Precise large deviation probabilities for a heavy-tailed random walk [1]

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Large deviations for a heavy-tailed iid sequence

- We define heavy tails by regular variation of the tails.

- Assume that (X_t) is iid regularly varying, i.e. there exists an $\alpha > 0$, constants $p, q \geq 0$ with $p + q = 1$ and a slowly varying function L such that

$$P(X > x) \sim p \frac{L(x)}{x^\alpha} \quad \text{and} \quad P(X \leq -x) \sim q \frac{L(x)}{x^\alpha} \quad \text{as } x \to \infty.$$

- Define the partial sums

$$S_n = X_1 + \cdots + X_n \,, \quad n \geq 1 \,,$$

and assume $EX = 0$ if $E|X|$ is finite.

- **Large deviations** refer to sequences of rare events $\{b_n^{-1}S_n \in A\}$, i.e. $P(b_n^{-1}S_n \in A) \to 0$ as $n \to \infty$.
- For example, if $EX = 0$ and A is bounded away from zero then $P(n^{-1}S_n \in A) \to 0$ as $n \to \infty$, e.g. $P(|S_n| > \delta n) \to 0$.
- Then the following relations hold for $\alpha > 0$ and suitable sequences $b_n \uparrow \infty$[2]

$$\lim_{n\to\infty} \sup_{x \geq b_n} \left| \frac{P(S_n > x)}{n\, P(|X| > x)} - p \right| = 0\,.$$

- For fixed n and $x \to \infty$, the result is a trivial consequence of regular variation (subexponentiality); e.g. Feller (1971).

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[2] A.V. Nagaev (1969), S.V. Nagaev (1979), Cline and Hsing (1998), Heyde (1967)

- If $p > 0$, the result can be written in the form

$$\lim_{n\to\infty} \sup_{x \ge b_n} \left| \frac{P(S_n > x)}{P(M_n > x)} - 1 \right| = 0\,,$$

where $M_n = \max(X_1, \ldots, X_n)$.

- If $\alpha > 2$ one can choose $b_n = \sqrt{an \log n}$, where $a > \alpha - 2$, and for $\alpha \in (0, 2]$, $b_n = n^{1/\alpha + \delta}$ for any $\delta > 0$.

- In particular, one can always choose $b_n = \delta\, n$, $\delta > 0$, provided $E|X| < \infty$.

- For $\alpha > 2$ and $\sqrt{n} \le x \le \sqrt{an \log n}$, $a < \alpha - 2$, the probability $P(S_n - ES_n > x)$ is approximated by the tail of a normal distribution.

- **A functional (Donsker) version for multivariate regularly varying summands holds.** Hult, Lindskog, M., Samorodnitsky (2005).

- **Then, for example,** $P(\max_{i\le n} S_i > b_n) \sim c_{\max}\, n\, P(|X| > b_n)$ **provided** $b_n^{-1} S_n \stackrel{P}{\to} 0$.

- **The iid heavy tail large deviation heuristics:** ***Large values of the random walk occur in the most natural way: due to a single large step.***

- **This means: In the presence of heavy tails it is very unlikely that two steps X_i and X_j of the sum S_n are large.**

- **These results are in stark contrast with large deviation probabilities when X has exponential moments (Cramér-type large deviations). Then $P(|S_n - ES_n| > \varepsilon n)$ decays exponentially fast to zero.**[3]

[3] Cramér-type large deviations are usually more difficult to prove than heavy-tailed large deviations.

Ruin probabilities for an iid sequence

- Assume the conditions of Nagaev's Theorem: (X_t) iid regularly varying with index $\alpha > 1$ and $EX = 0$.
- For fixed $\mu > 0$ and any $u > 0$, consider the ruin probability

$$\psi(u) = P(\sup_{n\geq 1} (S_n - \mu\, n) > u)\,.$$

- It is in general impossible to calculate $\psi(u)$ exactly and therefore most results on ruin study the asymptotic behavior of $\psi(u)$ when $u \to \infty$.
- If the sequence (X_t) is iid it is well known[4] that

$$\psi(u) \sim \frac{u\, P(X > u)}{\mu\,(\alpha - 1)} \sim \frac{1}{\mu} \int_u^\infty P(X > x)\, dx\,, \quad u \to \infty\,.$$

[4] Embrechts and Veraverbeke (1982), also for subexponentials.

- There is a direct relation between large deviations and ruin:

$$
\begin{aligned}
uP(X > u)(1+\mu)^{-\alpha} &\sim P(S_{[u]} > [u]\,(1+\mu)) \\
&\le P(\sup_{n\ge 1}(S_n - \mu\, n) > u) \\
&\approx P(\sup_{M^{-1}\,u\le n\le M\,u}(S_n - \mu\, n) > u) \\
&\approx P(S_{[u]} > u)\,. \\
&\sim u\,P(X > u)
\end{aligned}
$$

- Lundberg (1905) and Cramér (1930s) proved that $\psi(u)$ decays exponentially fast if X has exponential moments.

Examples of regularly varying stationary sequences

Linear processes.

- Examples of linear processes are ARMA processes with iid noise (Z_t), e.g. the AR(p) and MA(q) processes

$$X_t = Z_t + \varphi_1 X_{t-1} + \cdots + \varphi_p X_{t-p}\,,$$
$$X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}\,.$$

- Linear processes constitute the class of time series which have been applied most frequently in practice.
- Linear processes are regularly varying with index α if the iid noise (Z_t) is regularly varying with index α.

- **Linear processes**

$$X_t = \sum_j \psi_j Z_{t-j}, \quad t \in \mathbb{Z},$$

with iid regularly varying noise (Z_t) with index $\alpha > 0$ and $EZ = 0$ if $E|Z|$ is finite:[5]

$$\frac{P(X > x)}{P(|Z| > x)} \sim \sum_j |\psi_j|^\alpha (p\, I_{\psi_j > 0} + q\, I_{\psi_j < 0}) = \|\psi\|_\alpha^\alpha, \quad x \to \infty.$$

- **Regular variation of X is in general not sufficient for regular variation of Z.** **Jacobsen, M., Samorodnitsky, Rosiński (2009, 2011)**

[5] Davis, Resnick (1985); M., Samorodnitsky (2000) under conditions which are close to those in the 3-series theorem.

Solutions to stochastic recurrence equation.

- For an iid sequence $((A_t, B_t))_{t\in\mathbb{Z}}$, $A > 0$, the stochastic recurrence equation

$$X_t = A_t X_{t-1} + B_t\,, \quad t \in \mathbb{Z}\,,$$

has a unique stationary solution

$$X_t = B_t + \sum_{i=-\infty}^{t-1} A_t \cdots A_{i+1} B_i\,, \quad t \in \mathbb{Z},$$

provided $E \log A < 0$, $E|\log|B|| < \infty$.

- The sequence (X_t) is regularly varying with index α which is the unique positive solution to $EA^\kappa = 1$ (given this solution exists) Kesten (1973), Goldie (1991) and

$$P(X > x) \sim c_\infty^+ x^{-\alpha}\,, \quad P(X \le -x) \sim c_\infty^- x^{-\alpha}\,, \quad x \to \infty\,.$$

- **The GARCH(1, 1) process**[6] **satisfies a stochastic recurrence equation: for an iid standard normal sequence (Z_t)**

$$\sigma_t^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2 \,.$$

The process $X_t = \sigma_t Z_t$ is regularly varying with index α satisfying $E(\alpha_1 Z^2 + \beta_1)^{\alpha/2} = 1$.

[6] Bollerslev (1986)

Large deviations for a regularly varying linear process

- Assume (Z_t) iid, regularly varying with index $\alpha > 1$ and $EZ = 0$, hence $EX = 0$.
- Consider the linear process

$$X_t = \sum_j \psi_j Z_{t-j}, \quad t \in \mathbb{Z}.$$

- Let $m_\psi = \sum_j \psi_j$ and $\|\psi\|_\alpha^\alpha = \sum_j |\psi_j|^\alpha (p\, I_{\psi_j > 0} + q\, I_{\psi_j < 0})$.
- Then M., Samorodnitsky (2000)

$$\lim_{n\to\infty} \sup_{x \ge b_n} \left| \frac{P(S_n > x)}{n\, P(|X| > x)} - \frac{p\,(m_\psi)_+^\alpha + q\,(m_\psi)_-^\alpha}{\|\psi\|_\alpha^\alpha} \right| = 0\,.$$

- The threshold b_n is chosen as in the iid case.

Ruin probabilities for a regularly varying linear process

- Assume (Z_t) iid, regularly varying with index $\alpha > 1$ and $EZ = 0$, hence $EX = 0$.

- Also assume $\sum_j |j\psi_j| < \infty$, excluding long range dependence.

- Then for $\mu > 0$ M., Samorodnitsky (2000)

$$\begin{aligned}\psi(u) &= P(\sup_{n\geq 1}(S_n - \mu\, n) > u) \\ &\sim \frac{p\,(m_\psi)_+^\alpha + q\,(m_\psi)_-^\alpha}{\|\psi\|_\alpha^\alpha}\,\frac{u\,P(X>u)}{\mu\,(\alpha-1)} \\ &\sim \frac{p\,(m_\psi)_+^\alpha + q\,(m_\psi)_-^\alpha}{\|\psi\|_\alpha^\alpha}\,\psi_{\mathrm{ind}}(u)\,, \quad u\to\infty\,.\end{aligned}$$

- The proof is purely probabilistic.

- The constants $\|\psi\|_\alpha^\alpha$ and $p\,(m_\psi)_+^\alpha + q\,(m_\psi)_-^\alpha$ are crucial for *measuring the dependence in the linear process* (X_t) *with respect to large deviation behavior and the ruin functional.*

- A quantity of interest in this context is related to the maximum functional $M_n = \max(X_1, \ldots, X_n)$.

- Assume $P(|X| > a_n) \sim n^{-1}$. Then, for $x > 0$,[7]

$$-x^\alpha \log P(a_n^{-1} M_n \le x) \to \frac{p\ \max_j(\psi_j)_+^\alpha + q\ \max_j(\psi_j)_-^\alpha}{\|\psi\|_\alpha^\alpha}\,.$$

- The right-hand expression is the extremal index of (X_t) and measures the degree of extremal clustering in the sequence.

[7] Rootzén (1978), Davis, Resnick (1985)

Large deviation probabilities for solutions to stochastic recurrence equations

- Assume Kesten's conditions for the stochastic recurrence equation $X_t = A_t X_{t-1} + B_t$, $t \in \mathbb{Z}$, and $A > 0$. Then for some $\alpha > 0$, constants $c_\infty^\pm \ge 0$ such that $c_\infty^+ + c_\infty^- > 0$

$$P(X \le -x) \sim c_\infty^- x^{-\alpha} \quad \text{and} \quad P(X > x) \sim c_\infty^+ x^{-\alpha}, \quad x \to \infty.$$

- Then Buraczewski, Damek, M., Zienkiewicz (2011) if $c_\infty^+ > 0$

$$\lim_{n\to\infty} \sup_{b_n \le x \le e^{s_n}} \left| \frac{P(S_n - ES_n > x)}{n\,P(X > x)} - c_\infty \right| = 0,$$

where $b_n = n^{1/\alpha}(\log n)^M$, $M > 2$, for $\alpha \in (1,2]$, and $b_n = c_n n^{0.5} \log n$, $c_n \to \infty$, for $\alpha > 2$, c_∞ corresponds to the case $B = 1$, and $s_n/n \to 0$.

- **Write $\Pi_{ij} = A_i \cdots A_j$. Then $X_i = \Pi_{1i} X_0 + \widetilde{X}_i$, where**

$$\widetilde{X}_i = \Pi_{2i} B_1 + \Pi_{3i} B_2 + \cdots + \Pi_{ii} B_{i-1} + B_i\,, \quad i \geq 1\,,$$

and

$$S_n = X_0 \sum_{i=1}^{n} \Pi_{1i} + \sum_{i=1}^{n} \widetilde{X}_i\,.$$

The summands $\widetilde{X}_i$ are chopped into distinct parts of length $\log x$ and sums are taken over disjoint blocks of length $\log x$. Then Nagaev-Fuk and Prokhorov inequalities for independent summands apply.

- The condition $s_n/n \to 0$ is essential.

- Also notice that Embrechts and Veraverbeke (1982)

$$P(X_0 \sum_{i=1}^{n} \Pi_{1i} > x) \le P(X_0 \sum_{i=1}^{\infty} \Pi_{1i} > x) \sim c\, x^{-\alpha} \log x\,.$$

- Then

$$\frac{P(X_0 \sum_{i=1}^{n} \Pi_{1i} > x)}{n\, P(X > x)} \quad \text{"}\le\text{"} \quad \frac{x^{-\alpha} \log x}{n\, x^{-\alpha}} = \frac{\log x}{n}\,.$$

Ruin probabilities for solutions to stochastic recurrence equations

- Under Kesten's conditions for the stochastic recurrence equation $X_t = A_t X_{t-1} + B_t$, $t \in \mathbb{Z}$, with $A, B > 0$, for $\mu > 0$,

$$\psi(u) = P(\sup_{n\geq 1} (S_n - ES_n - \mu n) > u)$$

$$\sim c_\infty \frac{u P(X > u)}{\mu (\alpha - 1)}$$

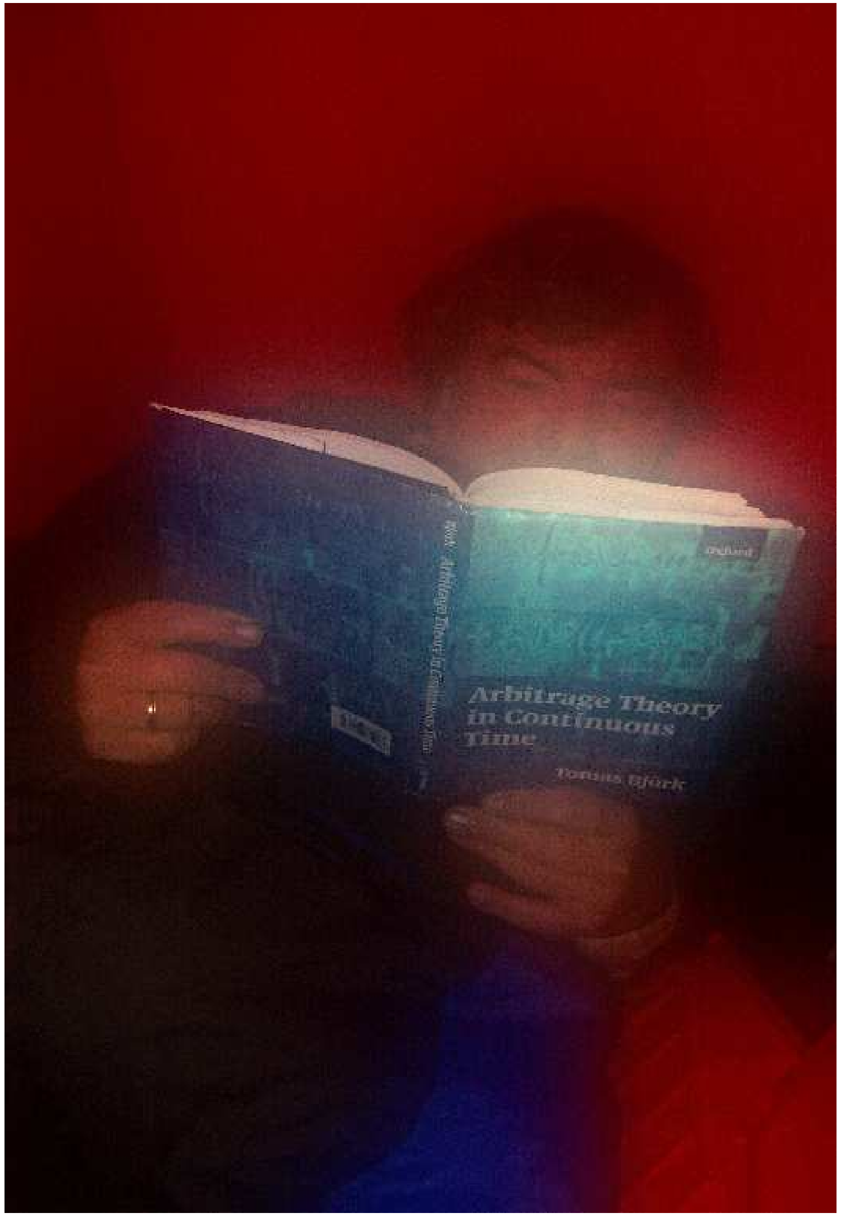
$$\sim c_\infty \psi_{\text{ind}}(u), \quad u \to \infty,$$

with Goldie's constant

$$c_\infty = \frac{E[(AX+1)^\alpha - (AX)^\alpha]}{\alpha E A^\alpha \log A} = \frac{E(AX+U)^{\alpha-1}}{EA^\alpha \log A}.$$

- The **extremal index** of the sequence (X_t) de Haan, Rootzén, Resnick, de Vries (1989)

$$-x^{\alpha} \log P(a_n^{-1} M_n \le x) \to c\,\alpha \int_1^{\infty} P(\max_{n\ge 1} \Pi_{1n} \le y^{-1})\, y^{-\alpha-1} dy\,.$$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Some other examples

- In the case of dependent stationary (X_t), assuming regular variation conditions with some index $\alpha < 2$, Jakubowski (1993,1997), Davis, Hsing (1995) show the existence of a sequence (c_n) such that $c_n^{-1} S_n \stackrel{P}{\rightarrow} 0$ and $\lim_{n\to\infty} \dfrac{P(S_n > c_n)}{n\,P(|X| > c_n)}$ exists and is positve.
- Konstantinides, M. (2005) prove precise Nagaev-type large deviations and ruin bounds for the solution to the stochastic recurrence equation $X_t = A_t X_{t-1} + B_t$ in the non-Kesten case when (B_t) is iid regularly varying with index $\alpha > 1$ and $EA^\alpha < 1$.
- In this case, the B-sequence determines the tail behavior of (S_n) and the A-sequence gets averaged.
- The results are similar to the linear process case.

- M., Samorodnitsky (2000) **prove precise ruin bounds for an ergodic stationary symmetric α-stable (sαs) sequence**

$$X_t = \int_E f_t(x)\, M(dx)\,, \quad t \in \mathbb{Z}\,,$$

where M is an sαs random measure with control measure μ and $\alpha \in (1,2)$.

- **Then (X_t) is regularly varying with index α and in particular**

$$P(X > x) = P(X \leq -x) \sim c_0\, x^{-\alpha}\,, \quad x \to \infty\,.$$

- **Conditions on (f_t) ensuring ergodicity and stationarity were proved by** Rosiński (1995).

- Tail bounds (large deviations) are trivial: for any $x = x_n \to \infty$, and $m_n = \left(\int_E \left| \sum_{i=1}^n f_t(x) \right|^\alpha \mu(dx) \right)^{1/\alpha}$, for an s$\alpha$s random variable M_0,

$$
\begin{aligned}
P(S_n > x) &= P\Big(\int_E \sum_{t=1}^n f_t(x) M(dx) > x \Big) \\
&= P(m_n M_0 > x) \\
&\sim m_n^\alpha \, P(M_0 > x)\,.
\end{aligned}
$$

- By ergodicity, $m_n = o(n)$.

- If $m_n = o(n^\beta)$ (mixing), some $\beta \in (0,1)$, bounds of the type $\psi(u) \sim u^{1-\alpha+\gamma} L(u)$ for $\gamma \leq \alpha - 1$ are possible.

- **For continuous-time processes $(S_t)_{t\geq 0}$ with stationary increments proof techniques for ruin can often be translated to other subadditive functionals acting on the sample paths of a random walk with negative drift without too much extra work.**

  Braverman, M., Samorodnitsky (2002).

- **Subadditivity of a functional f acting on the sample paths means**

$$f(\mathrm{x}+\mathrm{y}) \leq f(\mathrm{x}) + f(\mathrm{y})\,.$$

- **Examples: the supremum functional, the length of the period until the process is eventually negative, the length of the period the process is positive.**

Til lykke