Loss rate for a general Lévy process with downward periodic barrier

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Joint work with P. Świątek

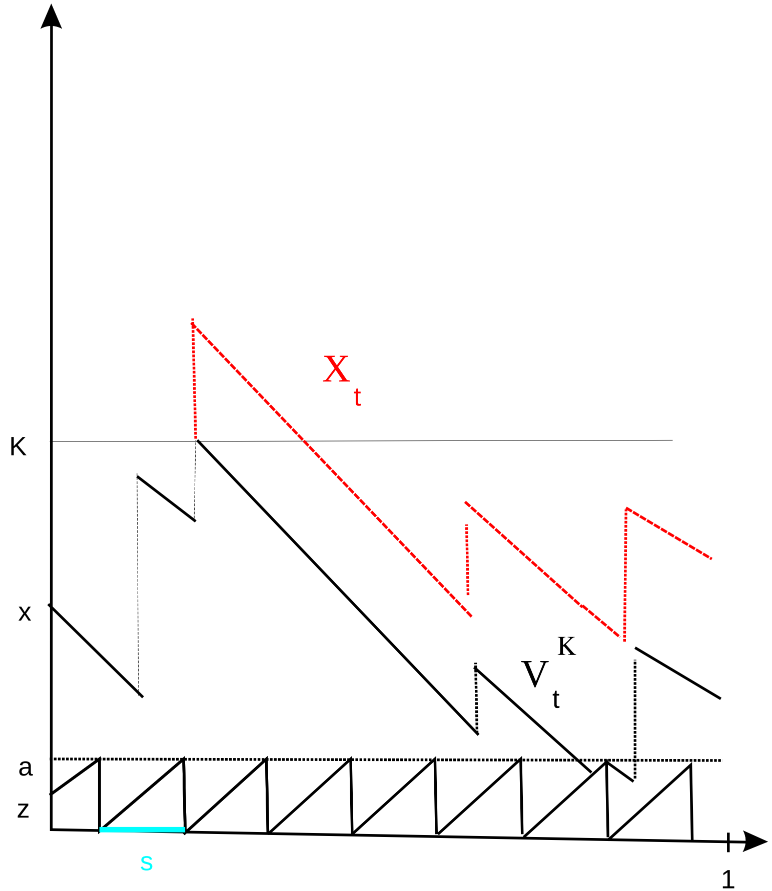
August 2011

Based on Soren's idea
as it is in many of my papers...

Extraordinariness

Snails and central station

Reflected Lévy process



Reflected Lévy process

X_t - general Lévy process which is not subordinator

models netto input to buffer of finite capacity K fluid queue

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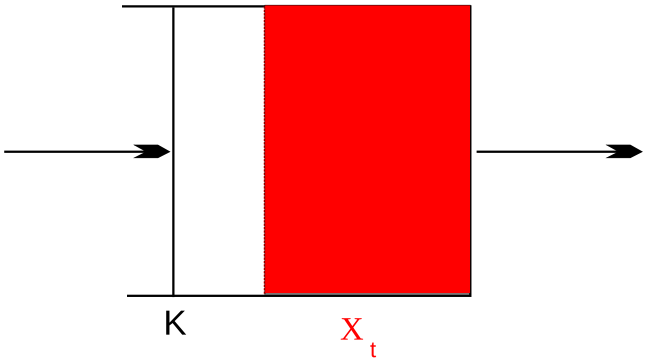

$$V_t = X_t - \sup_{0\le s\le t}\left(\max\left(\min\{X_s - K, \inf_{u\in[0,t]} X_u\}, \inf_{u\in[s,t]} X_u\right)\right)$$

$$\stackrel{D}{=} \sup_{s\in[0,t]} \max\{X_t - X_s, \inf_{u\in[s,t]}(K + X_t - X_u)\}$$

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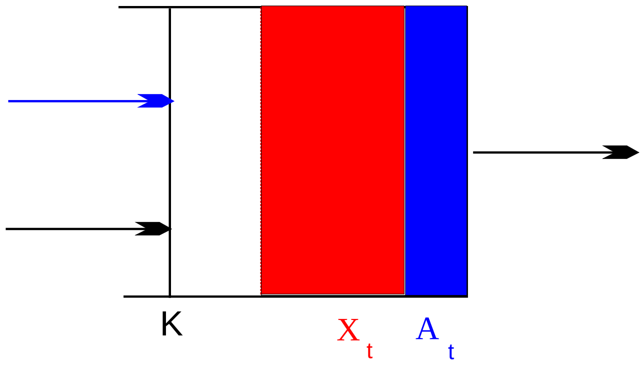
A_t

models additional input which is not available on liquit basis



Reflected Lévy process

X_t - general Lévy process which is not subordinator

models netto input to buffer of finite capacity K fluid queue

Downward periodic barrier

$$A_t = \varphi(t + U)$$

models additional input which is not available on liquit basis for

$\varphi(t)$ nonnegative periodic function with period s

$$U \stackrel{D}{=} \mathrm{U}[0, s]$$

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$\varphi(t)$ nonnegative function

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Reflected process:

$$V_t^K = X_t + L_t^A - L_t^K$$

being solution of respective Skorohod problem

Loss rate

X_t - general Lévy process which is not subordinator

Downward periodic barrier $A_t = \varphi(t+U)$

Reflected process:

$$V_t^K = X_t + L_t^A - L_t^K$$

with the stationary measure

$$\pi_K$$

Loss rate:

$$l^K = \mathbb{E} L_1^K$$

where expectation is taken for the stationary V_t^K

Goals: identification of l^K in terms of Lévy triple and finding its asymptotics as $K \to \infty$

Special case

Asmussen & Pihlsgard (2010):

$$A(t) \equiv 0$$

Denote $l^{K,0}$ loss rate in this case. Then:

$$l^{K,0} \leq l^K \leq l^{K-a,0}$$

and

$$\frac{1}{K} \log l^K = -\gamma$$

for some γ.

We want to get more explicit asymptotoics and exact expressions.

The ultimate goal is better understanding reflection for more general (lower and upper) barriers (both possibly tending to infinity).

Invariant measure of lower barrier

Assume that: $\varphi \in \mathcal{C}^1(\mathrm{int}\, J_k)$ is invertible on some disjoint intervals J_k satisfying $\cup_{k=1}^n J_k = [0, s]$ with $\varphi'(x) \neq 0$ for $x \in \mathrm{int}\, J_k$

Lemma 1. Process $A_t = \varphi(t + U)$ has invariant measure:

$$\xi(dy) = \sum_{k=1}^{n} \frac{1}{s} \left| h_k'(y) \right| \mathbf{1}_{\varphi(\mathrm{int} J_k)}(y)\, dy$$

where h_k is a inverse of φ on $\mathrm{int}\, J_k$.

Example 1

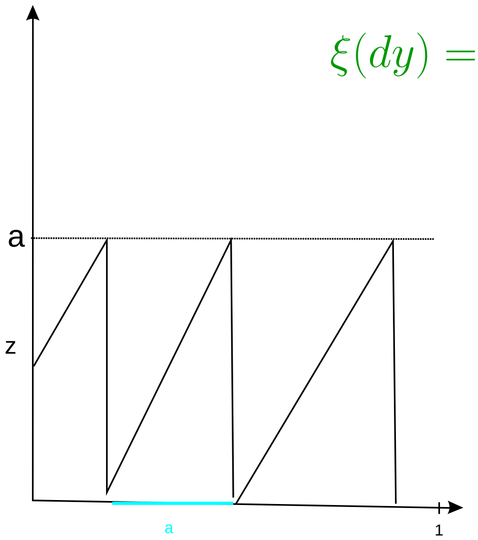
Saw-like lower boundary modeling constant intensity input (with rate 1 for simplicity)

$$\varphi(t) = t \bmod a$$

with $0 < a < K$.

In this case $n = 1$, $J_1 = [0, a]$, $s = a$ and

$$\xi(dy) = \frac{dy}{a} \mathbf{1}_{(y \in [0,a])}$$



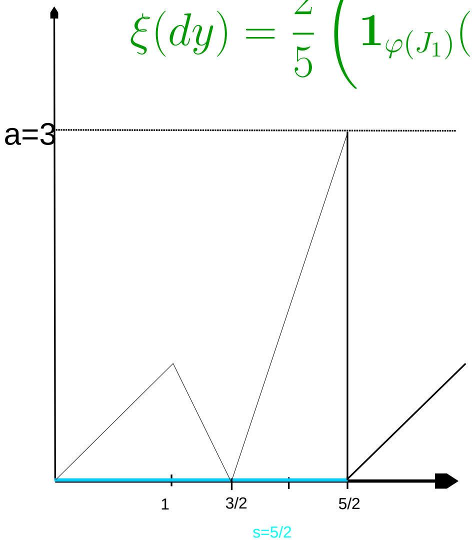
Example 2

Few lines with different slopes

$$\varphi(t) = \begin{cases} t & \text{for } t \in [0,1) \\ 1 - 2(t-1) & \text{for } t \in [1, \frac{3}{2}) \\ 3\left(t - \frac{3}{2}\right) & \text{for } t \in [\frac{3}{2}, \frac{5}{2}) \end{cases}$$

In this case $n = 3$, $J_1 = [0,1]$, $J_2 = [1, \frac{3}{2})$, $J_3 = [\frac{3}{2}, \frac{5}{2})$, $s = \frac{5}{2}$ and

$$\xi(dy) = \frac{2}{5}\left(\mathbf{1}_{\varphi(J_1)}(y) + \frac{1}{2}\mathbf{1}_{\varphi(J_2)}(y) + \frac{1}{3}\mathbf{1}_{\varphi(J_3)}(y)\right) dy$$



Stationary distribution

Lemma 2. Stationary distribution V_∞^K

$$\pi_K(x,\infty) = \mathbb{P}(V_\infty^K \geq x) = \int_0^a \underbrace{\sum_{k=1}^n \mathbb{P}(X_\tau \geq \widehat{A}_\tau^{-z,k} + x) p_k(z)}_{\overline{\pi}_K^z(x)} \xi(dz)$$

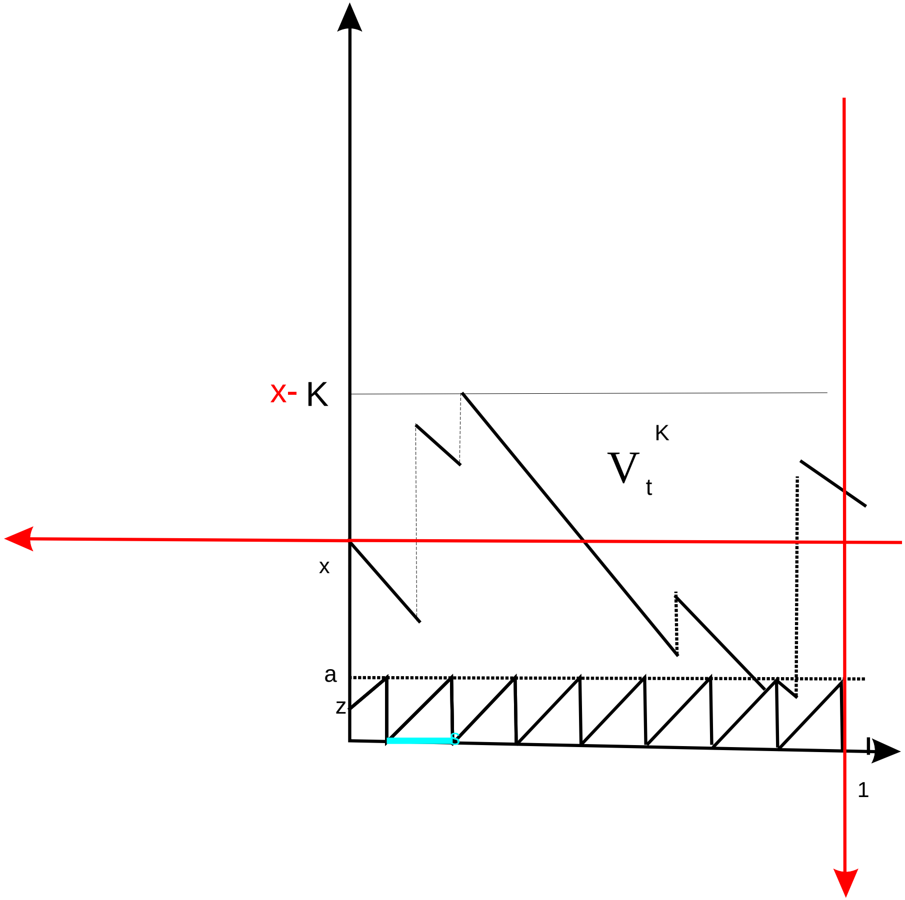
where

$$\tau = \inf\{t \geq 0 : X_t \notin [x - K, \widehat{A}_t^{-z,k} + x)\}$$

$$\widehat{A}_t^{-z,k} = -\varphi(h_k(z) - t) \quad \text{for } z \in \varphi(J_k)$$

$$p_k(z) = \mathbb{P}(U \in J_k | \varphi(U) = z) = \frac{|h_k'(z)| \mathbf{1}_{\varphi(J_k)}(z)}{\sum_{j=1}^n |h_j'(z)| \mathbf{1}_{\varphi(J_j)}(z)}$$

Duality



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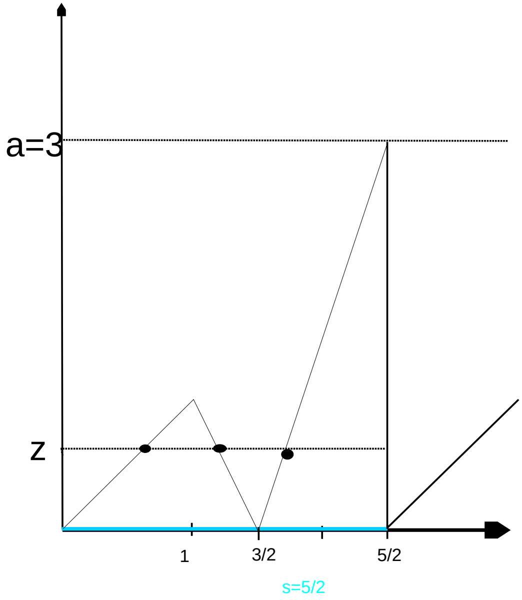


$$p_1(z) = \frac{1}{1+\frac{1}{2}+\frac{1}{3}} = \frac{6}{11}, \quad z \in [0,1]$$

$$p_2(z) = \frac{\frac{1}{2}}{1+\frac{1}{2}+\frac{1}{3}} = \frac{3}{11}, \quad z \in \left[1, \frac{3}{2}\right]$$

$$p_3(z) = \frac{\frac{1}{3}}{1+\frac{1}{2}+\frac{1}{3}} = \frac{2}{11}, \quad z \in \left[\frac{3}{2}, \frac{5}{2}\right]$$

Main result

Theorem 1. If $\int_1^\infty y\nu(dy) = \infty$, then $l^K = \infty$, and otherwise

$$l^K = \mathbb{E}X_1 \left[\frac{1}{K - \mathbb{E}A_0} \int_0^K x\pi_K(dx) - \frac{\mathbb{E}A_0}{(K - \mathbb{E}A_0)} \right] + \frac{\sigma^2}{2(K - \mathbb{E}A_0)} + \frac{1}{2(K - \mathbb{E}A_0)} \int_0^a \int_z^K \int_{-\infty}^\infty \varphi_K(x, y, z)\nu(dy)\pi_K^z(dx)\xi(dz)$$

where

$$\varphi_K(x, y, z) = \begin{cases} -(x-z)^2 - 2y(x-z) & \text{if } y \le -x+z \\ y^2 & \text{if } -x+z < y < K-x \\ 2y(K-x) - (K-x)^2 & \text{if } y \ge K-x \end{cases}$$

and ν is a Lévy measure of X
Idea of the proof will be given on next slides.

Kella-Whitt martingale

Step 1.

$$M_t = \kappa(\alpha)\int_0^t e^{\alpha V_s^K} ds + e^{\alpha V_0^K} - e^{\alpha V_t^K} + \alpha \int_0^t e^{\alpha A_s} dL_s^{A,c}$$
$$+ \sum_{0\le s\le t} e^{\alpha A_s}(1 - e^{-\alpha \Delta L_s^A}) - \alpha e^{\alpha K} L_t^{K,c} + e^{\alpha K} \sum_{0\le s\le t} (1 - e^{\alpha \Delta L_s^K})$$

is a zero-mean martingale, where

$$\kappa(\alpha) = \log \mathbb{E} \exp\{\alpha X(1)\}$$

is a Laplace exponent of X.

Step 2.

Take $t = 1$, start V_0^K according to the stationary distribution and use expansion

$$e^{\alpha x} = 1 + \alpha x + \frac{(\alpha x)^2}{2} + \frac{(\alpha x)^3}{6} e^{\theta \alpha x}, \quad \theta \in [-1, 1]$$

Kella-Whitt martingale

Step 2.

This produces:

$$
\begin{aligned}
\alpha(1-e^{\alpha K}+\alpha\mathbb{E}(A_0))l^K \;=\; & -\kappa(\alpha)\mathbb{E}e^{\alpha V_0^K}+\alpha\mathbb{E}X_1-\alpha e^{\alpha K}\bar{l}_j^K+\alpha\bar{l}_j^A \\
& +\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}L_s^K)^2+\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}L_s^A)^2 \\
& -e^{\alpha K}\mathbb{E}\sum_{0\le s\le 1}(1-e^{\alpha\overline{\Delta}L_s^K}) \\
& -\mathbb{E}\sum_{0\le s\le 1}e^{\alpha A_s}(1-e^{-\alpha\overline{\Delta}L_s^A}) \\
& +\alpha^2\mathbb{E}(A_0)\mathbb{E}X_1+\alpha^2\mathbb{E}\sum_{0\le s\le 1}A_s\overline{\Delta}L_s^A+o(\alpha^2)
\end{aligned}
$$

as $\alpha \downarrow 0$

We split ΔL_t^K into two parts, $\underline{\Delta}L_t^K$ and $\overline{\Delta}L_t^K$, corresponding to $\Delta X_s \in [0, L]$ and $\Delta X_s \in (L, \infty)$, respectively

We split ΔL_t^A into two parts, $\underline{\Delta}L_t^A$ and $\overline{\Delta}L_t^A$, corresponding to $\Delta X_s \in [-L, 0]$ and $\Delta X_s \in (-\infty, -L)$, respectively

Identifying terms - Lévy triple

Step 3.

$$\begin{aligned}
\alpha(1-e^{\alpha K}+\alpha\mathbb{E}(A_0))l^K \;=\;& -\kappa(\alpha)\mathbb{E}e^{\alpha V_0^K}+\alpha\mathbb{E}X_1-\alpha e^{\alpha K}\bar{l}_j^K+\alpha\bar{l}_j^A \\
&+\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}L_s^K)^2+\frac{\alpha^2}{2}\mathbb{E}\sum_{0\le s\le 1}(\underline{\Delta}L_s^A)^2 \\
&-e^{\alpha K}\mathbb{E}\sum_{0\le s\le 1}(1-e^{\alpha\overline{\Delta}L_s^K}) \\
&-\mathbb{E}\sum_{0\le s\le 1}e^{\alpha A_s}(1-e^{-\alpha\overline{\Delta}L_s^A}) \\
&+\alpha^2\mathbb{E}(A_0)\mathbb{E}X_1+\alpha^2\mathbb{E}\sum_{0\le s\le 1}A_s\overline{\Delta}L_s^A+o(\alpha^2).
\end{aligned}$$

Summing

Step 4.
After sending L to ∞ we get:

$$
\begin{aligned}
\alpha(1-e^{\alpha K}+\alpha\mathbb{E}A_0)l^K \;=\; & -\mathbb{E}X_1\alpha^2\int_0^K x\pi_K(dx)-\frac{\sigma^2\alpha^2}{2} \\
& -\frac{\alpha^2}{2}\int_0^a\int_z^K\int_{-x+z}^{K-x} y^2\nu(dy)\pi_K^z(dx)\xi(dz) \\
& +\frac{\alpha^2}{2}\int_0^a\int_z^K\int_{K-x}^{\infty}((x-K)^2+2y(x-K))\nu(dy) \\
& +\frac{\alpha^2}{2}\int_0^a\int_z^K\int_{-\infty}^{-x+z}((x-z)^2+2y(x-z))\nu(dy) \\
& +\alpha^2\mathbb{E}A_0\mathbb{E}X_1+o(\alpha^2)
\end{aligned}
$$

The proof follows by dividing both sides of above equation by $\alpha(1-e^{\alpha K}+\alpha\mathbb{E}A_0)$ and sending α to 0

Cramér asymptotics

Assume that jump measure ν is non-lattice and there exists $\gamma > 0$ such that

$$\kappa(\gamma) = 0$$

with $\kappa^{'}(\gamma) < \infty$ (hence $EX_1 < 0$)

Define:

$$\left.\frac{d\mathbb{P}^{\gamma}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\gamma X_t}$$

$$\tau_z^A(x) = \inf\{t \geq 0 : X_t \geq \widehat{A}_t^{-z} + x\}, \qquad \tau_{-z}^{-} = \inf\{t \geq 0 : X_t < -z\}$$

$$\tau^A(x) = \inf\{t \geq 0 : X_t \geq \widehat{A}_t^{\xi} + x\}$$ where $\widehat{A}_t^{\xi} = \int_0^{\infty} A_t^{-y}\,\xi(dy)$

$$B^A(x) = X_{\tau^A(x)} - x$$

- overshoot of the dual of the downward periodic barrier

Cramér asymptotics

We will write $f(K) \sim g(K)$ when $\lim_{K\to\infty} f(K)/g(K) = 1$

Theorem 2.

$$l^K \sim De^{-\gamma K}$$

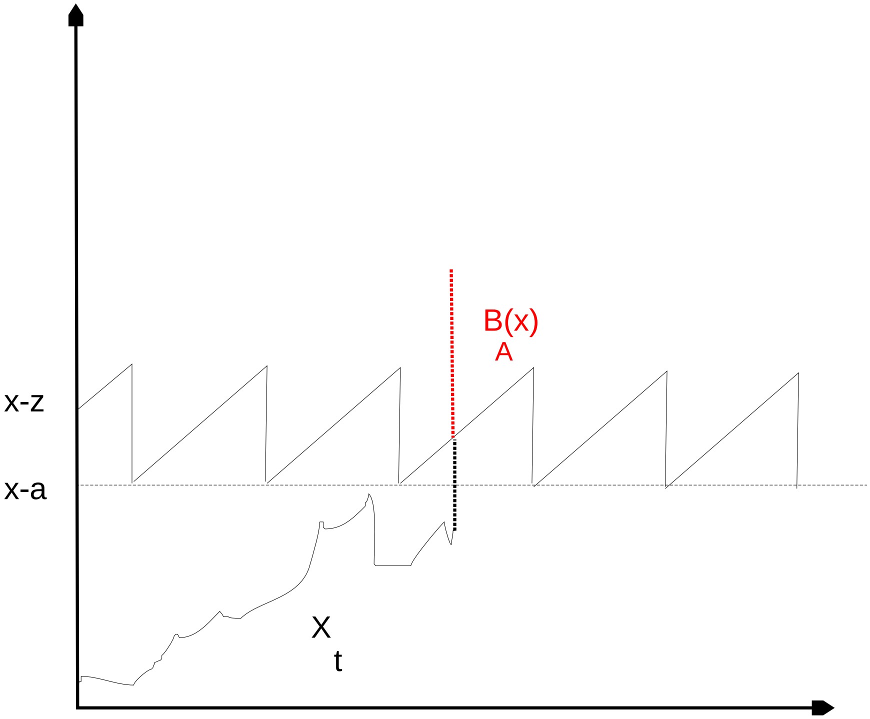
where

$$\begin{aligned} D \;=\; & -\mathbb{E}X_1 C_\gamma + \mathbb{E}^\gamma e^{-\gamma B^A(\infty)} \int_0^\infty e^{\gamma x}\mathbb{P}^\gamma(\tau^-_{-x} = \infty) \int_x^\infty \left(1 - e^{\gamma(y-x)}\right)\, \nu(dy) \\ & + \int_{-\infty}^0 \left(y + \gamma^{-1}\left(1 - e^{\gamma y}\right)\right)\, \nu(dy) \\ & + \int_0^\infty \int_0^{a\wedge x} \mathbb{P}(\tau^A_z(x) < \infty) \int_{-\infty}^{-x+z} \left(1 - e^{\gamma(x+y-z)}\right)\, \nu(dy)\,\xi(dz)\,dx \end{aligned}$$

with

$$C_\gamma = \mathbb{E}e^{\gamma A_s}, \qquad B^A(x) \xrightarrow{D} B^A(\infty)$$

Asymptotic overshoot



The proof is based on the exponential change of measure and regenerative arguments.

Exponential claims

$$X_t = \sum_{i=1}^{N_t} \sigma_i - t$$

with $\{\sigma_i\}_{\{i \geq 1\}}$ being i.i.d. Exp(μ)
and
N_t being a Poisson process with intensity $\lambda < \mu$

$$\varphi(t) = t \bmod a.$$

Then we have:

$$D = \frac{1}{a}\left(e^{a(\mu-\lambda)} - 1\right)\frac{\mu-\lambda}{\mu}\frac{\lambda}{\mu}$$

Heavy-tailed asymptotics

Assume:

$$EX_1 < 0$$

Define:

$$\nu_I(x) = \int_x^\infty \nu(y, \infty)\, dy$$

Theorem 3. (Andersen (2011)) If ν_I is subexponential and one of the following conditions holds:

(i) $EX_1^2 < \infty$ and $\int_K^\infty \nu_I(y)\, dy / \nu_i(x) = \mathrm{O}(K)$,

(ii) $\nu(K, \infty) \sim L(K) K^{-\alpha}$ for locally bounded slowly varying function L and $0 < \alpha < 2$,

then

$$l^K \sim \nu_I(K)$$

The proof is based on Theorem 1 and finding appropriate bounds.

Centered Lévy process - asym.

Assume:

$$EX_1 = 0$$

Theorem 4. (Andersen & Asmussen (2010)) (i) If $EX_1^2 < \infty$ then

$$l_K \sim \frac{1}{2K} \int_{-\infty}^{\infty} y^2 \, \nu(dy) + \frac{\sigma^2}{2K}$$

where σ is a Gaussian coefficient.

Centered Lévy process - asym.

Theorem 4. (Andersen & Asmussen (2010)) (ii) If for $1 < \alpha < 2$ and slowly varying functions L_1 and L_2:

$$\nu(x,\infty) = L_1(x)x^{-\alpha}, \quad \nu(-\infty,x) = L_2(x)|x|^{-\alpha}$$

such that

$$\lim_{x\to\infty} \frac{L_1(x)}{L_1(x)+L_2(x)} = d := \frac{\beta+1}{2}, \qquad \lim_{x\to\infty} L_0(x)^\alpha(L_1(x)+L_2(x)) = 1$$

for some slowly varying function L_0, then

$$l^K \sim \frac{\zeta}{K^{\alpha-1}L_0(K)^\alpha}$$

for

$$\zeta = \frac{cB(2-\alpha\rho,\alpha\rho) + dB(2-\alpha(1-\rho),\alpha(1-\rho))}{B(\alpha\rho,\alpha(1-\rho))(2-\alpha)(\alpha-1)}$$

and $c = 1-d$, $\rho = P(X_t > 0) = \frac{1}{2} + (\pi\alpha)^{-1}\arctan(\beta\tan(\pi\alpha/2))$
The proof is done by approximation.

THANK YOU

for Your Attention !