Exchangeability and infinite divisibility

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Overview

- Exchangeability of infinitely divisible laws
- Matrices that preserve exchangeability
- Mappings arising from time series or stochastic integrals that preserve exchangeability

Permutations on the integers

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- notice that $P_\pi (X_1, \ldots, X_d)^T = (X_{\pi^{-1}(1)}, \ldots, X_{\pi^{-1}(d)})^T$
- permutation matrices are orthogonal, i.e.

$$P_\pi P_\pi^T = Id_d, \quad P_\pi^T = P_\pi^{-1} = P_{\pi^{-1}}.$$

Exchangeability of random vectors and measures

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  A measure μ on $(\mathbb{R}^d, \mathcal{B}_d)$ is exchangeable, if $P_\pi \mu = \mu$ for all permutations $\pi \in [d]$.

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A matrix $\Sigma \in \mathbb{R}^{d\times d}$ *commutes with permutations* if

$$P_\pi \Sigma = \Sigma P_\pi \quad \forall\, \pi \in [d].$$

Exchangeable infinitely divisible distributions

Theorem

Let μ be an infinitely divisible distribution on $\mathbb{R}^d$ with characteristic exponent Ψ_μ and characteristic triplet (A, ν, γ), i.e.
$\widehat{\mu}(z) = \exp(\Psi_\mu(z))$ *with*

$$\Psi_\mu(z) = -\frac{1}{2} z^T A z + i\gamma^T z + \int_{\mathbb{R}^d} (e^{ix^T z} - 1 - ix^T z \mathbf{1}_{|x| \leq 1}) \, \nu(dx).$$

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(i) μ *is exchangeable.*
(ii) $\Psi_\mu(P_\pi z) = \Psi_\mu(z)$ *for all* $z \in \mathbb{R}^d$ *and* $\pi \in [d]$.
(iii) The Gaussian covariance matrix A commutes with permutations, the Lévy measure ν is exchangeable and $\gamma_i = \gamma_j$ for all $i, j \in \{1, \ldots, d\}$, where γ_i denotes the i'th component of γ.

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$$\nu(B) = \int_{\mathbb{S}} \lambda(d\xi) \int_0^\infty \mathbb{1}_B(r\xi) \frac{dr}{r^{1+\alpha}}, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

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Corollary

Let μ be an α-stable distribution with characteristic triplet $(0, \nu, (\gamma_1, \ldots, \gamma_d)^T)$, where $\alpha \in (0, 2)$. Then μ is exchangeable if and only if the spherical part λ of ν is exchangeable and if $\gamma_i = \gamma_j$ for all $i, j \in \{1, \ldots, d\}$.

Lévy copulas: definition; Tankov (2003), Cont and Tankov (2004)

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$$U_\nu(x_1,\ldots,x_d) := \begin{cases} \nu([x_1,\infty) \times \ldots \times [x_d,\infty)), & (x_1,\ldots,x_d) \neq \mathbf{0}, \\ \infty, & (x_1,\ldots,x_d) = \mathbf{0}. \end{cases}$$

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- define further the marginal tail integrals U_{ν_i} by

$$U_{\nu_i}(x_i) = U_\nu(0,\dots,0,x_i,0,\dots,0) = \begin{cases} \nu_i([x_i,\infty)), & x_i\in(0,\infty], \\ \infty, & x_i = 0. \end{cases}$$

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  - C is d-increasing

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- the Lévy copula is uniquely determined on $U_{\nu_1}([0,\infty]) \times \ldots \times U_{\nu_d}([0,\infty])$
- if $\nu_1,\ldots,\nu_d$ are one-dim. Lévy measures on $[0,\infty)$ and if C is a positive Lévy copula, then the rhs of (3) defines the tail integral of a Lévy measure ν on $[0,\infty)^d$ with margins $\nu_1,\ldots,\nu_d$ and Lévy copula C, see Cont and Tankov (2004).

Lévy copulas and Lévy measures with stable margins

- Barndorff-Nielsen and L. (2007):
  if C is a positive Lévy copula, then there exists a unique Lévy measure ν_C on $[0,\infty)^d$ such that

  $$\nu_C([x_1^{-1},\infty)\times\ldots\times[x_d^{-1},\infty)) = C(x_1,\ldots,x_d) \quad \forall\, x_1,\ldots,x_d \in [0,\infty] \tag{4}$$

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  which has unit 1-stable margins.
- conversely, to any Lévy measure ν_C on $[0, \infty)^d$ with unit 1-stable margins, the left-hand side of (4) defines a positive Lévy copula.

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- Theorem (Exchangeability and Lévy copulas)

  (i) A positive Lévy copula C is exchangeable if and only if the Lévy measure ν_C with unit 1-stable margins defined by (4) is exchangeable.

  (ii) Let ν be a Lévy measure on $[0,\infty)^d$ with marginal Lévy measures $\nu_1,\ldots,\nu_d$. If $\nu_1 = \ldots = \nu_d$ and if an associated positive Lévy copula C exists which is exchangeable, then ν is exchangeable.

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Further, $\mathcal{E}_d^0$ can be characterized as

$$\mathcal{E}_d^0 = \{A \in \mathbb{R}^{d\times d} : \exists\, a, b \in \mathbb{R}, a \neq 0, a \neq -db, \pi \in [d] \\ \text{such that } A = aP_\pi + bJ_d\}.$$

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(ii) Let $C \in \mathbb{R}^{d\times d}$. Then $\mathcal{L}(CX)$ is exchangeable for every exchangeable normal distribution $\mathcal{L}(X)$ on $\mathbb{R}^d$ if and only if C can be represented in the form $C = AQ$, where $A \in \mathbb{R}^{d\times d}$ commutes with permutations and $Q = (q_{ij})_{i,j=1,\ldots,d} \in \mathbb{R}^{d\times d}$ is an orthogonal matrix that satisfies $\sum_{j=1}^d q_{ij} = \sum_{j=1}^d q_{1j}$ for all $i \in \{1,\ldots,d\}$.

Exchangeability preserving transformations

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two classes of measures on $\mathbb{R}^d$ and $G : \mathcal{M}_1 \to \mathcal{M}_2$ a mapping. We say that G

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(ii) commutes with permutations if $P_\pi \mu \in \mathcal{M}_1$ for all $\mu \in \mathcal{M}_1$ and $\pi \in [d]$, and

$$P_\pi G(\mu) = G(P_\pi \mu) \quad \forall\, \mu \in \mathcal{M}_1, \quad \pi \in [d].$$

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Theorem

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two classes of measures on $\mathbb{R}^d$. Then every mapping $G : \mathcal{M}_1 \to \mathcal{M}_2$ that commutes with permutations is exchangeability preserving.

Theorem

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two classes of measures on $\mathbb{R}^d$ and $G : \mathcal{M}_1 \to \mathcal{M}_2$ an injective mapping that commutes with permutations. Then its inverse $G^{-1} : G(\mathcal{M}_1) \to \mathcal{M}_1$ also commutes with permutations, in particular G^{-1} is exchangeability preserving.

Convolution with an exchangeable distribution I

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  Then ρ is exchangeable, μ is not exchangeable, and

  $$\rho * \mu = \mathcal{L}((X_1 + X_3, X_2 + X_4)^T)$$

  is exchangeable.

- the convolution of two non-exchangeable distributions can be exchangeable

Exchangeability of stationary solution of AR(1) equation

Theorem

Let $\Phi \in \mathbb{R}^{d\times d}$ such that all eigenvalues of Φ lie in $\{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{M}_1$ be the set of all probability distributions $\mathcal{L}(X)$ on $\mathbb{R}^d$ with $\mathbb{E}\log^+|X| < \infty$ and $\mathcal{M}_2$ be the set of all probability distributions on $\mathbb{R}^d$. Consider the mapping

$$G_\Phi : \mathcal{M}_1 \to \mathcal{M}_2, \quad \mathcal{L}(Z_0) \mapsto \mathcal{L}\left(\sum_{k=0}^{\infty} \Phi^k Z_{-k}\right), \quad t \in \mathbb{Z},$$

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It associates to every i.i.d. noise sequence $(Z_k)_{k\in\mathbb{Z}}$ the marginal stationary distribution of the causal multivariate AR(1) process

$$Y_t - \Phi Y_{t-1} = Z_t, \quad t \in \mathbb{Z}, \tag{5}$$

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(ii) Let $\mathcal{M}_1'$ be the subset of all i.d. $\mu \in \mathcal{M}_1$, and let $G_\Phi' := G_\Phi|_{\mathcal{M}_1'}$. If Φ commutes w.p., then G_Φ' commutes w.p., G_Φ' is injective, and the inverse $(G_\Phi')^{-1} : G_\Phi(\mathcal{M}_1') \to \mathcal{M}_1'$ commutes w.p.

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- Similar results hold for stationary solutions of random recurrence equations $Y_t - \Phi_t Y_{t-1} = Z_t, \quad t \in \mathbb{Z}$.

Exchangeability preserving integrals of Lévy processes

Theorem

Let $L = (L_t^\mu)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Lévy process with distribution μ, and $f = (f(t))_{t\geq 0}$ an $\mathbb{R}^{d\times d}$-valued deterministic function. Let $\mathcal{M}_1$ be the set of all distributions μ on $\mathbb{R}^d$ for which $\mathcal{L}(\int_0^\infty f(t)\,dL_t^\mu)$ is defineable and consider the mapping

$$G : \mathcal{M}_1 \to \mathcal{M}_2, \quad \mu \mapsto \mathcal{L}\left(\int_0^\infty f(t)\,\mathrm{d}L_t^\mu\right),$$

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(i) Suppose that f takes only values in the set of exchangeability preserving matrices. Then G is exchangeability preserving.
(ii) Suppose that f takes only values in the set of matrices that commute with permutations. Then G commutes with permutations. If additionally G is injective, then the inverse $G^{-1} : G(\mathcal{M}_1) \to \mathcal{M}_1$ also commutes with permutations, so that in this case, $\int_0^\infty f(t)\,\mathrm{d}L_t^\mu$ is exchangeable if and only if μ is exchangeable.

Self-decomposable distributions

Let $c > 0$ be fixed. A distribution σ on $\mathbb{R}^d$ is self-decomposable if and only if it can be represented as an integral

$$\sigma = \mathcal{L}\left(\int_0^\infty e^{-ct}\, dL_t^\mu\right)$$

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Remark

Extensions to A-decomposable distributions (with a matrix A, under certain conditions), can also be formulated.

The Upsilon transform

For an infinitely divisible distribution μ, the Upsilon transform is defined by

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log\frac{1}{t}\, dL_t^{\mu}\right)$$

(Barndorff–Nielsen and Thorbjørnsen, 2004). It defines a bijection onto the Goldie–Steutel–Bondesson class (Barndorff–Nielsen, Maejima, Sato, 2006).

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Remark

More general Upsilon transforms have been defined in Barndorff–Nielsen, Rosinski and Thorbjørnsen (2008). These commute with permutations, and they are injective if a certain cancellation property for the multiplicative convolution holds. In that case, similar results can be obtained.

Happy birthday, Ole