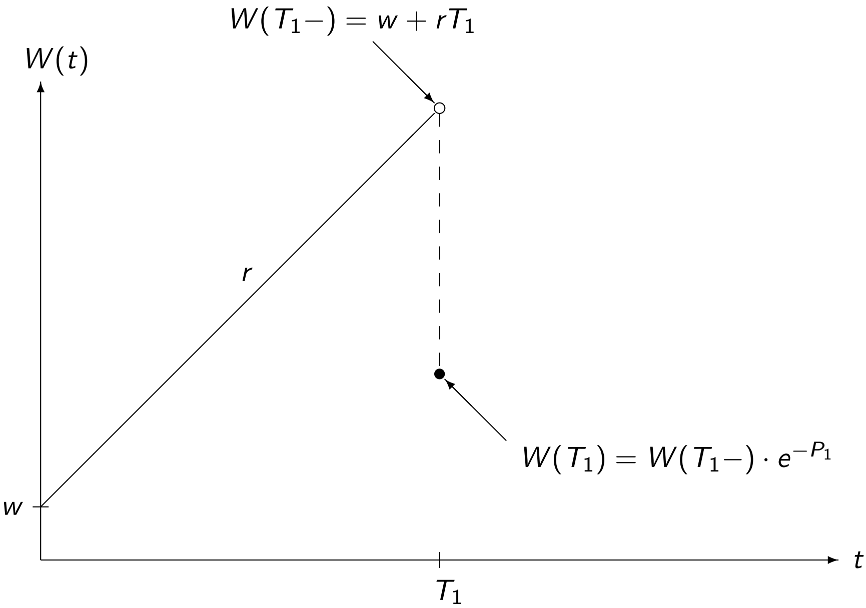
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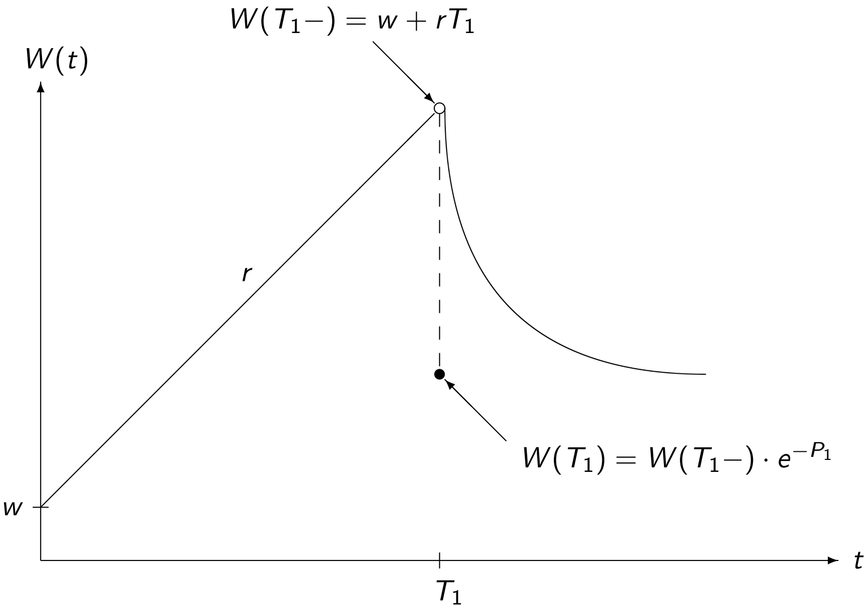
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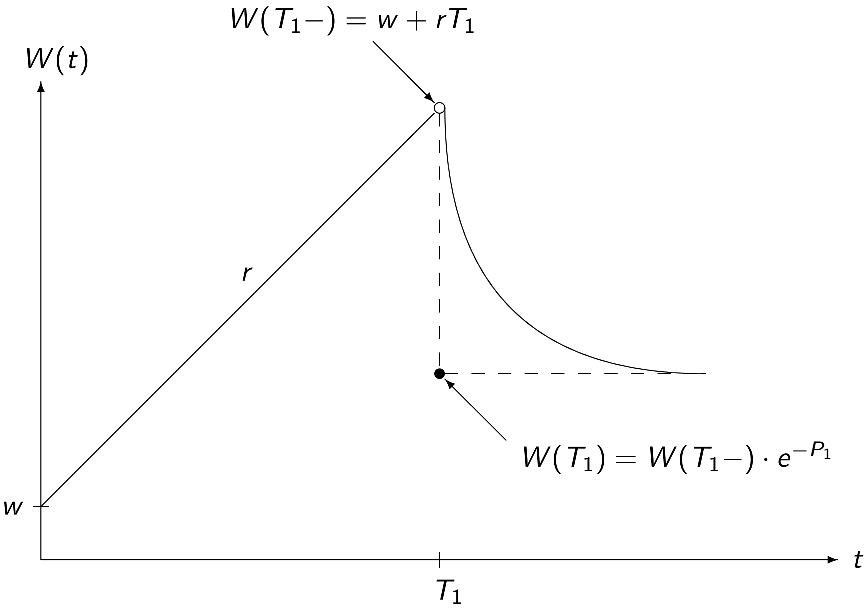

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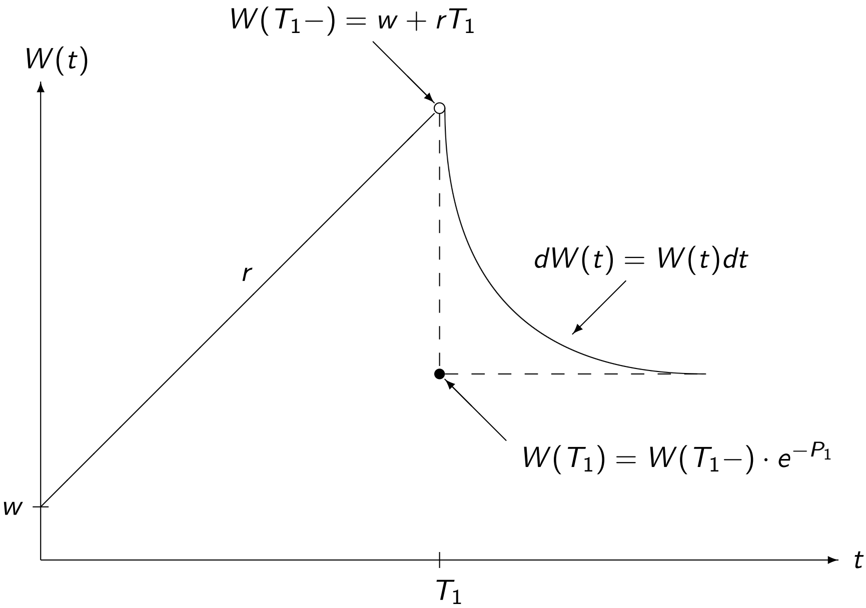
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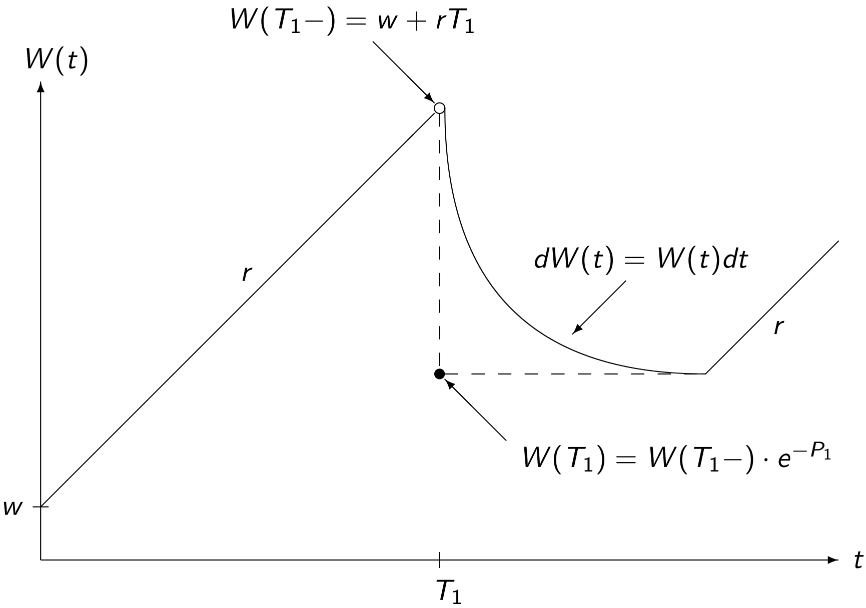
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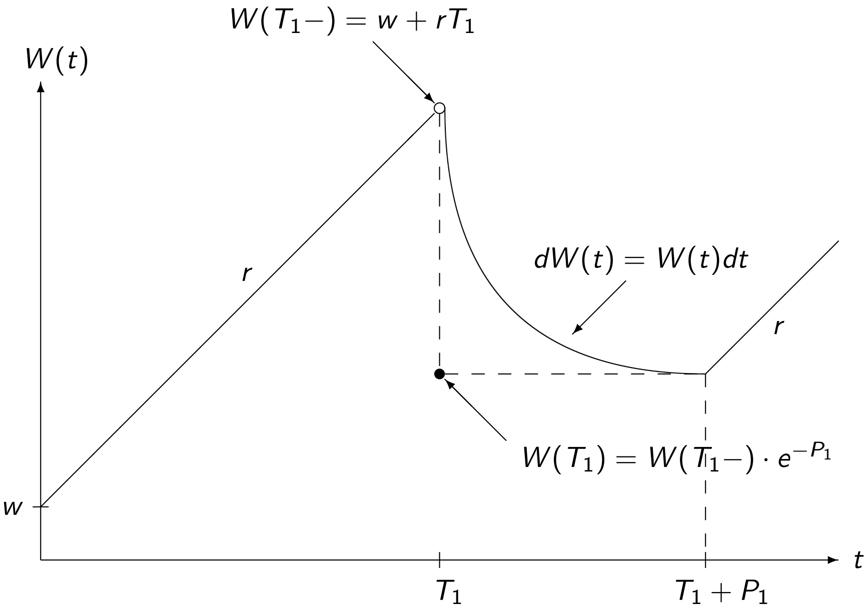
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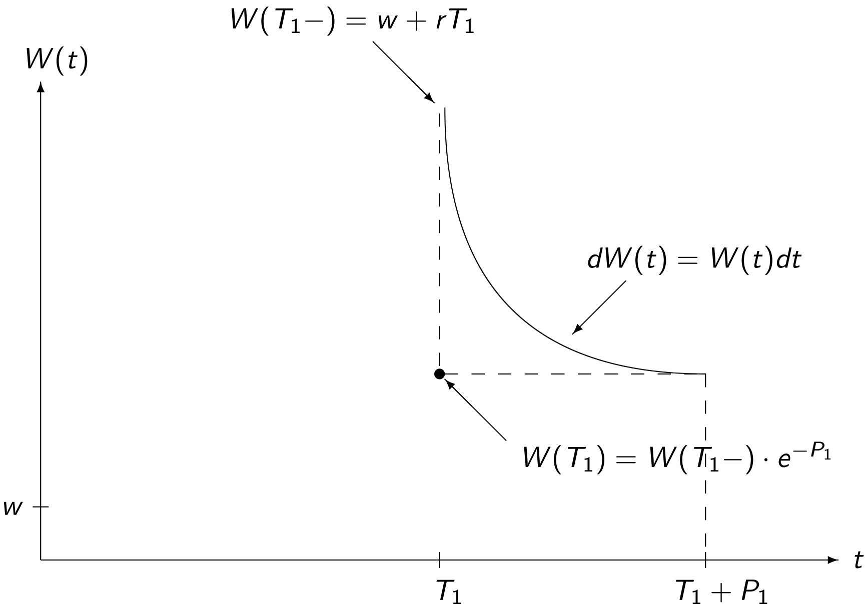
W(t)
r
dW(t) = W(t)dt
r
W(T1) = W(T1−) · e−P1
w
t
T1
T1 + P1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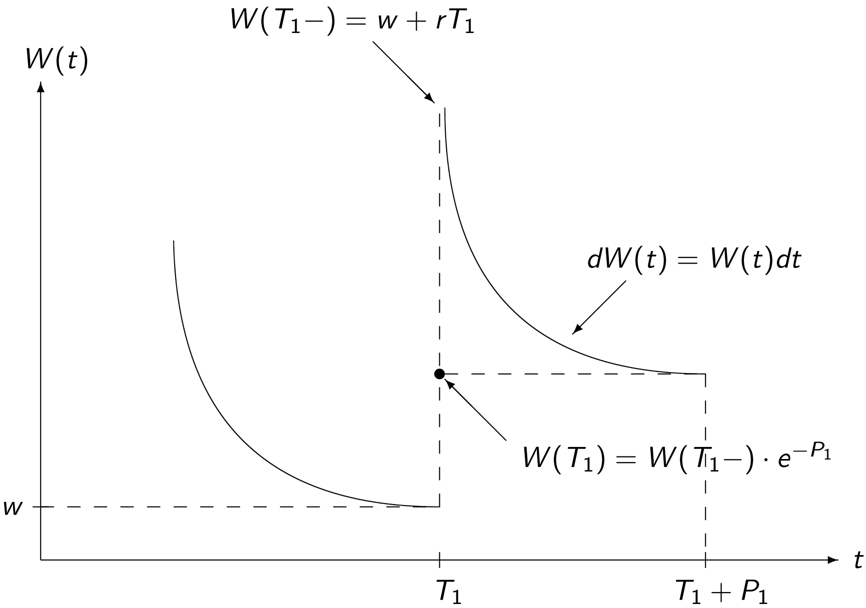
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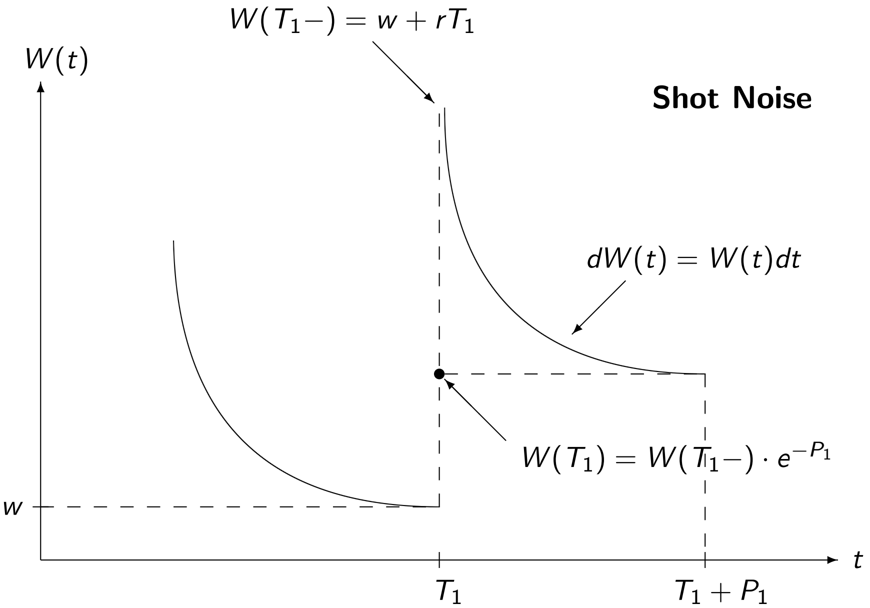
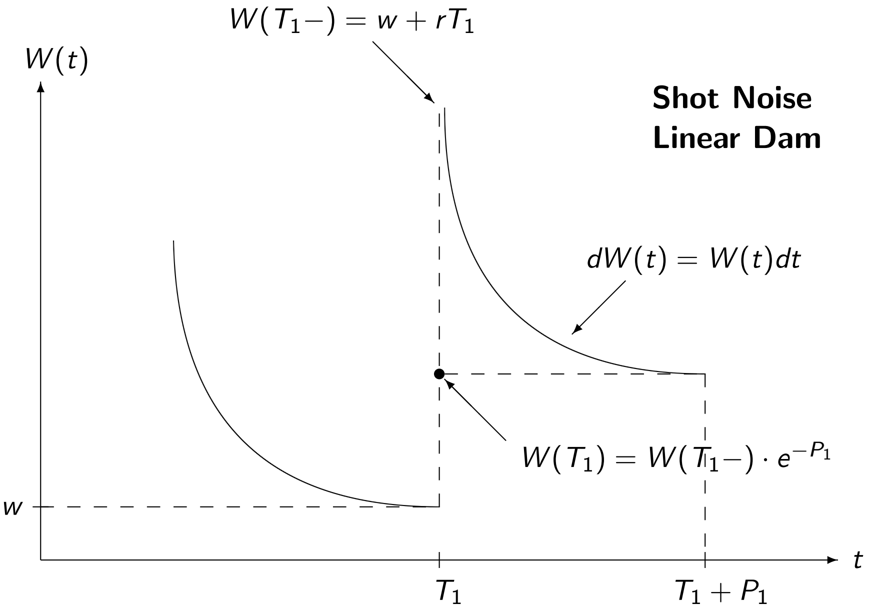

W(T1−) = w + rT1
W(t)
Shot Noise
dW(t) = W(t)dt
W(T1) = W(T1−) · e^−P1
w
t
T1
T1 + P1



$W(T_1-) = w + rT_1$
$W(t)$
Shot Noise
Linear Dam
$dW(t) = W(t)dt$
$W(T_1) = W(T_1-) \cdot e^{-P_1}$
w
t
T_1
$T_1 + P_1$

- Kaspi, **K** and Perry. - 1997, QUESTA **24**, 37-57.

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- We prefer to solve the shot noise problem

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- At state change epochs of J from i to $j \neq i$, $\Delta X(t) \sim G_{ij}$.
- $W(t) = W(0) + X(t) - \int_0^t r(J(s)) \cdot W(s)ds$ $(r(i) \geq 0)$

- Asmussen and K - 1996: Rate modulation in dams and ruin problems. J. Appl. Probab. **33** 523-535.

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- Asmussen and K - 2000: A multi-dimensional martingale for Markov additive processes and its applications. Adv. Appl. Probab. 32 376-393.

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3. J is irreducible
4. $r(i) > 0$ for some i.

Theorem

Under conditions $1-4$ *a unique stationary distribution for the joint (Markov) process* (W, J) *exists; and it is also the limiting distribution, which is independent of initial conditions.*

From Asmussen&K(2000), the following is a zero mean martingale:

$$\int_0^t e^{-\alpha W(s)} \mathbf{1}_{J(s)} \mathrm{d}s \cdot F(\alpha) \quad + \quad e^{-\alpha W(0)} \mathbf{1}_{J(0)} - e^{-\alpha W(t)} \mathbf{1}_{J(t)}$$

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- $\mathbf{1}_i = (0, \ldots, \underset{\substack{\uparrow \\ i}}{1}, \ldots, 0)$

Thus, if (W^*, J^*) has the stationary distribution:

$$Ee^{-\alpha W^*}\mathbf{1}_{J^*}F(\alpha) = \alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}Ee^{-\alpha W^*}\mathbf{1}_{J^*}r(J^*)$$

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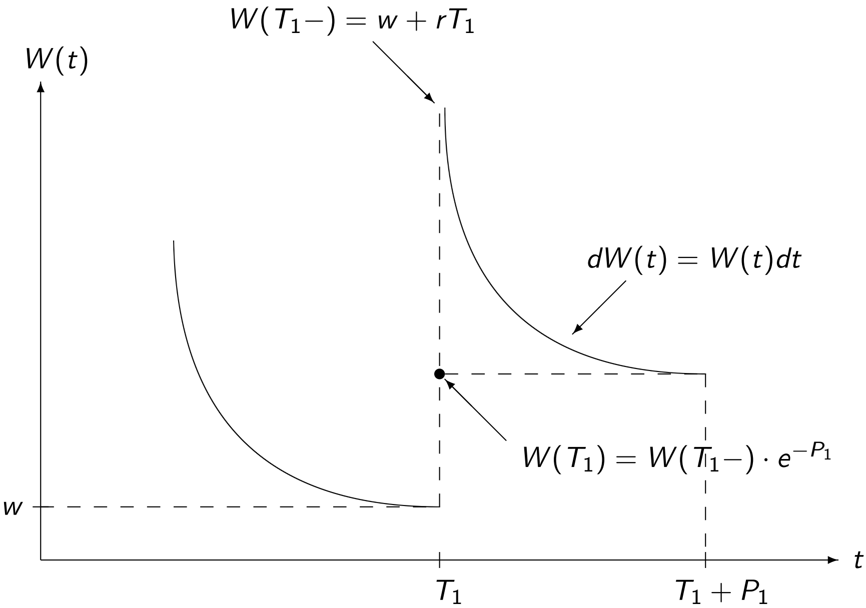
- $E(W^*)^n 1_{\{J^*=i\}}$ computable for all $n \geq 1$.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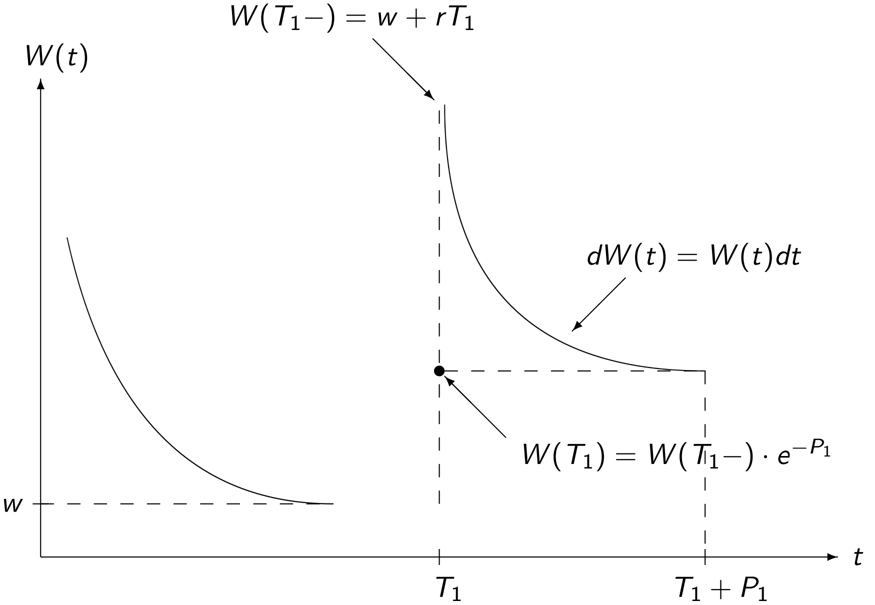
$T_1 + P_1$



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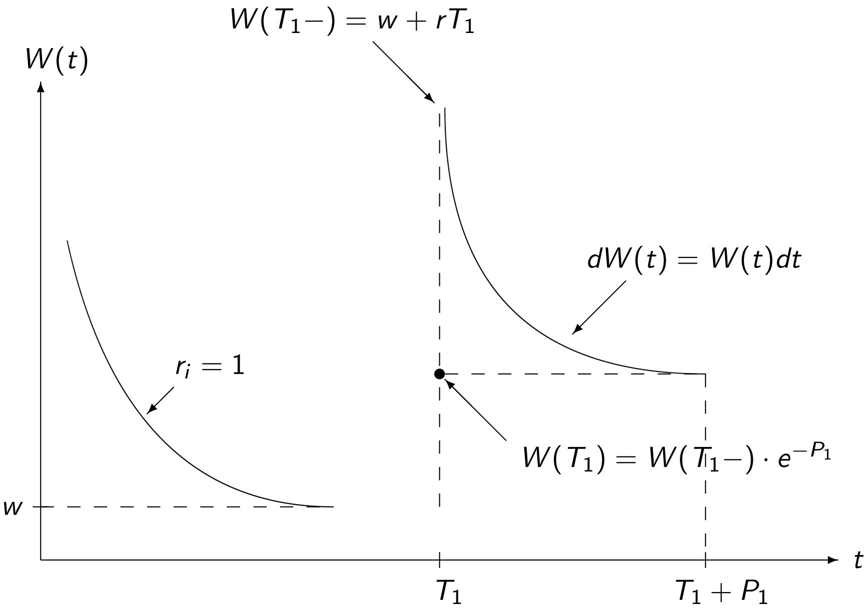
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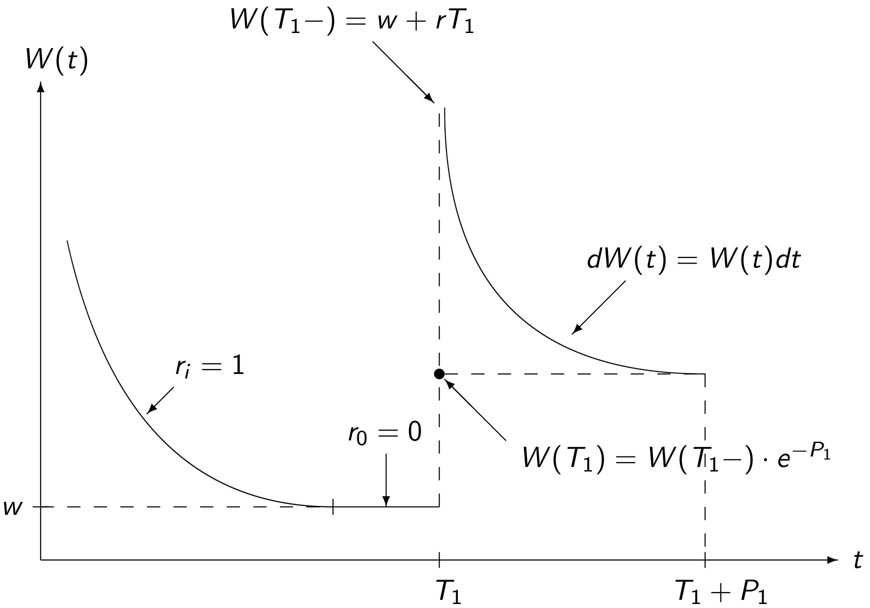
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- $\tilde{G}_{ij} = 1$ for $1 \leq i \leq K,\ 0 \leq j \leq K$

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- If $P[P_n > t] = \beta^T e^{-St}\mathbf{1}$ then

$$Q = \begin{pmatrix} -1 & \beta \\ -S\mathbf{1} & S \end{pmatrix}$$

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The stationary LST of the shot noise process is:

$$w(\alpha) = \frac{\sum_{i=1}^{K} w_i(\alpha)}{1-\pi_0} = \frac{\mathbf{1}^T \mathbf{w}(\alpha)}{1-\pi_0}$$

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- $\tilde{\pi}_i = \frac{\pi_i}{1-\pi_0}$ is stationary for $\tilde{Q}$.

One may check that with $\mu_i^n = E[(W^*)^n | J^* = i]$, $\mu_k = \int x^k G(dx)$

$$\frac{q_{0j}}{q_0} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\mu_{k+1}}{k+1} \sum_{i=1}^{K} \pi_i \mu^w_{n-1-k,i} q_{i0} = \sum_{i=1}^{K} \pi_i \mu^w_{n,i} \left(\delta_{ij} - \frac{\tilde{q}_{ij}}{n} \right)$$

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  $\pi_0 \sum_{k=0}^{\infty} \frac{1}{1+\tilde{a}} \left(\frac{\tilde{a}}{1+\tilde{a}} \right)^k \tilde{G}^k(\alpha)$
  Then if $Y_k \sim G$ are i.i.d. and $N \sim Geom((1+\tilde{a})^{-1})$, then
- $E \left(\sum_{k=1}^{N} Y_k \right)^n = \frac{m_n^w}{\pi_0}$

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  The unconditional moment:
- $\frac{1}{1-\pi_0} \sum_{j=1}^{K} \pi_j \mu^w_{n,j} = \frac{m^w_n}{(1-\pi_0)\tilde{a}} \sum_{j=1}^{K} \tilde{a}_j$

When $K = 2$, the solution is in terms of a hypergeometric function.

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- $\displaystyle \int_0^\infty e^{-\alpha x} f_{gc}(x)\mathrm{d}x = \frac{EP}{\mu}\int_0^\infty e^{-\alpha x} x f_{sn}(x)\mathrm{d}x = -\frac{EP}{\mu}\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_0^\infty e^{-\alpha x} f_{sn}(x)\mathrm{d}x$
- nth moment for growth collapse is $\frac{EP}{\mu} \cdot (n+1)$st moment for shot noise.

$\ddot{\smile}$ ...about two months from now $\ddot{\smile}$

HAPPY BIRTHDAY
DEAR SØREN