Pricing CoCos with a market trigger

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The debt instrument known as *Contingent Convertible (CoCo)* has been introduced in the market to enhance more stability.

1. Upon the appearance of a trigger mechanism, related with the insolvency of the issuer, a *CoCo* is converted into a predefined number of shares (or it suffers a write-down) .
2. An important feature of this contract is that conversion is mandatory, as opposed to convertible bonds, where conversion is a choice that the investor has.

In 2007 a financial crisis, originated in the U.S. home loans market, quickly spread to other markets, sectors and countries, forcing the Federal Reserve and the European Central Bank to intervene in response to the collapse of the interbank market. This gave rise, in 2010, to new regulation rules, known as Basel III, that would change the financial landscape.

This is when *CoCos* started to play an important role. Basel III, among other regulating measures, proposed the inclusion of *CoCos* as part of Additional Tier 1 Capital.

It is a controversial issue if *CoCos* are a stabilizing security

Koziol, C., Lawrenz, J.: Contingent convertibles. solving or seeding the next banking crisis? Journal of Banking & Finance **36**(1), 90–104 (2012).

show that, under certain modelling assumptions, if *CoCos* are part of the capital structure of the company equity holders can take more risky strategies, trying to maximize the value of their shares.

Dewatripont, M., Tirole, J.: Macroeconomic shocks and banking regulation. Journal of Money, Credit and Banking **44**(s2), 237–254 (2012).

defend that *CoCos* is an appropriate solution that does not lead to moral hazard provided that conversion is tied to exogenous macroeconomic shocks.

There is also disagreement about how to establish the trigger event. It is perhaps the most controversial parameter in a *CoCo*. Some advocate conversion based on book values, like the different capital ratios used in Basel III. Others defend market triggers like the market value of the equity. So far the *CoCos* issued by the private sector are based on accounting ratios. From a modelling point of view and sometimes depending on the trigger chosen for the conversion, usually a low level of a certain index related with the asset, the debt or the equity of the firm, one can follow an intensity approach or a structural approach to model the trigger.

For an intensity approach when modelling the conversion time, see for instance:

- De Spiegeleer, J., Schoutens, W.: Pricing contingent convertibles: a derivatives approach. Journal of Derivatives 20(2), 27-36 (2012)
- Cheridito, P., Xu, Z.: A reduced form CoCo model with deterministic conversion intensity. Preprint (2013)

For a structural approach:

- Brigo, D., Garcia, J., Pede, N.: Coco bonds valuation with equity and credit-calibrated first passage structural models. Preprint (2013).
- Chen, N., Glasserman, P., Nouri, B., Pelger, M.: CoCos, bail-in, and tail risk. OFR Working paper. U.S. Department of the Treasury (2013)

One argument against accounting triggers is that monitoring is not continuous, there is always a delay in the information. Moreover, in the recent crisis these triggers did not provide any signal of distress in troubled banks. On the contrary, when using market triggers, there exist the risk of market manipulations of the equity price trying to force the conversion or undesirable phenomena like the *death-spiral effect*.

De~Spiegeleer, J., Schoutens, W.: Steering a bank around a death spiral: Multiple trigger cocos. Wilmott **2012**(59), 62–69 (2012)

authors propose a system of multiple triggers to avoid the death spiral, whereas in

Corcuera, J. M., De~Spiegeleer, J., Jönsson, H., Fajardo, J., Shoutens, W., Valdivia, A.: Close form pricing formulas for coupon cancellable cocos. Journal of Banking & Finance 42 (2014) 339-331.

a system of coupon cancellations is proposed in order to alleviate this effect.

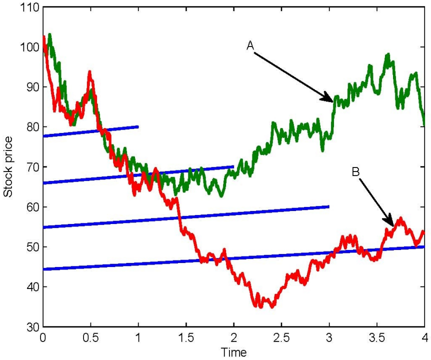

Figure: Illustration of a Coupon Cancellable CoCo with four payment dates ($T_1 = 1$, $T_2 = 2$, $T_3 = 3$, and $T_4 = 4$) with individual barriers $l^i(t) = L_i \exp(-r(T_i - t))$ ($L_1 = 80$, $L_2 = 70$, $L_3 = 60$, and $L_4 = 50$), and a risk-free rate of 3%.

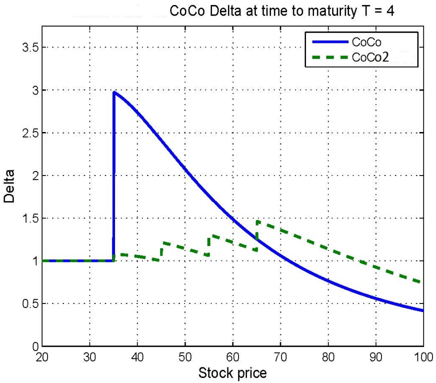

Figure: Illustration of the delta of a Coupon Cancellable Coco (CoCo2) versus the delta of a traditional CoCo for time to maturity $T = 4$ years. It pays annual coupons (15%). The conversion takes place at $S = 35$, and c_i-coupon cancelations at $S = 35 + 10(m - i)$, the conversion price $C_p = 100$ and the spot price $S_0 = 100$. The risk-free rate equals 3% and the volatility 40%.

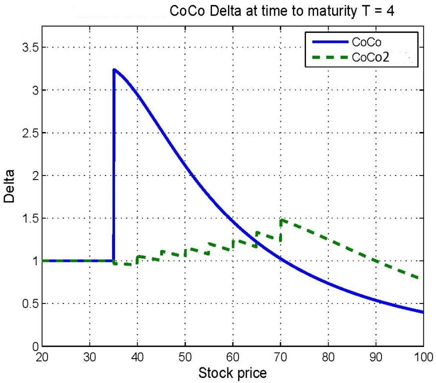

Figure: Illustration of the delta of a Coupon Cancellable Coco (Coco2) versus the delta of a traditional CoCo for time to maturity $T = 4$ years. It pays semiannual coupons (7.5%). The conversion takes place at $S = 35$ and c_i-coupon cancelation at $S = 35 + 5(m - i)$, the conversion price $C_p = 100$ and the spot price $S_0 = 100$. The risk-free rate equals 3% and the volatility equals $\sigma = 40\%$.

The definition of a *CoCo* requires the specification of the following parameters.

- K *Face value* of the *CoCo*.
- C_p *Conversion price*: the prefixed price of the share, for the investor, in case of conversion.
- T *Maturity of the CoCo.*
- $(t_i, c_i)_{i=1}^m$ *Coupon structure:* defines the time t_i at which an amount c_i is payed as coupon, $i = 1, ..., m$.
- τ_c *Conversion time*: the random time that defines when the *CoCo* conversion takes place. In other words, τ_c defines when the trigger mechanism takes place.

The contract has final payoff given by

$$K + \sum_{i=1}^{m} c_i \exp\left(\int_{t_i}^{T} r_u \mathrm{d}u\right) \mathbf{1}_{\{t_i < \tau_c\}} + \left(\frac{K}{C_p} S_T - K\right) \mathbf{1}_{\{\tau_c \leq T\}}.$$

The difference among the contributions in the literature are the way they model the conversion time, τ_c, the evolution of the stock $(S_t)_{t \geq 0}$ and the interest rates, $(r_t)_{t \geq 0}$.

All the mentioned papers consider a fixed maturity of the bond. However bonds often do not just have a legal maturity but can have also different call dates. In such cases, the bond can be called back by the issuer at these dates prior to the legal maturity. This is even more the case with perpetual instruments. In

Corcuera, J.M., Fajardo, J., Schoutens, W., Valdivia, A.: CoCos with extension risk: A structural approach. Social Science Research Network Working Paper Series (2014). http://dx.doi.org/10.2139/ssrn.2540625

we try to incorporate this risk of extension of the maturity of the contract into the price of CoCo bonds.

Corporate bonds with extension risk

Let $\pi(t; c, K, T)$ be the price, at time t, of a corporate bond with face value K that pays a coupon c and with maturity time T. Then we are going to consider a contract that pays a coupon c_1 before T_1 and at T_1 can pay K or to postpone the payment and to continue until T_2 but paying a higher coupon c_2 before T_2 and to pay the face value K at T_2. To postpone or not the payment depends on which is better for the issuer of the contract. In other words at time T_1 the payoff is

$$\min\{K, \pi(T_1; c_2, K, T_2)\} = K - (K - \pi(T_1; c_2, K, T_2))_+ .$$

T_1 is named the *call date* or *call time*, and the contract can be seen as consisting of a long position in a corporate bond with a coupon c_1, face value K and maturity time T_1, and a short position in a put option with strike K, maturity time T_1 and as underlying a corporate bond with a coupon c_2 (in the period (T_1, T_2)) face value K and maturity time T_2. We shall call this bond *a bond with extension risk* and we shall denote its price at time t by $\pi(t; c_1, c_2, K, T_1, T_2)$.

We can have a more complex contract with more call dates. For instance, consider now two call dates T_1, T_2, and a time T_3 that indicates the maturity of the contract. In the period $(0, T_1)$ this new contract, or bond, pays a coupon c_1, and K at T_1 or it postpones the latter payment depending on the condition $K > \pi(T_1; c_2, c_3, K, T_2, T_3)$ is satisfied or not. Then if $K \leq \pi(T_1; c_2, c_3, K, T_2, T_3)$ the contract continues and it pays c_2 in (T_1, T_2), and K at T_2 if $K > \pi(T_2; c_3, K, T_3)$ otherwise it pays the coupon c_3 and K at T_3.

Now again this contract can be seen as a long position in a bond with coupon c_1 a face value K and maturity time T_1 and a short position in a put option with strike K, maturity time T_1 and as underlying a *bond with extension risk* call date T_2 and maturity T_3, face value K and coupons c_2, c_3.

Suppose that, for any fixed c_2, K and T_2, $\pi(t; c_2, K, T_2) = f(S_t)$ where f is a continuous non-decreasing function and $(S_t)_{t\geq 0}$ is a positive process with *relative* independent increments, w.r.t. a risk neutral measure $\mathbb{P}^*$, then $\pi(t; c_1, c_2, K, T_1, T_2)$ will be an increasing function of S_t: in fact the payoff at T_1 is a non-decreasing function of S_{T_1} given by

$$\min\{K, \pi(T_1; c_2, K, T_2)\} = \min\{K, f(S_{T_1})\} := g(S_{T_1}),$$

Using the tilde for indicating discounted prices,

$$\begin{aligned}
&\tilde{\pi}(t; c_1, c_2, K, T_1, T_2) \\
&= c_1 \tilde{\pi}(t, T_{c_1}) + \mathbb{E}^* \left[\left. \frac{g(S_{T_1})}{B_{T_1}} \right| \mathcal{F}_t \right] = c_1 \tilde{\pi}(t, T_{c_1}) + \mathbb{E}^* \left. \left[\frac{g\left(\frac{S_{T_1}}{S_t} x\right)}{B_{T_1}} \right] \right|_{x=S_t} \\
&= c_1 \tilde{\pi}(t, T_{c_1}) + h(t, S_t), \; 0 \leq t \leq T_1
\end{aligned}$$

where, for shorten notation, we use $\tilde{\pi}(t, T_{c_1})$ instead of $\tilde{\pi}(t; 0, 1, T_{c_1})$, T_{c_1} is the time where the coupon c_1 is paid and

$$h(t, S_t) := \mathbb{E}^* \left. \left[\frac{g\left(\frac{S_{T_1}}{S_t} x\right)}{B_{T_1}} \right] \right|_{x=S_t},$$

that is a non decreasing function of S_t. $\mathbb{E}^*$ indicates the expectation with respect to $\mathbb{P}^*$.

So, if we consider two call dates T_1, T_2 and a maturity time T_3 we can write the payoff of this contract as

$$c_i \text{ in } (T_{i-1}, T_i) \text{ if } S_{T_0} < M_0, ..., S_{T_{i-1}} < M_{i-1}, i \geq 1$$
$$K \text{ at } T_i \text{ if } S_{T_1} < M_1, ..., S_{T_{i-1}} < M_{i-1}, S_{T_i} > M_i, i \geq 1$$

for certain constants M_i that depend on K and $c_{i+1}, .., c_3$ and the price formula $\pi(T_i; c_{i+1}, ..., c_3, K, T_{i+1}, ..., T_3)$:

$$M_i = \max \left\{ S_{T_i} \ : \ \pi(T_i; c_{i+1}, ..., c_3, K, T_{i+1}, ..., T_3) < K \right\}, i = 1, 2.$$

and the convention $M_0 = +\infty, M_3 = 0$.

Note that there is a backward procedure, in the sense that if we know how to obtain $\pi(T_2; c_3, K, T_3)$, a corporate bond without extension risk, we can calculate

$$M_2 = \max\{S_{T_2} \ : \ \pi(T_2; c_3, K, T_3) < K\}$$

and then we can obtain

$$\pi(T_1; c_2, c_3, K, T_2, T_3),$$

and from here we obtain M_1.

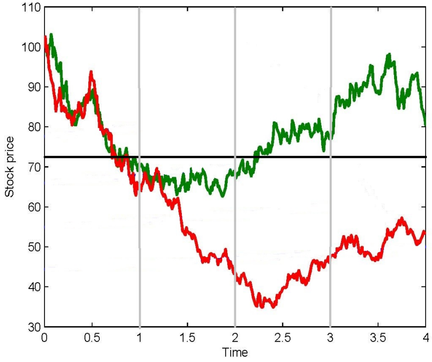

Figure: Illustration of a bond with callable dates

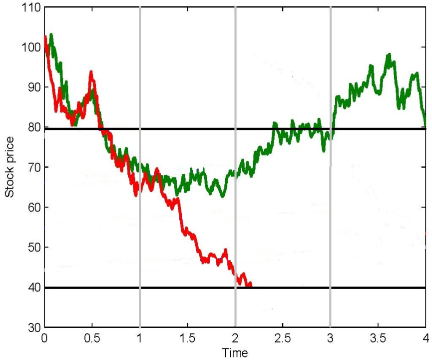

Figure: Illustration of a CoCo with callable dates

CoCos with Extension Risk

Then we will have that the payoff of this contract can be written as

$$c_{ij}\mathbf{1}_{\{\tau_{ij}>T_{ij},S_{T_{ij}}>M_{ij},S_{T_0}<M_0,\dots,S_{T_{i-1}}<M_{i-1}\}} \text{ at times } T_{ij}, i \geq 1,$$

$$K\mathbf{1}_{\{\tau_c>T_i,S_{T_0}<M_0,\dots,S_{T_{i-1}}<M_{i-1},S_{T_i}>M_i\}} \text{ at } T_i, i \geq 1,$$

$$\frac{K}{C_p}S_{\tau_c}\mathbf{1}_{\{\lceil\tau_c\rceil\leq T_N,S_{T_0}<M_0,\dots,S_{\lceil\tau_c\rceil-1}<M_{\lceil\tau_c\rceil-1}\}} \text{ at } \tau_c.$$

where $\lceil\tau_c\rceil$ is the element of $\{T_1, T_2, \dots, T_N\}$ such that $\lceil\tau_c\rceil - 1 < \tau_c \leq \lceil\tau_c\rceil$, and by convention $M_0 = \infty$ and $M_N = 0$.

So, on $\{t < \tau_c \wedge \tau\}$, where τ is the *call time*,

$$\begin{aligned}
\tilde{V}_t = & \sum_{i,j:T_{ij}>t} c_{ij}\mathbb{P}^* \left(\tau_{ij} > T_{ij}, S_{T_{ij}} > M_{ij}, S_{T_0} < M_0, ..., S_{T_{i-1}} < M_{i-1} \middle| \mathcal{F}_t \right) \\
& + \sum_{i:T_i>t} K\mathbb{P}^* \left(\tau_c > T_i, S_{T_0} < M_0, ..., S_{T_{i-1}} < M_{i-1}, S_{T_i} > M_i \middle| \mathcal{F}_t \right) \\
& + \frac{K}{C_p}\mathbb{E}^* \left[S_{\tau_c} \mathbf{1}_{\{\lceil \tau_c \rceil \le T_N, S_{T_0} < M_0, ..., S_{\lceil \tau_c \rceil - 1} < M_{\lceil \tau_c \rceil - 1}\}} \middle| \mathcal{F}_t \right].
\end{aligned}$$

Let us assume that all credit events are defined by the movements of $(S_t)_{t\geq 0}$ with respect to some critical barriers, in such a way that

$$\tau_c = \inf\{t \geq 0 \ : \ S_t \leq S^*\} \qquad \text{and} \qquad \tau_{ij} = \inf\{t \geq 0 \ : \ S_t \leq S^*_{ij}\}, \tag{1}$$

for a series of given constants S^* and $\{S^*_{ij}, \ i = 1, \ldots, N, j = 1, \ldots, m\}$, and with the usual convention of $\inf \emptyset = \infty$.

The following simple lemma give us a useful characterization of the discounted price of a CoCo with extension risk, in terms of the *share measure* $\mathbb{P}^{(S)}$. Recall that $\mathbb{P}^{(S)}$ is the risk-neutral probability measure, obtained by taking the share price as numéraire.

The discounted price of a CoCo with extension risk is given by

$$\begin{aligned}\tilde{V}_t = & \sum_{i,j:T_{ij}>t}^{N,m} c_{ij}\mathbb{P}^*\left(\underline{U}_{T_{ij}} > \log\frac{S^*_{ij}}{S^*}, U_{T_{ij}} > \log\frac{M_{ij}}{S^*}, U_{T_0} < \log\frac{M_0}{S^*}, \dots, U_{T_{i-1}} < \log\frac{M_{i-1}}{S^*}\,\middle|\, \mathcal{F}_t\right) \\ & + \sum_{i:T_i>t}^{N,m} K\mathbb{P}^*\left(\underline{U}_{T_i} > 0, U_{T_0} < \log\frac{M_0}{S^*}, .., U_{T_{i-1}} < \log\frac{M_{i-1}}{S^*}, U_{T_i} > \log\frac{M_i}{S^*}\,\middle|\, \mathcal{F}_t\right) \\ & + \frac{KS_t}{C_p}\sum_{i=1}^{N}\mathbb{P}^{(S)}\left(\underline{U}_{T_i} > 0, U_{T_0} < \log\frac{M_0}{S^*}, \dots, U_{T_{i-1}} < \log\frac{M_{i-1}}{S^*}, U_{T_i} \geq \log\frac{M_i}{S^*}\,\middle|\, \mathcal{F}_t\right),\end{aligned} \tag{2}$$

where $(U_t := \log\frac{S_t}{S^*})_{t\geq 0}$ and $\underline{U}_t := \inf_{0\leq u\leq t} U_u$.
This result shows that the computation of the price in (2) boils down to computating the joint distribution of $(\underline{U}_{T_n}, U_{T_1}, ..., U_{T_n})$.

Let us assume that the share price $(S_t)_{t\geq 0}$ is given as in the Black-Scholes model, *i.e.*,

$$\mathrm{d}S_t = S_t(r\mathrm{d}t + \sigma \mathrm{d}W_t^*),$$

where $(W_t^*)_{t\geq 0}$ follows a $\mathbb{P}^*$-Brownian motion. Further, let us assume that the critical barriers appearing in (1) are equal, that is, $S_{ij}^* = S^*$. Under these assumptions, the process $(U_t)_{t\geq 0}$ defined in (2) satisfies

$$U_t = \log \frac{S_t}{S^*} = \log \frac{S_0}{S^*} + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^*, \qquad t \geq 0.$$

Further, an application of the Girsanov theorem tells us that the dynamics of $(U_t)_{t\geq 0}$ under $\mathbb{P}^*$ and $\mathbb{P}^{(S)}$ differ only by the sign of its drift. Indeed, under $\mathbb{P}^{(S)}$ we have

$$U_t = \log \frac{S_0}{S^*} - \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{(S)}, \qquad t \geq 0.$$

Let $(W_t)_{t\geq 0}$ be a Brownian motion with respect to a probability measure $\mathbb{P}$, and denote its natural filtration by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. Let $(U_t := u_0 + \mu t + \sigma W_t)_{t\geq 0}$, $u_0 > 0$, and consider its first-passage time to the level zero, *i.e.*,

$$\tau := \inf\{t \geq 0 \ : \ U_t = 0\},$$

with the usual convention of $\inf \emptyset = \infty$.

The joint distribution of (τ, U_{T_1}) its well-known and, in particular, it allows to get, for any $a_1 \geq 0$, $T_1 > t$ and on the set $\{\tau > t\}$,

$$\begin{aligned}
&\mathbb{P}\left(\tau \geq T_1, U_{T_1} \geq a_1 \middle| \mathcal{F}_t\right) \\
&= \mathbb{P}\left(U_{T_1} \geq a_1 \middle| \mathcal{F}_t\right) - \mathrm{e}^{-2\mu\sigma^{-2}U_t}\mathbb{P}\left(U_{T_1} \leq -a_1 + 2\mu(T_1 - t) \middle| \mathcal{F}_t\right) \\
&= \Phi\left(\frac{-a_1 + U_t + \mu(T_1 - t)}{\sigma\sqrt{T_1 - t}}\right) - \mathrm{e}^{-2\mu\sigma^{-2}U_t}\Phi\left(\frac{-a_1 - U_t + \mu(T_1 - t)}{\sigma\sqrt{T_1 - t}}\right).
\end{aligned} \tag{3}$$

Now we extend this result in order to obtain an explicit expression for the joint finite-dimensional distributions of a drifted Brownian motion in different times and its infimum. The results are based in (3), an iterative procedure and the following lemmas.

Lemma

Consider the processes $(U_t)_{t\geq 0}$ *and* $(\eta_t^j)_{t\geq 0}$, $j = 1, \ldots, n$, *given by*

$$U_t := u_0 + \int_0^t \mu(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}W_s, \qquad t \geq 0,$$

$$\eta_t^j := \exp\left\{\int_0^{t\wedge T_j} \theta(s)\mathrm{d}W_s - \frac{1}{2}\int_0^{t\wedge T_j} \theta^2(s)\mathrm{d}s\right\}, \qquad t \geq 0,$$

where $\mu(s)$ *and* $\theta(s)$ *are deterministic and càdlàg functions. For every bounded function* F *and every* $0 \leq t \leq T_1 \leq T_n$ *the following equivalence holds true*

$$\mathbb{E}\left[e^{\int_0^{T_j} \theta(s)\mathrm{d}W_s - \frac{1}{2}\int_0^{T_j} \theta^2(s)\mathrm{d}s} F(U_{T_1}, \ldots, U_{T_n}) \middle| \mathcal{F}_t \right] =$$

$$\eta_t^j \mathbb{E}\left[F\left(U_{T_1} + \int_t^{T_1\wedge T_j} \sigma(s)\theta(s)\mathrm{d}s, \ldots, U_{T_n} + \int_t^{T_n\wedge T_j} \sigma(s)\theta(s)\mathrm{d}s \right) \middle| \mathcal{F}_t \right].$$

Lemma

Assume $\mu(s) \equiv \mu$. *On* $\{\tau > t\}$, *for* $a_1 \geq 0$ *and* $B_i \in \mathcal{B}(\mathbb{R})$, $i = 2, ..., n$,

$$\begin{aligned} &\mathbb{P}\left(\tau \geq T_1, U_{T_1} < a_1, U_{T_2} \in B_2, ..., U_{T_n} \in B_n \middle| \mathcal{F}_t\right) \\ &= \mathbb{P}\left(0 < U_{T_1} < a_1, U_{T_2} \in B_2, ..., U_{T_n} \in B_n \middle| \mathcal{F}_t\right) \\ &- \mathrm{e}^{-2\mu\sigma^{-2}U_t}\mathbb{P}\left(-a_1 < \bar{U}_{T_1} < 0, \bar{U}_{T_2} \in -B_2, ..., \bar{U}_{T_n} \in -B_n \middle| \mathcal{F}_t\right), \end{aligned}$$

with $\bar{U}_s = U_s - 2\mu(s-t), s \geq t$.

From this lemmas we have the following result.

Theorem

Let $a_1, \ldots, a_n$ be non-negative constants. On $\{\tau > t\}$, the following equation holds true

$$\mathbb{P}\left(\tau \geq T_n, U_{T_1} < a_1, \ldots, U_{T_{n-1}} < a_{n-1}, U_{T_n} > a_n \middle| \mathcal{F}_t\right)$$
$$= \mathbb{P}\left(-a_1 < U_{T_1} < a_1, \ldots, -a_{n-1} < U_{T_{n-1}} < a_{n-1}, U_{T_n} > a_n \middle| \mathcal{F}_t\right)$$
$$- \mathrm{e}^{-2\mu\sigma^{-1}U_t}\mathbb{P}\left(-a_1 < \bar{U}_{T_1} < a_1, \ldots, -a_{n-1} < \bar{U}_{T_{n-1}} < a_{n-1}, \bar{U}_{T_n} < -a_n \middle| \mathcal{F}_t\right)$$

where $\bar{U}_{T_j} = U_{T_j} - 2\mu(T_j - t)$, $j = 1, \ldots, n$.

The case of an infinite horizon can be treated using the results in

Peskir, G. and Shiryaev, A. N. (2006). Optimal Stopping and Free-Boundary Problems. Birkhäuser, Basel.

The details can be found in

Corcuera, J.M., Fajardo, J., Schoutens, W., Valdivia, A.: CoCos with extension risk: A structural approach. Social Science Research Network Working Paper Series (2014). http://dx.doi.org/10.2139/ssrn.2540625

CoCos under short-term uncertainty

Linking credit events to the movements of a *fully observable* (*i.e.*, $\mathbb{F}$-adapted) process $(U_t)_{t\geq 0}$ is certainly one of the most appealing features of structural models. Indeed, this *full observability* assumption gives rise to clear and analytically tractable models as we have seen in the previous sections. When considering contingent capital contracts such as *CoCos*, however, this assumption seems arguable since in most cases regulatory capital depends on the balance sheets of the issuer, and those sheets are updated only at series of predetermined dates $(t_j)_{t\in\mathbb{N}}$. Thus we are interested in considering the following *partial observability* assumption.

The fundamental process $(U_t)_{t\geq 0}$ is fully observable only at predetermined dates $(t_j)_{t\in\mathbb{N}}$. And the process $(U_t)_{t\geq 0}$ is related to the share price, in such a way that the correlation between the noises driving the share price and $(U_t)_{t\geq 0}$ is equal to ρ.
So far we assumed

$$\mathrm{d}S_t = S_t(r\mathrm{d}t + \sigma \mathrm{d}W_t^*),$$

and the cancellation of the j-th coupon is triggered as soon as the process

$$\mathrm{d}U_t := \mathrm{d}\log\frac{S_t}{\ell_t} = -\frac{1}{2}\sigma^2\mathrm{d}t + \sigma\mathrm{d}W_t^*,$$

wherer $(\ell_t)_{t\geq 0}$ is the barrier process, crosses the critical value $\log\frac{S_j^*}{L_j}$, $j = 1, ..., m$, whereas conversion coincides with the last coupon cancellation. In this new setting

$$\mathrm{d}U_t(\rho) := -\frac{1}{2}\sigma^2\mathrm{d}t + \sigma\mathrm{d}W_t^\rho := -\frac{1}{2}\sigma^2\mathrm{d}t + \sigma\mathrm{d}(\rho W_t^* + \sqrt{1-\rho^2}Z_t),$$

Further, the time at which the j-th may be cancelled is given by

$$\tau_j(\rho) := \inf\left\{ t \geq 0 \ : \ U_t(\rho) \leq \log \frac{S_j^*}{L_j} \right\}.$$

A full information flow corresponds to

$$\mathcal{G}_t := \sigma(W_s^*, Z_s, \ 0 \leq s \leq t) = \mathcal{F}_t^{W^*} \vee \mathcal{F}_t^{Z}, \qquad t \geq 0,$$

whereas, setting $\lfloor t \rfloor := \min\{t_j \in \{0, t_1, t_2, ...\} \ : \ t_j \leq t < t_{j+1}\}$, the partial information correspons to

$$\widetilde{\mathcal{F}}_t := \mathcal{F}_t^{W^*} \vee \sigma(Z_s, \ 0 \leq s \leq \lfloor t \rfloor) = \mathcal{F}_t^{W^*} \vee \mathcal{F}_{\lfloor t \rfloor}^{Z}, \qquad t \geq 0.$$

We define two auxiliar processes

$$\zeta_t := \sigma\sqrt{1-\rho^2}(t-\lfloor t\rfloor)Z_t \text{ and } \xi_t := \rho\log\frac{S_t}{S_{\lfloor t\rfloor}} + \rho\log\frac{\ell_t}{\ell_{\lfloor t\rfloor}}, \;\; t \geq 0.$$

These processes have an important role in the computations since they appear implicitely in $(U_t(\rho))_{t\geq 0}$ according to the factorization

$$U_t(\rho) = \left(U_{\lfloor t\rfloor}(\rho) + \xi_t\right) + \zeta_t.$$

It is apparent that the term between parentheses belongs to $\widetilde{\mathcal{F}}_t = \mathcal{F}_t^{W^*} \vee \mathcal{F}_{\lfloor t\rfloor}^Z$. On the other hand, ζ_t is independent of $\widetilde{\mathcal{F}}_t$, and it is normally distributed with zero mean and variance

$$\nu^2(t) := (1-\rho^2)(t-\lfloor t\rfloor)^2\sigma^2.$$

The standard deviation $\nu(t)$ represents a key quantity within this framework. Indeed, on the one hand, it actually encodes the two new features of our model: the factor $1-\rho^2$ measures how close $(S_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ are to being completely correlated; whereas the factor $t-\lfloor t\rfloor$ measures the elapsed time from the last information update.

For every $x \in \mathbb{R}$ define the random time $\tau_x(\rho) := \inf\{s \geq 0 \ : \ U_s(\rho) \leq x\}$. Then, for every $0 \leq t \leq T$, the following equation holds true on $\{\tau > \lfloor t \rfloor\}$

$$\begin{aligned}\mathbb{P}^*(\tau_x(\rho) \quad > \quad & T|\widetilde{\mathcal{F}}_t) = \mathbb{E}^*\left[\Phi\left(-D_- + \frac{\zeta_t}{\sigma\sqrt{T-t}}\right)\right] \\ & - \mathrm{e}^{-(U_{\lfloor t \rfloor}(\rho)-x)}\left(\frac{S_{\lfloor t \rfloor}\ell_{\lfloor t \rfloor}}{S_t \ell_t}\right)\mathbb{E}^*\left[\mathrm{e}^{-\zeta_t}\Phi\left(D_+ - \frac{\zeta_t}{\sigma\sqrt{T-t}}\right)\right],\end{aligned}$$

where the expectations above are restricted to the values

$$D_\pm = \frac{x - U_{\lfloor t \rfloor}(\rho) - \xi_t \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Moreover,

$$\mathbb{E}^*\left[\Phi\left(-D_- + \frac{\zeta_t}{\sigma\sqrt{T-t}}\right)\right] = \int_{\mathbb{R}} \Phi\left(-D_- + \frac{z\nu(t)}{\sigma\sqrt{T-t}}\right)\phi(z)\mathrm{d}z$$

and

$$\begin{aligned} & \mathbb{E}^*\left[\mathrm{e}^{-\zeta_t}\Phi\left(D_+ - \frac{\zeta_t}{\sigma\sqrt{T-t}}\right)\right] \\ = \quad & \mathrm{e}^{\frac{1}{2}\nu^2(t)}\int_{\mathbb{R}} \Phi\left(D_+ - \frac{z\nu(t)-\nu^2(t)}{\sigma\sqrt{T-t}}\right)\phi(z)\mathrm{d}z, \end{aligned}$$

where Φ and ϕ stand, respectively, for the standard Gaussian distribution and density functions.

CoCo prices can be obtained in our current setting once we compute expressions of the form

$$\mathbb{P}^*(\tau_j(\rho) > T_j, S_{T_j} > L_j | \widetilde{\mathcal{F}}_t), \text{ and } \mathbb{P}^{(S)}(\tau_m(\rho) > T_m, S_{T_m} > L_m | \widetilde{\mathcal{F}}_t).$$

It is worth noticing that the $\widetilde{\mathcal{F}}_t$-conditional joint distribution of $(\tau_j(\rho), S_{T_j}) = (\inf_{s \leq T_j} U_s(\rho), S_{T_j})$ cannot be computed directly from previous results since the entries of this vector are driven by two different (though correlated) Brownian motions.

Let $(W_t^1)_{t\geq 0}$ and $(W_t^2)_{t\geq 0}$ be two independent Brownian motions, and let $\mathbb{F}=(\mathcal{F}_t:=\sigma(W_s^1,W_s^2,\ s\leq t))_{t\geq 0}$. For $\sigma_1,\sigma_2>0$, $\mu_1,\mu_2\in\mathbb{R}$ and $0\leq\rho\leq 1$, consider the correlated Brownian motions given by

$$B_t^1:=\sigma_1 W_t^1+\mu_1 t,\ \text{and}\ B_t^2:=\sigma_2(\rho W_t^1+\sqrt{1-\rho^2}W_t^2)+\mu_2 t,\ \ t\geq 0.$$

For every $c,L\in\mathbb{R}$ and $0\leq t\leq T$ define $\tau_c^1=:\inf\{s\geq 0\ :\ B_s^1\leq c\}$ and

$$x:=\frac{-c-B_t^1+2\mu_1 t}{\sigma_1\sqrt{T-t}},\qquad y:=\frac{\mu_1}{\sigma_1}\sqrt{T-t},$$

$$d_\pm^1=\frac{c-B_t^1\pm\mu_1(T-t)}{\sigma_1\sqrt{T-t}}\qquad\text{and}\qquad d_\pm^2=\frac{L-B_t^2\pm\mu_2(T-t)}{\sigma_2\sqrt{T-t}}.$$

Then, on $\{\tau_c^1>t\}$, we have

(i) for $\rho = 1$

$$\begin{aligned} & \mathbb{P}(\tau_c^1 > T, B_T^2 \geq L | \mathcal{F}_t) \\ = \; & \Phi\left(-d_-^1 - \frac{L - \mu_2 T}{\sigma_2\sqrt{T-t}} - \frac{\mu_1 T}{\sigma_1\sqrt{T-t}}\right) \\ & -\mathrm{e}^{-2\mu_1\sigma_1^{-2}(B_t^1 - c)}\Phi\left(d_+^1 - \frac{L - \mu_2 T}{\sigma_2\sqrt{T-t}} - \frac{\mu_1 T}{\sigma_1\sqrt{T-t}}\right); \end{aligned}$$

(ii) for $0 < \rho < 1$

$$\begin{aligned} & \mathbb{P}(\tau_c^1 > T, B_T^2 \geq L | \mathcal{F}_t) \\ = \; & \Phi\left(-d_-^1\right) - 2\Phi\left(d_-^2\right) - \mathrm{e}^{-2\mu_1\sigma_1^{-2}(B_t^1 - c)}\Phi\left(d_+^1\right) \\ & -\mathrm{e}^{2xy}\Psi_\rho\left(-x-y, -2\rho x + d_-^2\right) + \Psi_\rho\left(x-y, d_-^2\right); \end{aligned}$$

(iii) and for $\rho = 0$

$$\mathbb{P}(\tau_c^1 > T, B_T^2 \geq L | \mathcal{F}_t) = \left(\Phi\left(-d_-^1\right) - \mathrm{e}^{-2\mu_1\sigma_1^{-2}(B_t^1 - c)}\Phi\left(d_+^1\right)\right)\Phi\left(-d_-^2\right),$$

Ψ_ρ stand for the bivariate normal distribution of two random variables with standard normal distribution and correlation ρ. Similar results holds for $\rho \in [-1, 0)$.

From here we can compute

$$\mathbb{P}^*(\tau_j(\rho) > T_j, S_{T_j} > L_j|\widetilde{\mathcal{F}}_t), \text{ and } \mathbb{P}^{(S)}(\tau_m(\rho) > T_m, S_{T_m} > L_m|\widetilde{\mathcal{F}}_t).$$

and to derive a formula for the price of Coco under this framework. More details can be found in

J.M. Corcuera, A. Valdivia. Cocos under short-term uncertainty. Preprint. 2015