Dynamic no good deal bounds: linear and convex price systems

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Outlines

1. No-good-deal bounds

2. Frictionless markets: linear price systems

3. Market with friction: convex pricing

1. No-good-deal bounds

Consider a model for a *frictionless*, generally *incomplete* market where *trading is continuous* over a time horizon $[0, T]$ $(T > 0)$.

Then the price $x_{st}(X)$ of any *financial position* X matured at t and purchased at s is a *linear* operator.

We study "fair" prices. These are characterized by an *equivalent martingale measure* (EMM). In fact this measure allows the representation of (discounted) linear prices in terms of conditional expectations with respect to the given information flow.
In incomplete markets an EMM is not unique and any EMM gives a fair price evaluation.

Surely, one can decide to select one EMM among the infinite available according to some optimality criteria or by some other specific argument, e.g. the statistical structure-preserving.

Also one can characterize EMMs that are in some sense “reasonable” in the sense that the pricing measure embodies some wished properties of the actual prices themselves.
This is the case for the *no-good-deal (NGD) pricing measures*.

No-good-deal bounds were introduced simultaneously by Cochrane and Saa Requejo (2000) and Bernardo and Ledoit (2000). The idea is to consider EMMs that not only rule out arbitrage possibilities, but also those deals that are “too good to be true”.

Static setting definition. The EMM $Q \sim P$ is a no good deal pricing measure at level δ if the Sharpe ratio is bounded for all X:

$$-\delta \le \frac{E[X] - E_Q[X]}{\sqrt{Var(X)}} \le \delta.$$

A glimpse to the literature:
– Cochrane and Saa Requejo (2000), Björk and Slinko (2006) start from a specific model for traded assets. The study leads to an upper good deal price process. The result depends on the chosen dynamics.
– Klöppel and Schweizer (2007) have a utility-based approach to restrict the set of EMMs providing bounds that are in some sense related to the Sharpe ratio. The approach depends on the shape of the densities of the EMM in the Lévy filtration of the prices.
– For a discussion on links between risk measures and no good deal pricing see e.g. Jaschke and Küchler (2001).

Goal: Study NGB for continuous dynamic trading in a setting independent of the specific underlying models of assets and choice of information flow.
We will consider both a frictionless market, linear pricing, and in markets with frictions, then we consider convex prices.
For this we take an axiomatic approach to price processes inspired by risk measures.

- In dN and Eide (2010) these time-consistent price systems where studied in L_p for $p \in [1, \infty)$ and a version of the fundamental theorem of asset pricing with pricing EMMs having bounds on the density was suggested.
- In Bion-Nadal (2009) a characterization of time-consistent dynamic risk processes is given for the L_∞ setting.

Here we give conditions for the existence of a NGD pricing measure, which corresponds to the NGD bounds on the linear prices.

Later, we study the NGD bounds for convex prices in connection with the NGD pricing measures.
Example of a convex price system can be given by risk-indifference pricing.

Framework

Information. $(\Omega, \mathcal{F}, P)$ complete. Right continuous P-augmented filtration $\mathbb{F} = \{\mathcal{F}_t \subseteq \mathcal{F}\,, t \in [0, T]\}$

Claims. For any time t, all market claims that are payable at time t constitute a linear sub-space:

$$L_t \subseteq L_p(\mathcal{F}_t): \qquad \begin{array}{l} L_t \subseteq L_T, \quad 1 \in L_t, \\ \forall A \in \mathcal{F}_t, \quad \forall X \in L_t, \quad 1_A X \in L_t. \end{array}$$

The corresponding cone of the non-negative elements is marked by the superscript $+$. We set:

$$\|X\|_p := \begin{cases} (E[|X|^p])^{1/p}, & p \in [1, \infty), \\ esssup|X|, & p = \infty. \end{cases}$$

N.B. In $p = \infty$, the L_∞ space with weak* topology is not metrizable.

Note that in a *complete* market $L_t = L_p(\mathcal{F}_t)$ for all $t \in [0, T]$. However, in general we have *incompleteness*, i.e. $L_t \subsetneq L_p(\mathcal{F}_t)$ for some $t \in [0, T]$.

A *numéraire* R_t, $t \in [0, T]$, is fixed in the market. Here $R_t \equiv 1$. Then prices and discounted prices will coincide.

2. Frictionless markets: linear price systems

Definition. For any $s, t \in [0, T]$, $s \leq t$, the operator $x_{st}(X)$, $X \in L_t$, with values in $L_p(\mathcal{F}_s)$ is a *linear price operator* if it is

- *monotone*, i.e. for any X', $X'' \in L_t$,

$$x_{st}(X') \geq x_{st}(X''), \quad X' \geq X'',$$

- *additive*, i.e. for any X', $X'' \in L_t$,

$$x_{st}(X' + X'') = x_{st}(X') + x_{st}(X'')$$

- *$\mathcal{F}_s$-homogeneous*, i.e.

$$x_{st}(\lambda X) = \lambda x_{st}(X)$$

  for all $X \in L_t$ and $\mathcal{F}_s$-measurable multipliers λ such that $\lambda X \in L_t$,

- $x_{st}(1) = 1$.

N.B. As a consequence of the above we also have that:

$$x_{st}(0) = 0, \quad x_{tt}(X) = X$$

A *linear price system* is a whole *time-consistent, right-continuous* family of price operators $x_{st}(X)$, $X \in L_t$, $0 \leq s \leq t \leq T$. If $p = \infty$, we consider prices that are also *continuous from above*.

Definition. Let $\mathcal{T} \subseteq [0, T]$. The family x_{st}, $s, t \in \mathcal{T} : s \leq t$, of price operators $x_{st}(X)$, $X \in L_t$, is *time-consistent* if, for all $s, u, t \in \mathcal{T}$: $s \leq u \leq t$,

$$x_{st}(X) = x_{su}\big(x_{ut}(X)\big),$$

for all $X \in L_t$ such that $x_{ut}(X) \in L_u$.

Definition. The family of price operators x_{st}, $0 \leq s \leq t \leq T$, is *right-continuous at s* if, for every $X \in L_t$,

$$x_{s't}(X) \longrightarrow x_{st}(X), \quad s' \downarrow s,$$

in L_p if $p \in [1, \infty)$ and $P - a.s.$ if $p = \infty$.

Definition. For $p = \infty$. Let $s \leq t$. The price operator $x_{st}(X)$, $X \in L_t$, is *continuous from above* at $X \in L_t$ if

$$x_{st}(X_n) \downarrow x_{st}(X) \ P - a.s., \qquad \text{for } X_n \downarrow X, \ P - a.s.$$

Observation The existence of an EMM is characterized by the possiblity of representing the whole linear price system as a conditional expectation:

$$x_{st}(X) = E_Q\left[X|\mathcal{F}_s\right], \quad \forall X \in L_t.$$

Naturally, this also entails the possibility of extending the price operators to be defined on the whole $L_p(\mathcal{F}_t)$ space.

Hence, our approach is to characterize the existence of a Q allowing the representation above and embedding the no good deal constrains.
Our results rely on:
– a representation theorem for linear price systems
– a sandwich preserving extension theorem for linear operators.

Representation theorem

The representation works if the prices are defined on the whole $L_t = L_p(\mathcal{F}_t)$!

Theorem [dN and Eide (2010), Bion-Nadal and dN (2013)].

Consider the linear price system:

$$x_{st}(X), \quad X \in L_p(\mathcal{F}_t), \quad 0 \le s \le t \le T.$$

Then there exists a $Q \sim P$ such that $\frac{dQ}{dP} = f \in L_q^+(\mathcal{F}_T)$ $(\frac{1}{q} + \frac{1}{p} = 1)$ with $f > 0$ P-a.s. which allows the representatation

$$x_{st}(X) = E_Q[X|\mathcal{F}_s] = E\Big[X \frac{f}{E[f|\mathcal{F}_s]} |\mathcal{F}_s\Big]$$

for all $s \le t$ and $X \in L_p(\mathcal{F}_t)$.

Sandwich extension theorem

In order to consider the case $L_t \subsetneq L_p(\mathcal{F}_t)$ we need the following results on extension theorems for operators.

To keep the setting general, we consider the following setup:

- $\mathcal{A} \subseteq \mathcal{B}$ and $p \in [1, \infty]$.
- let $M : L_p^+(\mathcal{B}) \longrightarrow L_p^+(\mathcal{A})$ be a sublinear operator
- let $m : L_p^+(\mathcal{B}) \longrightarrow L_p^+(\mathcal{A})$ be a superlinear operator.

Let $x : L \longrightarrow L_p(\mathcal{A})$ be a monotone linear operator satisfying the *sandwich condition*:

$$m(X) \leq x(X) \leq M(X), \quad X \in L^+,$$

where $L^+ = \{X \in L : \ X \geq 0\}$ or, equivalently,

$$m(Z) + x(X) \leq M(Y)$$
$$\forall Y, Z \in L_p^+(\mathcal{B}),\ X \in L : \quad Z + X \leq Y.$$

Theorem [Bion-Nadal and dN (2013); Albeverio, dN, Rozanov (2005)]

Let L be a linear subspace of $L_p(\mathcal{B})$.

Let M be a sublinear majorant operator, if $p = \infty$ also *regular*, i.e. $M(X_n) \longrightarrow 0 \; P-a.s.$ for any $X_n \downarrow 0 \; P-a.s.$

Let m be a superlinear minorant.

Let $x : L \longrightarrow L_p(\mathcal{A})$ be a linear operator satisfying the sandwich condition:

$$m(Z) + x(X) \leq M(Y)$$
$$\forall Y, Z \in L_p^+(\mathcal{B}),\ X \in L : \quad Z + X \leq Y.$$

Then x admits a monotone linear extension $\hat{x}$ on $L_p(\mathcal{B})$ and, if $p = \infty$, it is continuous from above. This extension is sandwich preserving:

$$m(Z) + \hat{x}(X) \leq M(Y)$$
$$\forall Y, Z \in L_p^+(\mathcal{B}),\ X \in L_p(\mathcal{B}) : \quad Z + X \leq Y.$$

or, equivalently,

$$m(X) \leq \hat{x}(X) \leq M(X), \quad X \in L_p^+(\mathcal{B}).$$

Fundamental theorem of asset pricing

Definition. The family M_{st}, $s, t \in [0, T]$, $s \leq t$, of $\mathcal{F}_s$-homogeneous, sublinear operators $M_{st} : L_p^+(\mathcal{F}_t) \to L_p^+(\mathcal{F}_s)$ is *weak time-consistent* if, for every $X \in L_p^+(\mathcal{F}_t)$,

$$M_{rs}(M_{st}(X)) \leq M_{rt}(X), \quad \forall r \leq s \leq t, \tag{2.1}$$

and

$$M_{st}(X) = lim_{t' \downarrow t} M_{st'}(X). \tag{2.2}$$

Analogously for the family m_{st}, $s, t \in [0, T]$, $s \leq t$, of $\mathcal{F}_s$-homogeneous, superlinear operators $m_{st} : L_p^+(\mathcal{F}_t) \to L_p^+(\mathcal{F}_s)$.

Remark. Every time-consistent family M_{st} of $\mathcal{F}_s$-homogeneous, sublinear operators such that (3.3) is satisfied is weak time-consistent.

Theorem [Bion-Nadal and dN (2013)]

Let M_{st}, $s, t \in [0, T]$, $s \leq t$, be a weak time-consistent family of regular $\mathcal{F}_s$-homogeneous, sublinear operators $M_{st} : L_p^+(\mathcal{F}_t) \to L_p^+(\mathcal{F}_s)$.

Let m_{st}, $s, t \in [0, T]$, $s \leq t$, be a weak time-consistent family of $\mathcal{F}_s$-homogeneous, superlinear operators $m_{st} : L_p^+(\mathcal{F}_t) \to L_p^+(\mathcal{F}_s)$.

Assume that $m_{0,T}$ is non-degenerate, i.e. $m_{0T}(X) > 0 \ P - a.s.$ for every $X > 0$ and that, for every $X \in L_p^+(\mathcal{F}_t)$, for every sequence s_n decreasing to s, we have

$$(2.3) \qquad M_{st}(X) \geq \liminf M_{s_n t}(X); \qquad m_{st}(X) \leq \limsup m_{s_n t}(X)$$

Let

$$x_{st}(X),\ X \in L_t,\ 0 \leq s \leq t \leq T,$$

be a time-consistent and right-continuous family of price operators. Suppose that the following sandwich condition is satisfied:

$$(2.4) \qquad m_{st}(X) \leq x_{st}(X) \leq M_{st}(X), \quad X \in L_t^+,$$

Then there exists a probability measure $Q \sim P$:

$$Q(A) = \int_A f(\omega) P(d\omega), \quad A \in \mathcal{F},$$

with $f \in L_q^+(\mathcal{F})$ and $E[f|\mathcal{F}_0] = 1$ such that

$$(2.5) \qquad m_{st}(X) \leq E_Q[X|\mathcal{F}_s] \leq M_{st}(X), \quad X \in L_\infty^+(\mathcal{F}_t).$$

and allowing the representation:

$$x_{st}(X) = E_Q[X|\mathcal{F}_s] = E\Big[X \frac{f}{E[f|\mathcal{F}_s]} |\mathcal{F}_s\Big], \quad X \in L_t,$$

for all price operators.

Obeservation. If $m_{0,T}$ is degenerate, then we have to work with the feasibility property for the prices: Assume that there exists a probability measure $\bar{Q} \sim P$ such that

$$(2.6) \qquad E_{\bar{Q}}[X|\mathcal{F}_s] \leq x_{s,T}(X), \qquad X \in L_T$$

No good deal EMMs

$p = 2$

Two equivalent definitions:

Static setting definition. The martingale measure $Q \sim P$ is a NGD pricing measure at level δ if the Sharpe ratio is bounded for all X:

$$-\delta \leq \frac{E[X] - E_Q[X]}{\sqrt{Var(X)}} \leq \delta.$$

Remark that the relation above implies $|E_Q[X] - E[X]| \leq \delta\sqrt{E[X^2]}$. Hence $X \Rightarrow E_Q[X] - E[X]$ is linear, continuous in the norm $L_2(\Omega)$. Being $L_2(\Omega)$ dual of the Banach space $L_2(\Omega)$. We conclude that $\frac{dQ}{dP} \in L_2(\mathcal{F}_T)$.

Static setting definition. The martingale measure $Q \sim P$ is a NGD pricing measure at level δ if $\frac{dQ}{dP} \in L_2(\mathcal{F}_T)$ satisfies:

$$E\left[\left(\frac{dQ}{dP} - 1\right)^2\right] \leq \delta^2.$$

Dynamic setting. For every $s \leq t$, define the set of probability measures:

$$\mathcal{Q}_{st} = \left\{ Q << P \text{ on } \mathcal{F}_t \;|\quad Q_{|\mathcal{F}_s} = P_{|\mathcal{F}_s} \text{ and } \frac{dQ}{dP} = 1 + g_{st}, \, E[g_{st}^2 | \mathcal{F}_s] \leq \delta_{st}^2 \right\}$$

with $\delta_{st} > 0$.

Lemma. Assume that the family δ_{st}, $0 \leq s \leq t \leq T$, satisfies

$$(1 + \delta_{rt}) = (1 + \delta_{rs})(1 + \delta_{st}), \quad \forall r \leq s \leq t. \tag{2.7}$$

Then for every $Q_{rs} \in \mathcal{Q}_{rs}$ and $Q_{st} \in \mathcal{Q}_{st}$, the probability measure Q_{rt}:

$$\frac{dQ_{rt}}{dP} := \frac{dQ_{rs}}{dP} \frac{dQ_{st}}{dP} \in \mathcal{Q}_{rt}.$$

Moreover, for all $A \in \mathcal{F}_s$ and $Q_1, Q_2 \in \mathcal{Q}_{st}$, the measure Q_3:

$$\frac{dQ_3}{dP} = \frac{dQ_1}{dP} 1_A + \frac{dQ_2}{dP} 1_{A^c} \in \mathcal{Q}_{st}.$$

Example: Condition (2.7) is satisfied by $\delta_{st} := \delta^{t-s} - 1$ for some $\delta > 1$.

Dynamic setting definition. A martingale measure $Q \sim P$ is a *dynamic NGD pricing measure* if $\frac{dQ}{dP} \in L_2(\mathcal{F}_T)$ satisfies

$$E\Big[\big(\Big(\frac{dQ}{dP}\Big)_t\Big(\frac{dQ}{dP}\Big)_s^{-1} - 1\big)^2|\mathcal{F}_s\Big] \leq \delta_{st}^2,$$

for every $s \leq t$ and constants $\delta_{st} > 0$ satisfying (2.7). Here $\big(\frac{dQ}{dP}\big)_t := E\big[\frac{dQ}{dP}|\mathcal{F}_t\big]$.

Proposition. In the context of the previous lemma, with $\delta_{st} \to 0$, $t \downarrow s$, we define:

$$M_{st}(X) := \mathrm{esssup}_{Q\in\mathcal{Q}_{st}} E_Q[X|\mathcal{F}_s], \quad X \in L_2^+(\mathcal{F}_t),$$
$$m_{st}(X) := \mathrm{essinf}_{Q\in\mathcal{Q}_{st}} E_Q[X|\mathcal{F}_s], \quad X \in L_2^+(\mathcal{F}_t).$$

Then $M_{st}(X)$, $X \in L_2^+(\mathcal{F}_t)$, $s, t \in [0, T] : s \leq t$, is a weakly time-consistent, regular family of sublinear, monotone, $\mathcal{F}_s$-homogeneous operators. Moreover we have:

$$M_{st}(X) \geq \liminf_{n\to\infty} M_{s_n t}(X), \quad X \in L_2^+(\mathcal{F}_t).$$

Accordingly similar results hold for the minorant operators.

Theorem [Bion-Nadal and dN (2013)]

Let x_{st}, $0 \leq s \leq t \leq T$, be a right-continuous time-consistent family of price operators defined on the linear space of marketed financial assets L_t. Assume that the family x_{st} satisfies the following sandwich condition:

$$m_{st}(X) \leq x_{st}(X) \leq M_{st}(X), \quad X \in L_t^+,$$

where M_{st} and m_{st} are defined as in the result above.
Assume the feasibility for the prices.

Then there exists a dynamic NGD pricing measure Q such that

$$x_{st}(X) = E_Q(X|\mathcal{F}_s), \qquad X \in L_t.$$

In fact the RHS defines an extension

$$\hat{x}_{st}(X) = E_Q(X|\mathcal{F}_s), \qquad X \in L_p(\mathcal{F}_t),$$

of the price system to time-consistent family of linear price operators $\hat{x}_{st}$ defined on all $L_p(\mathcal{F}_t)$ with values in $L_p(\mathcal{F}_s)$, such that

$$m_{st}(X) \leq \hat{x}_{st}(X) \leq M_{st}(X), \quad X \in L_p^+(\mathcal{F}_T). \tag{2.8}$$

3. Market with friction: convex pricing

Definition. For any $s, t \in [0, T]$ with $s \leq t$, the operator $x_{s,t} : L_t \longrightarrow L_s$ is a convex price operator if it is:

- *monotone*, i.e. for any $X', X'' \in L_t$,

$$x_{s,t}(X') \geq x_{s,t}(X''), \quad X' \geq X'',$$

- *convex*, i.e. for any $X', X'' \in L_t$ and $\lambda \in [0, 1]$,

$$x_{s,t}\big(\lambda X' + (1-\lambda)X''\big) \leq \lambda x_{s,t}(X') + (1-\lambda)x_{s,t}(X'')$$

- *lower semi-continuous*, i.e. for any $X \in L_t$ and any sequence $(X_n)_n$ in L_t with limit X,

$$\liminf_{n\to\infty} x_{s,t}(X_n) \geq x_{s,t}(X)$$

- *weak $\mathcal{F}_s$-homogeneous*, i.e. for all $X \in L_t$

$$x_{s,t}(1_A X) = 1_A x_{s,t}(X), \quad A \in \mathcal{F}_s,$$

- *projection property*

$$x_{s,t}(f) = f, \quad f \in L_p(\mathcal{F}_s) \cap L_t.$$

In particular we have $x_{s,t}(0) = 0$ and $x_{s,t}(1) = 1$.

Definition. The family of operators $x_{s,t} : L_t \Longrightarrow L_s$, $s, t \in [0, T]$: $s \le t$, of the type above is a *(convex) price system* if the family is:

- time-consistent (on $[0, T]$), i.e. for all $s, t, u \in [0, T]$: $s \le t \le u$

  $$x_{s,u}(X) = x_{s,t}\big(x_{t,u}(X)\big), \tag{3.1}$$

  for all $X \in L_u$
- right-continuous, i.e. for all t, all $X \in L_t$, and all sequences $(s_n)_n$, $s < s_n \le t$, $s_n \downarrow s$, $x_{s,t}(X) = \lim_{n\to\infty} x_{s_n,t}(X)$, where the convergence is P a.s.

Remark. The time-consistency and the projection property yield:

$$x_{s,t}(X) = x_{s,T}(X), \quad X \in L_t$$

Conditions: Bounds, sandwich, and conditions

Condition on bounds The family $(m_{s,t}, M_{s,t})_{s,t\in[0,T]}$ satisfies

1. the family $(m_{s,t}, M_{s,t})_{s,t\in\mathcal{T}}$ are weak time-consistent families of super-linear, respectively sub-linear, weak $\mathcal{F}_s$-homogeneous operators such that $m_{s,t},\ M_{s,t} : L_p(\mathcal{F}_t)^+ \longrightarrow L_p(\mathcal{F}_s)^+$, and $M_{s,t}$ is regular if $p = \infty$.
2. $\mathrm{esssup}_{s\leq T}(M_{s,T}(X))$ belongs to $L_p(\mathcal{F}_T)^+$ for all $X \in L_p(\mathcal{F}_T)^+$;
3. for every $X \in L_p(\mathcal{F}_t)^+$,

   (3.2) $$m_{s,t}(X) = lim_{t'>t,t'\downarrow t} m_{st'}(X);$$

4. for every $X \in L_p(\mathcal{F}_t)^+$,

   (3.3) $$M_{s,t}(X) = lim_{t'>t,t'\downarrow t} M_{st'}(X);$$

5. for every $X \in L_p(\mathcal{F}_t)^+$,

   (3.4) $$m_{s,t}(X) \leq \limsup_{s'>s,s'\downarrow s} m_{s't}(X); \qquad M_{s,t}(X) \geq \liminf_{s'>s,s'\downarrow s} M_{s't}(X);$$

Sandwich condition The price operators $(x_{s,t})_{s,t\in\mathcal{T}}$ satisfies the *sandwich condition* when

$$m_{s,t}(Z) + x_{s,t}(X) \leq M_{s,t}(Y) \tag{3.5}$$

$$\forall X \in L_t \quad \forall\, Y, Z \in L_p(\mathcal{F}_t)^+ : \quad Z + X \leq Y,$$

for some families of operators $(m_{s,t})_{s,t\in\mathcal{T}}$ and $(M_{s,t})_{s,t\in\mathcal{T}}$ with $m_{s,t},\ M_{s,t} : L_p(\mathcal{F}_t)^+ \longrightarrow L_p(\mathcal{F}_s)^+$.

Feasibility conditions

- The price system $(x_{s,t})_{s,t\in[0,T]}$ satisfies the *feasibility* property if there exists a probability measure $\bar{Q}$ equivalent to P such that

$$E_{\bar{Q}}[X|\mathcal{F}_s] \leq x_{s,T}(X), \qquad X \in L_T \tag{3.6}$$

- The couple $(x_{s,t}, M_{s,t})_{s,t\in[0,T]}$ is feasible if for some probability measure $\bar{Q}$ equivalent to P, we have both (3.6) and

$$E_{\bar{Q}}[X|\mathcal{F}_s] \leq M_{s,T}(X), \quad X \in L_p(\mathcal{F}_T)^+. \tag{3.7}$$

N.B. Instead of the feasibility condition we can have the non-degeneracy of the lower bound. For this see [Bion-Nadal and dN 2014]. In the NGD bounds we cannot guarantee the non-degeneracy.

Convex sandwich extension and representation

Theorem [Bion-Nadal and dN 2015, 2014].
Let us consider a right-continuous time-consistent system of convex operators $(x_{s,t})_{s,t\in[0,T]}$ defined on $(L_t)_{t\in[0,T]}$ satisfying the sandwich condition with mM and the feasibility property.
Then there is a right-continuous, time-consistent, sandwich preserving extension $(\hat{x}_{s,t})_{s,t\in[0,T]}$ defined on the whole $(L_p(\mathcal{F}_t))_{t\in[0,T]}$.
Furthermore for all $0 \le s \le t \in [0,T]$ and all $X \in (L_p(\mathcal{F}_t))_{t\in[0,T]}$,

$$\hat{x}_{s,t}(X) \ge E_{\bar{Q}}(X|\mathcal{F}_s) \tag{3.8}$$

One of these extensions can be represented as

$$\hat{x}_{s,t}(X) = \operatorname{esssup}_{R\in\mathcal{R}^e}[E_R(X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(R)], \quad X \in L_p(\mathcal{F}_t), \tag{3.9}$$

$$\mathcal{R}^e := \{R \sim P :\ \hat{\alpha}_{0,T}(R) < \infty\}$$

Also for any $X \in L_p(\mathcal{F}_t)$, there exists $R_X \in \mathcal{R}^e$ such that

$$\hat{x}_{s,t}(X) = E_{R_X}(X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(R_X) \quad \forall s \le t.$$

For all $s \le t$, $\hat{\alpha}_{s,t}(R)$ is the minimal penalty associated to $\hat{x}_{s,t}$, i.e.

$$\hat{\alpha}_{s,t}(R) = \operatorname{esssup}_{X\in L_p(\mathcal{F}_t)}[E_R(X|\mathcal{F}_s) - \hat{x}_{s,t}(X)]. \tag{3.10}$$

Furthermore for all t and all X, $\hat{x}_{s,t}(X)_{0\le s\le t}$ admits a càdlàg version.

Representation of the extensions

Theorem For all given $s \leq t$, the maximal extension $\hat{x}_{s,t}$ admits the following equivalent representations

(3.11) $\quad \hat{x}_{s,t}(X) = \mathrm{esssup}_{R \in \mathcal{R}_{s,t}} [E_R(X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(R)], \quad X \in L_p(\mathcal{F}_t),$

with $\mathcal{R}_{s,t} = \{R \ll P \text{ on } \mathcal{F}_t,\ R_{|\mathcal{F}_s} = P_{|\mathcal{F}_s} \text{ and } E(\hat{\alpha}_{s,t}(R)) < \infty\}$

(3.12) $\quad \hat{x}_{s,t}(X) = \mathrm{esssup}_{R \in \mathcal{Q}^S_{s,t}} [E_R(X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(R)], \quad X \in L_p(\mathcal{F}_t),$

where $\mathcal{Q}^S_{s,t} = \{R \ll P \text{ on } \mathcal{F}_t,\ R_{|\mathcal{F}_s} = P_{|\mathcal{F}_s},\ \textit{and } \frac{dQ}{dP} \in \mathbb{D}^S_{s,t}\}$.

(3.13) $\quad \hat{x}_{s,t}(X) = \mathrm{esssup}_{R \in \mathcal{Q}^{S,e}} [E_R(X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(R)], \quad X \in L_p(\mathcal{F}_t),$

where $\mathcal{Q}^{S,e} = \{R \sim P : \frac{(\frac{dQ}{dP})_t}{(\frac{dQ}{dP})_s} \in \mathbb{D}^S_{s,t},\ \forall 0 \leq s \leq t \leq T\}$.

Here above,

$$\mathbb{D}^S_{s,t} := \Big\{ f \in L_q(\mathcal{F}_t)^+, E[f|\mathcal{F}_s] = 1;$$
$$m_{s,t}(X) \leq E[fX|\mathcal{F}_s] \leq M_{s,t}(X),\ \forall X \in L_p(\mathcal{F}_t) \Big\}.$$

Furthermore for all $X \in \mathcal{F}_t$, there exists $R_X \in \mathcal{Q}^{S,e}$ such that for all $0 \leq s \leq t \leq T$,

$$\hat{x}_{s,t}(X) = E_{R_X}(X|\mathcal{F}_s) - \hat{\alpha}_{s,t}(R_X) \tag{3.14}$$

The maximal extension satisfies the feasibility property.

Remark Let $m_{0,T}$ be non-degenerate. Then there exits a probability measure $Q_0 \sim P$ such that $\hat{\alpha}_{s,t}(Q_0) = 0$ for all $0 \leq s \leq t \leq T$. See [dN and BN 2014].
The existence of such an equivalent probability measure is fundamental to extend the price system in continuous time such that the price system becomes a càdlàg process for every X. Notice that from the representation of the minimal penalty it follows that the price system $(x_{s,t})_{s,t\in[0,T]}$ together with the upperbound $(M_{s,t})_{s,t\in[0,T]}$ satisfy the *feasibility* property with $\bar{Q} = Q_0$:

$$E_{Q_0}[X|\mathcal{F}_s] \leq x_{s,T}(X), \qquad X \in L_T,$$
$$E_{Q_0}[X|\mathcal{F}_s] \leq M_{s,T}(X), \quad X \in L_p(\mathcal{F}_t)^+.$$

This is a motivation to assume the feasibility as a necessary assumption for the convex price system.

Proofs

- The linear pricing systems propose the challenges of progressing from a finite discrete, then countable discrete, to the continuous identifying the right extension which will turn out to be time-constistent. In fact it is not enough to chose any concatenation of extended operators to guarantee the global time-consistency!
- The convex case, provides similar challenges, but at various level. Even at the countable discrete case the definition of the the extension is delicate: representation problems and characterization of the penalty. The feasibility assumption is used in the passage from countable to continuous.

No arbitrage for time consistent convex extensions

Let $(\Pi_t)_{t\in[0,T]}$ be $\mathbb{F}$-adapted semimartingales locally bounded such that $\Pi_t \in L_t$ for all t.

These processes may represent the core set of underlyings, but also the value processes of self-financing admissible strategies.

Proposition Any convex time consistent feasible price system $\left(x_{s,t}\right)_{s,t\in[0,T]}$ on $L_p(\mathcal{F}_t) = L_t$ admits the representation:

$$x_{s,T}(X) = \mathrm{esssup}_{R\in\mathcal{Q}^e,\ \alpha_{0,T}(R)<\infty}[E_R(X|\mathcal{F}_s) - \alpha_{s,T}(R)] \tag{3.15}$$

with $\alpha_{s,T}$ the minimal penalty for $x_{s,T}$.

Let $(\Pi_t)_{t\in[0,T]}$ be a process such that for all t, $\Pi_t \in L_p(\mathcal{F}_t)$. The following assertions are equivalent

1. For all $n \in \mathbb{Z}$, for all $s \le t$, $x_{s,t}(n\Pi_t) = n\Pi_s$.
2. Every probability measure R in (3.15) is an EMM for the process $(\Pi_t)_{t\in[0,T]}$.

The feasibility property can be put in relationship with the concept of No-static-free-lunch as introduced in [Bion-Nadal 2009].

Corollary. It the convex price operator satisfies some sandwich bounds for (m, M) with all the conditions above, including the feasibility. Then:

$$\hat{x}_{s,T}(X) = \mathrm{esssup}_{R\in\mathcal{Q}^{S,e},\ \hat{\alpha}_{0,T}(R)<\infty}[E_R(X|\mathcal{F}_s) - \hat{\alpha}_{s,T}(R)]$$

And for the $(\Pi_t)_t$ as above: $\hat{x}_{s,T}(X) = \mathrm{esssup}_{R\in\mathcal{Q}^{S,e}\cap\mathcal{M}(\Pi),\ \hat{\alpha}_{0,T}(R)<\infty}[E_R(X|\mathcal{F}_s) - \hat{\alpha}_{s,T}(R)]$

NGD convex price systems

We recall the definition of NGD bounds and NGD pricing measures.
Dynamic setting definition. A martingale measure $Q \sim P$ is a *dynamic NGD pricing measure* if $\frac{dQ}{dP} \in L_2(\mathcal{F}_T)$ satisfies

$$E\Big[\Big(\Big(\frac{dQ}{dP}\Big)_t\Big(\frac{dQ}{dP}\Big)_s^{-1} - 1\Big)^2|\mathcal{F}_s\Big] \leq \delta_{st}^2,$$

for every $s \leq t$ and constants $\delta_{st} > 0$ satisfying (2.7). Here $\left(\frac{dQ}{dP}\right)_t := E\left[\frac{dQ}{dP}|\mathcal{F}_t\right]$.

Proposition. In the context of the previous lemma, with $\delta_{st} \to 0$, $t \downarrow s$, we define:

$$M_{st}(X) := \mathrm{esssup}_{Q\in\mathcal{Q}_{st}} \mathrm{E_Q}[\mathrm{X}|\mathcal{F}_\mathrm{s}], \quad \mathrm{X} \in \mathrm{L}_2^+(\mathcal{F}_\mathrm{t}),$$
$$m_{st}(X) := \mathrm{essinf}_{Q\in\mathcal{Q}_{st}} \mathrm{E_Q}[\mathrm{X}|\mathcal{F}_\mathrm{s}], \quad \mathrm{X} \in \mathrm{L}_2^+(\mathcal{F}_\mathrm{t}).$$

One can verify that these bounds satisfy all the required conditions, there is degeneracy in general, hence the feasibilty condition comes in.

Theorem Let the price system $(x_{s,t})_{s,t\in[0,T]}$, defined on $(L_t)_{t\in[0,T]}$ satisfies the sandwich condition with the no-good-deal bounds.
Then the convex price system admits a sandwich preserving extension on the whole $(L_2(\mathcal{F}_t))_{t\in[0,T]}$. The maximal of such extensions admits representation:

$$(3.16) \qquad \hat{x}_{s,t}(X) = \mathrm{esssup}_{\mathrm{Q}\in\mathbb{Q}^{\mathrm{S,e}}}\left[\mathrm{E_Q}[\mathrm{X}|\mathcal{F}_\mathrm{s}] - \hat{\alpha}_{\mathrm{s,t}}(\mathrm{Q})\right]$$

given in terms of no-good-deal measures:

$$(3.17) \qquad \mathbb{Q}^{S,e} := \left\{Q \sim P : \left(\frac{dQ}{dP}\right)_t \left(\frac{dQ}{dP}\right)_s^{-1} \in \mathcal{D}_{s,t}\right\}.$$

Moreover, for all $X \in L^2(\mathcal{F}_t)$, there exists $Q_X \in \mathbb{Q}^{S,e}$ such that

$$(3.18) \qquad \hat{x}_{s,t}(X) = E_{Q_X}[X|\mathcal{F}_s] - \hat{\alpha}_{s,t}(Q_X).$$

Also we have equivalence between:

$$\mathcal{D}_{st} := \left\{1 + g_{s,t} : \ g_{st} \in L^2(\mathcal{F}_t),\ E[g_{s,t}|\mathcal{F}_s] = 0\ E\left[g_{st}^2|\mathcal{F}_s\right] \leq \delta_{st}^2\right\}$$

$$\mathbb{D}_{s,t}^S := \left\{f \in \mathbb{D}_{s,t},\ m_{s,t}(X) \leq E[fX|\mathcal{F}_s] \leq M_{s,t}(X),\ \forall X \in L_p(\mathcal{F}_t)\right\}.$$

NGD bounds and no-arbitrage

Fix $(\Pi_t)_t$ $\mathbb{F}$-adapted semimartingales locally bounded such that, for all t, $\Pi_t \in L_t$ and $x_{s,t}(\Pi_t) = \Pi_s$.

Proposition Let the price system $(x_{s,t})_{s,t\in[0,T]}$, defined on $(L_t)_{t\in[0,T]}$ satisfies the sandwich condition with the no-good-deal bounds.
The process $(\Pi_t)_{0\leq t\leq T}$, $\Pi_t \in L_t$, is an underlying model for the convex price system $(x_{s,t})_{s,t\in[0,T]}$ if and only if the maximal sandwich preserving extension $(\hat{x}_{s,t})_{s,t\in[0,T]}$ of $(x_{s,t})_{s,t\in[0,T]}$ on the whole $(L_2(\mathcal{F}_t))_{t\in[0,T]}$ admits the following representation for all $X \in L_2(\mathcal{F}_T)$.

$$\hat{x}_{s,T}(X) = \mathrm{esssup}_{\mathrm{Q}\in\mathbb{Q}^{\mathrm{S,e}}\cap\mathcal{M}(\Pi)}\left[\mathrm{E_Q}[\mathrm{X}|\mathcal{F}_\mathrm{s}] - \hat{\alpha}_{\mathrm{s,T}}(\mathrm{Q})\right] \tag{3.19}$$

given in terms of no-good-deal measures $\mathbb{Q}^{S,e}$ and of EMM for the process (Π_t) $\mathcal{M}(\Pi)$:

$$\mathbb{Q}^{S,e} := \left\{Q \sim P : \left(\frac{dQ}{dP}\right)_t \left(\frac{dQ}{dP}\right)_s^{-1} \in \mathbb{D}^S_{s,t}\right\}. \tag{3.20}$$

Moreover, for all $X \in L^2(\mathcal{F}_t)$, there exists $Q_X \in \mathbb{Q}^{S,e} \cap \mathcal{M}(\Pi)$ such that

$$\hat{x}_{s,t}(X) = E_{Q_X}[X|\mathcal{F}_s] - \hat{\alpha}_{s,t}(Q_X). \tag{3.21}$$

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