Volatility and Variation

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Dynamic stochastic phenomena frequently involve a significant element of randomness beyond the most basic types of stochastic innovations. Additional variations of this kind are referred to as *Volatility* or *Intermittency*.

Such ‘additional’ random fluctuations generally vary, in time and/or in space, in regard to both intensity and amplitudes.

This talk presents an overview of some recent and ongoing work on

How to model, measure and assess volatility/intermittency

Volatility/intermittency constitutes an important element in many scientific contexts.

- Turbulence [Kolmogorov-Obukhov (1962)]
- Finance
- Rain
- Nanoscale emitters

In this talk the connection to Turbulence will be in focus.

Ambit Stochastics

Namer for the theory and applications of *ambit fields* and *ambit processes*

While we focus here is on the turbulence context, ambit stochastics has also found roles in finance and biology.

Ambit fields

$$
\begin{aligned}
Y_t(x) &= \mu + \int_{A_t(x)} g(\xi, s; t, x)\, \sigma_s(\xi)\, L(\mathrm{d}\xi \mathrm{d}s) \\
&\quad + \int_{D_t(x)} q(\xi, s; t, x)\, a_s(\xi)\, \mathrm{d}\xi \mathrm{d}s.
\end{aligned}
$$

Here $A_t(x)$, and $D_t(x)$ are *ambit sets* (i.e. deterministic subsets of $\mathbb{R}^d \times \mathbb{R}$), g and q are deterministic (matrix) functions, $\sigma \geq 0$ is a stochastic field, and $L(\mathrm{d}\xi, \mathrm{d}s)$ is a homogeneous Lévy basis (i.e. an independently scattered random measure whose values are infinitely divisible and such that L is homogeneous in space-time).

The *volatility/intermittency* is embodied in σ.

Ambit processes

We shall, in particular, be interested in settings where the data consist in observation of the values of an ambit field along a curve $(x(\theta), t(\theta))$ in space-time $\mathbb{R}^d \times \mathbb{R}$.

Thus we consider processes of the form $X = \{X_\theta\}$ where $X_\theta = Y_{t(\theta)}(x(\theta))$.

We refer to such processes as *ambit processes*.

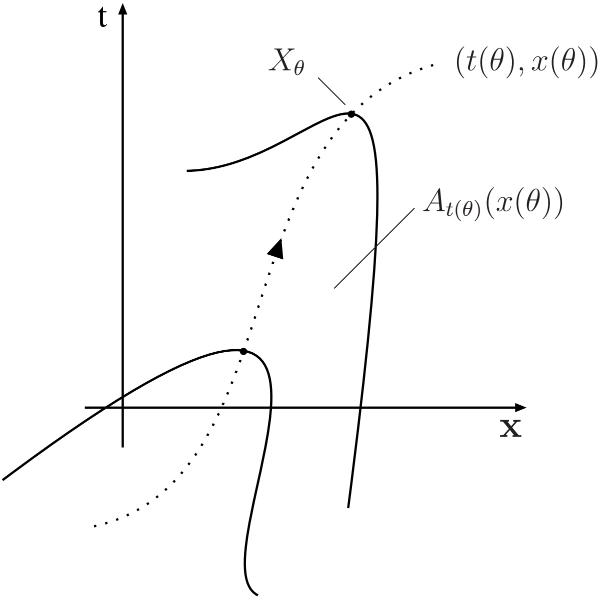

Figure: Ambit processes

Stationary settings

We will mainly consider settings which are stationary in time and sometimes also in space.

$$\begin{aligned} Y_t(x) &= \mu + \int_{A_t(0)+(x,0)} g(t-s,\xi,x)\,\sigma_s(\xi)\,L(\mathrm{d}\xi\mathrm{d}s) \\ &\quad + \int_{D_t(0)+(x,0)} q(t-s,\xi,x)\,a_s(\xi)\,\mathrm{d}\xi\mathrm{d}s. \end{aligned}$$

where σ and a are stochastic fields that are stationary at least in time.

Modelling turbulence

Let $Y_t(x)$ denote the velocity (d-dimensional; $d = 1, 2, 3$) at time t and at position $x \in \mathbb{R}^d$.

$$\begin{aligned} Y_t(x) &= \mu + \int_{A_0(x)+(0,t)} g(t-s,\xi;x)\,\sigma_s(\xi)\,W(\mathrm{d}\xi\mathrm{d}s) \\ &\quad + \int_{D_0(x)+(0,t)} q(t-s,\xi;x)\,a_s(\xi)\,\mathrm{d}\xi\mathrm{d}s. \end{aligned}$$

Encompassing stylised features of turbulence; by suitable choice of the defining elements. [[BNSch04], [BNBlSch04], [BNEgSch05], [BNSch07]]

The null-spatial setting

$\mathcal{BSS}$ processes – *Brownian semistationary processes*

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s B(\mathrm{d}s) + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s$$

where B is Brownian motion on $\mathbb{R}$, σ and a are stationary cadlag processes and g and q are deterministic continuous memory functions on $\mathbb{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$. It is sometimes convenient to indicate the formula for Y as

$$Y = g * \sigma \bullet B + q * a \bullet \mathrm{Leb}.$$

$\mathcal{LSS}$ processes:

$$Y = g * \sigma \bullet L + q * a \bullet \mathrm{Leb}$$

We consider the $\mathcal{BSS}$ processes to be the natural analogue, in stationarity related settings, of the class $\mathcal{BSM}$ of Brownian semimartingales.

$$Y_t = \int_0^t \sigma_s \mathrm{d}W_s + \int_0^t a_s \mathrm{d}s.$$

- The $\mathcal{BSS}$ processes are not in general semimartingales

Example Suppose $Y = g * \sigma \bullet B$ with $g(t) = t^{\nu-1}e^{-\lambda t}$ and $\sigma = 1.$, and where $\lambda > 0$ and $\nu > \frac{1}{2}$ (where the latter condition is required for the integral $g * B$ to exist). Then, by a theorem due to Knight, Y is a semimartingale if and only if either $\nu = 1$ or $\nu > \frac{3}{2}$. $\square$

- In the context of turbulence the most interesting cases are $\nu \in \left(1, \frac{3}{2}\right)$.

In general, the question of when an ambit process is a semimartingale is far from resolved. However, an important class of one-dimensional cases is covered by recent work of Basse-O'Connor and Pedersen [[BasPed09]].

Modelling of volatility/intermittency fields

The volatility/intermittency fields we shall consider are the form

$$\sigma_t(x) = V\left(\int_{C_t(x)} h(t,s,x,\xi)\, L(\mathrm{d}s\mathrm{d}\xi)\right)$$

where V is a smooth positive function and L is a homogeneous Lévy basis while $C_t(x)$ and $h(t,s,x,\xi)$ are deterministic.

Of special interest are cases where $\sigma_t(x)$ is stationary in t.

We mention some main settings.

Arithmetic case

$$\sigma_t^2(x) = L(C_t(x)) \tag{1}$$

for a positive homogeneous Lévy basis L and where
$C_t(x) = C_0(x) + (t, 0)$.

More generally,

$$\sigma_t^2(x) = \int_{C_t(x)} h(t-s, x, \xi)\, L(\mathrm{d}s\mathrm{d}\xi) \tag{2}$$

with $h \geq 0$.

Geometric case

$$\sigma_t(x) = e^{Z_{t^i}(x)} \qquad (3)$$

where

$$Z_t(x) = \int_{C_t(x)} h(t-s, x, \xi) L(\mathrm{d}s\mathrm{d}\xi). \qquad (4)$$

A specification of particular interest in the context of turbulence is

$$Z_t(x) = W(C_t(x)).$$

Specification of ambit regions

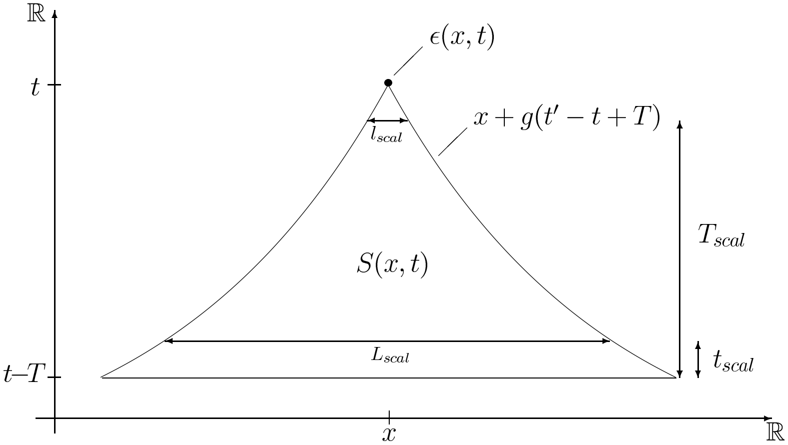

Figure:

[[BNEgSch05]]

Inference on volatility/intermittency

Key tools are *realised quadratic variation (RQV)* and, more generally, *realised multipower variation (RMPV)*.

Quadratic variation

The realised quadratic variation of a stochastic process X is denoted by $[X_\delta]$. Here δ is the time lag between observations, and $[X_\delta]$ and its normalised version $\overline{[X_\delta]}$ are, for $t > 0$, given by

$$[X_\delta]_t = \sum_{k=1}^{[t/\delta]} (X_{k\delta} - X_{(k-1)\delta})^2 \quad \text{and} \quad \overline{[X_\delta]}_t = \frac{\delta}{c(\delta)} [X_\delta]_t$$

where $c(\delta)$ is a positive constant, depending only on δ, whose specific form will be discussed below.

We are interested in the asymptotic behavior of $[X_\delta]$ and $\overline{[X_\delta]}_t$ for $\delta \to 0$.

Quadratic variation of ambit processes

In general, for ambit processes the realised quadratic variation does not converge but in many cases of interest a suitably normed version of RQV, i.e. $\overline{[X_\delta]}$, does have a limit in probability.

The limit behaviour of $[X_\delta]$ depends generally in a crucial way on an interplay between the shape of the ambit sets and the kernel function g.

We illustrate that by three examples. In the first two there is no spatial component.

Example Suppose that for some $0 < l < \infty$ we have $g(v) = e^{-\lambda v} 1_{(0,l)}(v)$, so that

$$Y_t = \int_{t-l}^{t} e^{-\lambda(t-s)} \sigma_s \mathrm{d}B_s.$$

This is a non-semimartingale case. Nevertheless,

$$[Y_\delta]_t \xrightarrow{p} \left(1 + e^{-2\lambda l}\right)^{-1} \sigma_t^{2+} - \left(1 + e^{2\lambda l}\right)^{-1} (\sigma_{t-l}^{2+} - \sigma_{-l}^{2+}).$$

Thus the ordinary RQV converges but the process Y is not *volatility memoryless*, that is we do not have $[Y_\delta]_t \xrightarrow{p} \sigma_t^{2+}$.

Example [[BNSch09]] Let $g(v) = v^{\alpha}(1-v)\,1_{(0,1)}(v)$ with $-\frac{1}{2} < \alpha$ (this inequality ensures existence of the stochastic integral $g * \sigma \bullet B$)

If $\alpha < 0$ then we are in the nonsemimartingale situation.

Then, for $c(\delta) = (1-2\gamma)^{-1}\,\delta^{1+2\alpha}$, we have

$$\overline{[X_{\delta}]_t} \xrightarrow{P} \sigma_t^{2+}.$$

Hence the process X is volatility memoryless.

Example [[BNG10]] Suppose now that

$$Y_t(x) = \int_{A+(x,t)} g(x-\xi, t-s)\, \sigma_s(\xi)\, W(\mathrm{d}\xi \mathrm{d}s)\,. \tag{5}$$

For a given smooth curve $\gamma = (\gamma_1, \gamma_2) : \mathbf{R} \to \mathbf{R}^2$ consider the process

$$X_\theta = Y(\gamma(\theta)) \quad \theta \geq 0.$$

Pick up of information on intermittency:

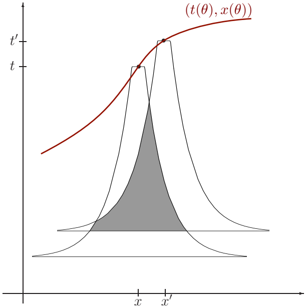

Figure: Increments

The figure shows a type of ambit sets that is of interest in turbulence studies and whose shape is motivated in Taylor's frozen field hypothesis (cf. [BNSch04]))

We are primarily interested in the case where the ambit set A is bounded with regular boundary curve and where the kernel function g is 'regular' on the interior of A.

We now introduce a probability measure π_δ which is determined by the kernel function g and whose behaviour as $\delta \to 0$ is of key importance for the probabilistic limit properties of $\overline{[X_\delta]}$.

Define π_δ by

$$\pi_\delta\left(\mathrm{d}u\mathrm{d}v\right) = \frac{\psi_\delta(u,v)}{c(\delta)} du\, dv$$

where

$$c(\delta) = \int_{\mathbf{R}^2} \psi_\delta(u,v) du\, dv$$

and where $\psi_\delta(u,v)$ is defined in terms of squared increments of g.

By construction, π_δ is a probability measure and all weak limit points of π_δ for $\delta \to 0$ will be probability measures concentrated on $-A$. Simple calculations together with the continuity assumption on the volatility field σ imply that in case the limit

$$\pi_\delta \overset{w}{\longrightarrow}_{\delta\to 0} \pi_0$$

exists for some probability measure π_0 then

$$E[\,\overline{[Y_\delta]}_t \,|\, \sigma] \rightarrow_{\delta\to 0} \int_{\mathbf{R}^2} \int_0^t \sigma^2_{\gamma_2(s)-v}(\gamma_1(s)-u)\, ds\, \pi_0(du\, dv).$$

We are particularly interested in conditions on A and g ensuring that the limit π_0 exists and is concentrated on $\partial(-A) = -\partial A$. In this case we have, as a main result, that

$$\overline{[Y_\delta]}_t \xrightarrow{p} \int_{\mathbf{R}^2} \int_0^t \sigma^2_{\gamma_2(s)+v}(\gamma_1(s) + u)\, ds\, \pi(du\, dv). \tag{6}$$

Here π denotes the image measure of π_0 under the transformation $(u, v) \mapsto (-u, -v)$. Observe that π is concentrated on ∂A.

Remark Note especially that π may be situated on ∂A even if the function g tends rather rapidly to 0 as its argument tends to the boundary.

Remark If A has a unique top point then the limit measure π exists and equals the delta measure at that point, in which case the process is *volatility memoryless*.

The validity of the result

$$\overline{[Y_\delta]}_t \xrightarrow{p} \int_{\mathbf{R}^2} \int_0^t \sigma^2_{\gamma_2(s)+v}(\gamma_1(s)+u)\, ds\, \pi(du\, dv)$$

has been established subject to specified regularity conditions on A, g and σ, including convexity of A, continuity of σ and independence of σ and W. And we have worked under the assumption that the curve γ is linear. [BNG10]

Thus the assumptions have been quite restrictive, but we believe that the conclusion holds in much greater generality.

But in any case, limit in probability can only be considered a first step. Namely:

It is important to establish CLT's for the normed RQV $\overline{[Y_\delta]}$ and, more generally, for normed *realised multipower variations (RMPV)*. Such results are, in particular, key for inference on g and σ.

Some first results in this direction have recently been established in the null-spatial setting of $\mathcal{BSS}$ processes. [[BNCP09], [BNCP10]]

We proceed to give an indication of those results.

We note first, however, that the theory of realised multipower variations was first developed in the semimartingale setting, motivated by problems in financial mathematics and financial econometrics. [[BNS02], [BNS03], [BNS04], [BNGJPS06], [BNGJS06], [J08a], [J08b]]

Unlike these semimartingale based results, essential for the derivations in ambit settings are the use of Malliavin calculus, in particular a new *CLT for Gaussian triangular arrays*.

Multipower Variation

Let X be a stochastic process in continuous time, observed over the interval $[0, t]$ at time points $0, \delta, 2\delta, \ldots$, where $\delta = n^{-1}$ for some positive integer n.

Realised multipower Δ-variations

A realised multipower Δ-variation of a stochastic process X is an object of the type

$$V_{\Delta}(Y, p_1, \ldots, p_k)_t^n = \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n X|^{p_j}$$

where $\Delta_i^n X = X_{i\delta} - X_{(i-1)\delta}$ and $p_1, \ldots, p_k \geq 0$.

Realised multipower $\Diamond$-variations

A realised multipower $\Diamond$-variation of a stochastic process X is an object of the type

$$V_{\Diamond}(Y, p_1, \ldots, p_k)_t^n = \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Diamond_{i+j-1}^n X|^{p_j}$$

where $\Diamond_i^n X = X_{i\delta} - 2X_{(i-1)\delta} + X_{(i-2)\delta}$ and $p_1, \ldots, p_k \geq 0$.

- Importance of 'diamond' variations

Now consider a $\mathcal{BSS}$ process

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s.$$

Let G be the *Gaussian core* of Y, i.e.

$$G_t = \int_{-\infty}^{t} g(t-s)W(\mathrm{d}s)$$

and let $\mathcal{G}$ be the σ-algebra generated by G.

Key quantities

Define r_n^{Δ} as the autocorrelation function of the Δ-increments of G, i.e.

$$r_n^{\Delta}(j) = \operatorname{cov}\left(\frac{\Delta_1^n G}{\tau_n^{\Delta}}, \frac{\Delta_{1+j}^n G}{\tau_n^{\Delta}}\right)$$

and $r_n^{\Diamond}$ as the autocorrelation function of the $\Diamond$-increments of G, i.e.

$$r_n^{\Diamond}(j) = \operatorname{cov}\left(\frac{\Diamond_1^n G}{\tau_n^{\Diamond}}, \frac{\Diamond_{1+j}^n G}{\tau_n^{\Diamond}}\right)$$

where

$$\left(\tau_n^{\Delta}\right)^2 = \mathrm{E}\left\{\left|\Delta_1^n G\right|^2\right\} \quad \text{and} \quad \left(\tau_n^{\Diamond}\right)^2 = \mathrm{E}\left\{\left|\Diamond_1^n G\right|^2\right\}.$$

Let π_δ^Δ be the measure on $\mathbb{R}_+$ defined by

$$\pi_\delta^\Delta(A) = \frac{\int_A (g(x-\delta) - g(x))^2 \mathrm{d}x}{\int_0^\infty (g(x-\delta) - g(x))^2 \mathrm{d}x}.$$

Note that π_δ^Δ is a probability measure on $\mathbb{R}_+$, and set

$$\bar{\pi}_\delta^\Delta(x) = \pi_\delta^\Delta(\{y : \ y > x\}).$$

This measure $\bar{\pi}_\delta$ has a crucial influence on the asymptotic behaviour of the realised multipower Δ-variations of Y.

Similarly, let $\pi_\delta^\Diamond$ be the measure on $\mathbb{R}_+$ defined by

$$\pi_\delta^\Diamond(A) = \frac{\int_A (g(x-2\delta) - 2g(x-\delta) + g(x))^2 \mathrm{d}x}{\int_0^\infty (g(x-2\delta) - 2g(x-\delta) + g(x))^2 \mathrm{d}x}.$$

Note that $\pi_\delta^\Diamond$ is a probability measure on $\mathbb{R}_+$, and set

$$\bar{\pi}_\delta^\Diamond(x) = \pi_\delta^\Diamond(\{y : \ y > x\}).$$

This measure $\bar{\pi}_\delta^\Diamond$ has a crucial influence on the asymptotic behaviour of the realised multipower $\Diamond$-variations of Y.

We are interested in the probabilistic limit behaviour of the *normalised* realised multipower Δ-variations

$$\bar{V}_{\Delta}(Y, p_1, \ldots, p_k)_t^n = \frac{1}{n\left(\tau_n^{\Delta}\right)^{p_+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j}$$

and of the *normalised* realised multipower $\Diamond$-variations

$$\bar{V}_{\Diamond}(Y, p_1, \ldots, p_k)_t^n = \frac{1}{n\left(\tau_n^{\Diamond}\right)^{p_+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Diamond_{i+j-1}^n Y|^{p_j}$$

Example of results obtained (under regularity conditions): [BNCP09]

Joint Central Limit Theorem:

$$\sqrt{n}\Big(\bar{V}_\Delta(Y, p_1^j, \ldots, p_k^j)_t^n - \rho^{(n)}_{p_1^j,\ldots,p_k^j} \int_0^t |\sigma_s|^{p_+^j} \mathrm{d}s\Big)_{1\le j\le d}$$

$$\xrightarrow{\mathcal{G}-st} \int_0^t Z_s^{1/2} \mathrm{d}B_s$$

where B is a d-dimensional Brownian motion that is independent of Y, and Z is a $d \times d$-dimensional process

$$Z_s^{ij} = \beta_{ij} |\sigma_s|^{p_+^i + p_+^j} , \qquad 1 \le i,j \le d \ .$$

Realised Variation Ratios (RVR)

$$RVR_{\triangle} = \frac{V_{\triangle}(Y,1,1)}{V_{\triangle}(Y,2)} \quad \text{and} \quad RVR_{\lozenge} = \frac{V_{\lozenge}(Y,1,1)}{V_{\lozenge}(Y,2)}$$

Roles in **Finance** and **Turbulence**

In particular:

- RVRs: Fingerprints of turbulence

Joint Central Limit Theorems for RVR's established in [BNCP10].

Exit remark:

To what extent can one create a

Stochastic Calculus for (Stationary) Ambit Processes?

Integration issues

Stochastic differentials?

"Ito Algebra"?

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