Optimal control for an insider

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Aarhus Conference of Probability, Statistics and their Applications,
Celebrating the Scientific Achievements of
Ole E. Barndorff-Nielsen
15-19 June 2015

Acknowledgments/thanks:

This talk is based on joint papers with

Olfa Draouil, University of Tunis El Manar.

This research was carried out with support of
Centre for Advanced Study (CAS),
at the Norwegian Academy of Science and Letters, within the research program
Stochastics in Environmental and Financial Economics (SEFE).

Mexico City University 1997 (?)

2. Introduction

In this talk we present a general method for solving *optimal insider control problems*, i.e. optimal stochastic control problems where the controller has access to information about the future of the system. This *inside information* in the control process puts the problem outside the context of semimartigale theory, and we therefore apply general *anticipating white noise calculus*, including *forward integrals* and *Hida-Malliavin calculus*. Combining this with the *Donsker delta functional* for the random variable Y which represents the inside information, we are able to prove both a sufficient and a necessary maximum principle for the optimal control of such systems.
We then apply this machinery to the problem of optimal portfolio for an insider in a jump-diffusion financial market, and we obtain explicit expressions for the optimal insider portfolio in several cases, extending results that have been obtained earlier (by other methods) in [PK], [BØ], [DMØP2] and [ØR1].

We now explain this in more detail:
The system we consider, is described by a stochastic differential equation driven by a Brownian motion $B(t)$ and an independent compensated Poisson random measure $\tilde{N}(dt, d\zeta)$, jointly defined on a filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions. We assume that the inside information is of *initial enlargement* type. Specifically, we assume that the inside filtration $\mathbb{H}$ has the form

$$\mathbb{H} = \{\mathcal{H}_t\}_{t\geq 0}, \text{ where } \mathcal{H}_t = \mathcal{F}_t \vee Y \tag{2.1}$$

for all t, where Y is a given $\mathcal{F}_{T_0}$-measurable random variable, for some $T_0 > T$ (both constants).

We assume that the value at time t of our insider control process $u(t)$ is allowed to depend on both Y and $\mathcal{F}_t$. In other words, u is assumed to be $\mathbb{H}$-adapted. Therefore it has the form

$$u(t,\omega) = u_1(t, Y, \omega) \tag{2.2}$$

for some function $u_1 : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $u_1(t, y)$ is $\mathbb{F}$-adapted for each $y \in \mathbb{R}$. For simplicity (albeit with some abuse of notation) we will in the following write u in stead of u_1.

Consider a controlled stochastic process $X(t) = X^u(t)$ of the form (2.3)

$$\begin{cases} dX(t) = b(t, X(t), u(t), Y)dt + \sigma(t, X(t), u(t), Y)dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t), u(t), Y, \zeta)\tilde{N}(dt, d\zeta); \quad t \geq 0 \\ X(0) = x, \quad x \in \mathbb{R}, \end{cases}$$

where $u(t) = u(t, y)_{y=Y}$ is our insider control and the (anticipating) stochastic integrals are interpreted as *forward integrals*, as introduced in [RV] (Brownian motion case) and in [DMØP1] (Poisson random measure case). A motivation for using forward integrals in the modelling of insider control is given in [BØ].

Let $\mathcal{A}$ be a given family of admissible $\mathbb{H}-$adapted controls u. The *performance functional* $J(u)$ of a control process $u \in \mathcal{A}$ is defined by

$$J(u) = \mathbb{E}[\int_0^T f(t, X(t), u(t))dt + g(X(T))] \tag{2.4}$$

We consider the problem to find $u^\star \in \mathcal{A}$ such that

$$\sup_{u \in \mathcal{A}} J(u) = J(u^\star). \tag{2.5}$$

We use the Donsker delta functional of Y to transform this anticipating system into a classical (albeit parametrised) adapted system with a non-classical performance functional. Then we solve this transformed system by using modified maximum principles.

Here is an outline of the content of the paper:

- In Section 2 we discuss properties of the Donsker delta functional and its conditional expectation and Hida-Malliavin derivatives.
- In Section 3 we present the general insider control problem and its transformation to a more classical problem.
- In Sections 4 and 5 we present a sufficient and a necessary maximum principle, respectively, for the transformed problem.
- Then in Section 6 we illustrate our results by applying them to optimal portfolio problems for an insider in a financial market.

List of notation:

- $F \diamond G =$ the Wick product of random variables F and G.
- $F^{\diamond n} = F \diamond F \diamond F \ldots \diamond F$ (n times). (The n'th Wick power of F).
- $\exp^{\diamond}(F) = \Sigma_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n}$ (The Wick exponential of F.)
- $D_t F =$ the Hida-Malliavin derivative of F at t with respect to $B(\cdot)$.
- $D_{t,z} F =$ the Hida-Malliavin derivative of F at (t, z) with respect to $N(\cdot, \cdot)$.
- $D.(\varphi^{\diamond}(F)) = ((\varphi)')^{\diamond}(F) \diamond D.F$,
  where $D. = D_t$ or $D. = D_{t,z}$.(The Wick chain rule.)
- $(\mathcal{S}), (\mathcal{S})^* =$ the Hida stochastic test function space and stochastic distribution space, respectively.
- $(\mathcal{S})_1, (\mathcal{S})_{-1} =$ the Kondratiev stochastic test function space and stochastic distribution space, respectively.
- $(\mathcal{S})_1 \subset (\mathcal{S}) \subset L^2(P) \subset (\mathcal{S})^* \subset (\mathcal{S})_{-1}$.

3. The Donsker delta functional

Definition

Let $Z : \Omega \to \mathbb{R}$ be a random variable which also belongs to $(\mathcal{S})^*$. Then a continuous functional

$$\delta_Z(.) : \mathbb{R} \to (\mathcal{S})^* \tag{3.1}$$

is called a *Donsker delta functional* of Z if it has the property that

$$\int_{\mathbb{R}} g(z)\delta_Z(z)dz = g(Z) \quad a.s. \tag{3.2}$$

for all (measurable) $g : \mathbb{R} \to \mathbb{R}$ such that the integral converges.

The Donsker delta functional is related to the *regular conditional distribution*. The connection is the following:

Define the *regular conditional distribution* with respect to $\mathcal{F}_t$ of a given real random variable Y, denoted by $Q_t(dy) = Q_t(\omega, dy)$, by the following properties:

- For any Borel set $\Lambda \subseteq \mathbb{R}$, $Q_t(\cdot, \Lambda)$ is a version of $\mathbb{E}[\mathbf{1}_{Y\in\Lambda}|\mathcal{F}_t]$
- For each fixed ω, $Q_t(\omega, dy)$ is a probability measure on the Borel subsets of $\mathbb{R}$.

It is well-known that such a regular conditional distribution always exists. See e. g. [B], page 79.
From the required properties of $Q_t(\omega, dy)$ we get the following formula

$$\int_{\mathbb{R}} f(y) Q_t(\omega, dy) = \mathbb{E}[f(Y)|\mathcal{F}_t] \tag{3.3}$$

Comparing with the definition of the Donsker delta functional, we obtain the following representation of the regular conditional distribution:

Theorem

Suppose $Q_t(\omega, dy)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Then the Donsker delta functional of Y, $\delta_Y(y)$, exists and we have

$$\frac{Q_t(\omega, dy)}{dy} = \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] \tag{3.4}$$

A general expression, in terms of Wick calculus, for the Donsker delta functional of an Itô diffusion with non-degenerate diffusion coefficient can be found in the amazing paper [LP]. See also [MP]. In the following we present more explicit formulas the Donsker delta functional and its conditional expectation and Hida-Malliavin derivatives, for Itô - Lévy processes:

3.1. The Donsker delta functional for a class of Itô - Lévy processes

Consider the special case when Y is a first order chaos random variable of the form

$$Y = Y(T_0); \text{ where}$$

(3.5)

$$Y(t) = \int_0^t \beta(s)dB(s) + \int_0^t \int_{\mathbb{R}} \psi(s,\zeta)\tilde{N}(ds,d\zeta); t \in [0, T_0]$$

for some deterministic functions β, ψ satisfying

$$\int_0^{T_0} \{\beta^2(t) + \int_{\mathbb{R}} \psi^2(t,\zeta)\nu(d\zeta)\}dt < \infty \text{ a.s.} \tag{3.6}$$

We also assume that the following holds thoughout this paper: For every $\epsilon > 0$ there exists $\rho > 0$ such that

$$\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} e^{\rho\zeta} d\nu(\zeta) < \infty. \tag{3.7}$$

In this case it is well known (see e.g. [MØP], [DiØ1], [DiØ2], and [DØP]) that the Donsker delta functional exists in $(\mathcal{S})^*$ and is given by

$$\begin{aligned}\delta_Y(y) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp^{\diamond}\Big[&\int_0^{T_0}\int_{\mathbb{R}}(e^{ix\psi(s,\zeta)}-1)\tilde{N}(ds,d\zeta)\\ &+\int_0^{T_0} x\beta(s)dB(s)\\ &+\int_0^{T_0}\{\int_{\mathbb{R}}(e^{ix\psi(s,\zeta)}-1-ix\psi(s,\zeta))\nu(d\zeta)\\ &+\frac{1}{2}x^2\beta^2(s)\}ds - ixy\Big]dx. \end{aligned} \tag{3.8}$$

We will need the following result:

Lemma

$$
\begin{aligned}
&\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\Big[\int_0^t \int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t x\beta(s)dB(s) \\
&+ \int_t^{T_0}\int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta))\nu(d\zeta)ds \\
&+ \int_t^{T_0} \frac{1}{2}x^2\beta^2(s)ds - ixy\Big] dx
\end{aligned}
\tag{3.9}
$$

Next, we need the following:

Lemma

(3.10)

$$\mathbb{E}[D_{t,z}\delta_Y(y)|\mathcal{F}_t]$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\Big[\int_0^t \int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t x\beta(s)dB(s)$$
$$+ \int_t^{T_0}\int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta))\nu(d\zeta)ds + \int_t^{T_0} \frac{1}{2}x^2\beta^2(s)ds - ixy\Big]$$
$$\times (e^{ix\psi(t,z)} - 1)dx.$$

Finally, we need the following result:

Lemma

(3.11)

$$
\begin{aligned}
&\mathbb{E}[D_t\delta_Y(y)|\mathcal{F}_t] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\Big[\int_0^t \int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t x\beta(s)dB(s) \\
&+ \int_t^{T_0}\int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta))\nu(d\zeta)ds \\
&+ \int_t^{T_0} \frac{1}{2}x^2\beta^2(s)ds - ixy\Big]x\beta(t)dx.
\end{aligned}
$$

3.2. The Donsker delta functional for a Gaussian process

Consider the special case when Y is a Gaussian random variable of the form

$$(3.12)\quad Y = Y(T_0); \text{ where } Y(t) = \int_0^t \beta(s)dB(s), \text{ for } t \in [0, T_0]$$

for some deterministic function $\beta \in \mathbf{L}^2[0, T_0]$ with

$$(3.13)\qquad \|\beta\|^2_{[0,T]} := \int_t^T \beta(s)^2 ds > 0 \text{ for all } t \in [0, T].$$

In this case it is well known that the Donsker delta functional is given by

$$(3.14)\qquad \delta_Y(y) = (2\pi v)^{-\frac{1}{2}} \exp^{\diamond}[-\frac{(Y-y)^{\diamond 2}}{2v}]$$

where we have put $v := \|\beta\|^2_{[0,T_0]}$. See e.g. [AaØU], Proposition 3.2.

Using the Wick rule when taking conditional expectation, using the martingale property of the process $Y(t)$ and applying Lemma 3.7 in [AaØU] we get

$$
\begin{aligned}
&\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] \\
&= (2\pi v)^{-\frac{1}{2}} \exp^{\diamond}[-\mathbb{E}[\frac{(Y(T_0)-y)^{\diamond 2}}{2v}|\mathcal{F}_t]] \\
&= (2\pi\|\beta\|^2_{[0,T_0]})^{-\frac{1}{2}} \exp^{\diamond}[-\frac{(Y(t)-y)^{\diamond 2}}{2\|\beta\|^2_{[0,T_0]}}] \\
&= (2\pi\|\beta\|^2_{[t,T_0]})^{-\frac{1}{2}} \exp[-\frac{(Y(t)-y)^{2}}{2\|\beta\|^2_{[t,T_0]}}].
\end{aligned}
\tag{3.15}
$$

Similarly, by the Wick chain rule and Lemma 3.8 in [AaØU] we get, for $t \in [0, T]$,

$$
\begin{aligned}
&\mathbb{E}[D_t \delta_Y(y) | \mathcal{F}_t] \\
&= -\mathbb{E}[(2\pi v)^{-\frac{1}{2}} \exp^{\diamond}[-\frac{(Y(T_0) - y)^{\diamond 2}}{2v}] \diamond \frac{Y(T_0) - y}{v} \beta(t) | \mathcal{F}_t] \\
&= -(2\pi v)^{-\frac{1}{2}} \exp^{\diamond}[-\frac{(Y(t) - y)^{\diamond 2}}{2v}] \diamond \frac{Y(t) - y}{v} \beta(t)
\end{aligned}
$$

(3.16)

$$
= -(2\pi \|\beta\|^2_{[t,T_0]})^{-\frac{1}{2}}) \exp[-\frac{(Y(t) - y)^2}{2\|\beta\|^2_{[t,T_0]})}] \frac{Y(t) - y}{v} \beta(t).
$$

3.4 The Donsker delta functional for a Poisson process

Next, assume that $Y = Y(T_0)$, such that

$$Y(t) = \tilde{N}(t) = N(t) - \lambda t,$$

where $N(t)$ is a Poisson process with intensity $\lambda > 0$.

In this case the Lévy measure is $\nu(d\zeta) = \lambda\delta_1(d\zeta)$ since the jumps are of size 1. Comparing with (3.8) and by taking $\beta = 0$ and $\psi = 1$, we obtain

(3.17)

$$\delta_Y(y) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp^{\diamond}\left[(e^{ix}-1)\tilde{N}(T_0) + \lambda T_0(e^{ix}-1-ix) - ixy\right]dx$$

By using the general expressions in Section 2.1, we get:

(3.18)

$$\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[ix\tilde{N}(t) + \lambda(T_0 - t)(e^{ix} - 1 - ix) - ixy\right]dx$$

and

$$\mathbb{E}[D_{t,1}\delta_Y(y)|\mathcal{F}_t]$$

(3.19)

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[ix\tilde{N}(t) + \lambda(T_0 - t)(e^{ix} - 1 - ix) - ixy\right](e^{ix} - 1)dx$$

The forward integral with respect to Brownian motion

The forward integral with respect to Brownian motion was first defined in the seminal paper [RV] and further studied in [RV1], [RV2]. This integral was introduced in the modeling of insider trading in [BØ] and then applied by several authors in questions related to insider trading and stochastic control with advanced information (see, e.g., [DMØP2]). The forward integral was later extended to Poisson random measure integrals in [DMØP1].

Definition

We say that a stochastic process $\phi = \phi(t), t \in [0, T]$, is *forward integrable* (in the weak sense) over the interval $[0, T]$ with respect to B if there exists a process $I = I(t), t \in [0, T]$, such that

$$(3.20) \quad \sup_{t \in [0,T]} \left| \int_0^t \phi(s) \frac{B(s+\epsilon) - B(s)}{\epsilon} ds - I(t) \right| \to 0, \quad \epsilon \to 0^+$$

in probability. In this case we write

$$(3.21) \qquad I(t) := \int_0^t \phi(s) d^- B(s), t \in [0, T],$$

and call $I(t)$ the *forward integral* of ϕ with respect to B on $[0, t]$.

The following results give a more intuitive interpretation of the forward integral as a limit of Riemann sums.

Lemma

Suppose ϕ is càglàd and forward integrable. Then

$$\int_0^T \phi(s)d^- B(s) = \lim_{\triangle t \to 0} \sum_{j=1}^{J_n} \phi(t_{j-1})(B(t_j) - B(t_{j-1})) \tag{3.22}$$

with convergence in probability. Here the limit is taken over the partitions
$0 = t_0 < t_1 < \ldots < t_{J_n} = T$ *of* $t \in [0, T]$ *with*
$\triangle t := \max_{j=1,\ldots,J_n}(t_j - t_{j-1}) \to 0, n \to \infty.$

Remark
From the previous lemma we can see that, if the integrand ϕ is $\mathcal{F}$-adapted, then the Riemann sums are also an approximation to the Itô integral of ϕ with respect to the Brownian motion. Hence in this case the forward integral and the Itô integral coincide. In this sense we can regard the forward integral as an extension of the Itô integral to a nonanticipating setting.

We now give some useful properties of the forward integral. The following result is an immediate consequence of the definition.

Lemma

Suppose ϕ is a forward integrable stochastic process and G a random variable. Then the product $G\phi$ is forward integrable stochastic process and

$$\int_0^T G\phi(t)d^- B(t) = G \int_0^T \phi(t)d^- B(t) \tag{3.23}$$

As a consequence of the above we get the following useful result:

Lemma

Let $\varphi(t, y)$ be an $\mathbb{F}$-adapted process for each $y \in \mathbb{R}$ such that the classical Itô integral

$$\int_0^T \phi(t, y) dB(t)$$

exists for each $y \in \mathbb{R}$. Let Y be a random variable. Then $\varphi(t, Y)$ is forward integrable and

$$\int_0^T \varphi(t, Y) d^- B(t) = \int_0^T \varphi(t, y) dB(t)_{y=Y}. \tag{3.24}$$

The next result shows that the forward integral is an extension of the integral with respect to a semimartingale.

Lemma

Let $\mathbb{G} := \{\mathcal{G}_t, t \in [0, T]\}(T > 0)$ *be a given filtration. Suppose that*

1. B *is a semimartingale with respect to the filtration* $\mathbb{G}$.
2. ϕ *is* $\mathbb{G}$*-predictable and the integral*

$$\int_0^T \phi(t)dB(t), \tag{3.25}$$

with respect to B, *exists.*

Then ϕ *is forward integrable and*

$$\int_0^T \phi(t)d^-B(t) = \int_0^T \phi(t)dB(t), \tag{3.26}$$

We now turn to the Itô formula for forward integrals. In this connection it is convenient to introduce a notation that is analogous to the classical notation for Itô processes.

Definition

A *forward process* (with respect to B) is a stochastic process of the form

$$(3.27) \qquad X(t) = x + \int_0^t u(s)ds + \int_0^t v(s)d^-B(s), \qquad t \in [0, T],$$

(x constant), where $\int_0^T |u(s)|ds < \infty$, $\mathbf{P}$-a.s. and v is a forward integrable stochastic process. A shorthand notation for (3.27) is that

$$(3.28) \qquad d^-X(t) = u(t)dt + v(t)d^-B(t).$$

Theorem

The one-dimensional Itô formula for forward integrals. Let

$$d^-X(t) = u(t)dt + v(t)d^-B(t) \tag{3.29}$$

be a forward process. Let $f \in \mathbf{C}^{1,2}([0,T] \times \mathbb{R})$ *and define*

$$Y(t) = f(t, X(t)), \quad t \in [0,T]. \tag{3.30}$$

Then $Y(t), t \in [0,T]$*, is also a forward process and*

(3.31)

$$d^-Y(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))d^-X(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))v^2(t)dt.$$

Similar definitions and results can be obtained in the Poisson random measure case. See [DMØP1].

4. The general insider optimal control problem

We now present a general method, based on the Donsker delta functional, for solving optimal insider control problems when the inside information is of *initial enlargement* type. Specifically, let us from now on assume that the inside filtration $\mathbb{H}$ has the form

$$\mathbb{H} = \{\mathcal{H}_t\}_{t\geq 0}, \text{ where } \mathcal{H}_t = \mathcal{F}_t \vee Y \tag{4.1}$$

for all t, where $Y \in L^2(\mathbf{P})$ is a given $\mathcal{F}_{T_0}$-measurable random variable, for some $T_0 > T$. We also assume that Y has a Donsker delta functional $\delta_Y(y) \in (\mathcal{S})^*$. We consider the situation when the value at time t of our insider control process $u(t)$ (required to be in a given set $\mathbb{U}$) is allowed to depend on both Y and $\mathcal{F}_t$. In other words, u is assumed to be $\mathbb{H}$-adapted. Therefore it has the form

$$u(t,\omega) = u_1(t, Y, \omega) \tag{4.2}$$

for some function $u_1 : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $u_1(t, y)$ is $\mathbb{F}$-adapted for each $y \in \mathbb{R}$. For simplicity (albeit with some abuse of notation) we will in the following write u in stead of u_1.

Consider a controlled stochastic process $X(t) = X^u(t)$ of the form (4.3)

$$\begin{cases} dX(t) = b(t, X(t), u(t), Y)dt + \sigma(t, X(t), u(t), Y)dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t), u(t), Y, \zeta)\tilde{N}(dt, d\zeta); \quad t \geq 0 \\ X(0) = x, \quad x \in \mathbb{R}, \end{cases}$$

where $u(t) = u(t, y)_{y=Y}$ is our insider control. We assume that the functions

$$\begin{aligned} b(t, x, u, y) &= b(t, x, u, y, \omega) : [0, T_0] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \mapsto \mathbb{R} \\ \sigma(t, x, u, y) &= \sigma(t, x, u, y, \omega) : [0, T_0] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \mapsto \mathbb{R} \\ \gamma(t, x, u, y, \zeta) &= b(t, x, u, y, \zeta, \omega) : [0, T_0] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \mapsto \mathbb{R} \end{aligned}$$

(4.4)

are given bounded C^1 functions with respect to x and adapted processes in (t, ω) for each given x, y, ζ. We interpret the stochastic integrals as forward integrals, and in the following we write $d^- B(t) = dB(t)$ and $\tilde{N}(d^- t, d\zeta) = \tilde{N}(dt, d\zeta)$.

Since $X(t)$ is $\mathbb{H}$-adapted, by the definition of the Donsker delta functional $\delta_Y(y)$ of Y we can write

$$X(t) = x(t, Y) = x(t, y)_{y=Y} = \int_{\mathbb{R}} x(t, y)\delta_Y(y)dy \tag{4.5}$$

for some y-parametrized process $x(t, y)$ which is $\mathbb{F}$-adapted for each y. Then, again by the definition of the Donsker delta functional and the properties of forward integration (see Appendix), we can write

$$
\begin{aligned}
X(t) &= x + \int_0^t b(s, X(s), u(s), Y) ds + \int_0^t \sigma(s, X(s), u(s), Y) dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, X(s), u(s), Y, \zeta) \tilde{N}(ds, d\zeta) \\
&= x + \int_0^t b(s, x(s, Y), u(s, Y), Y) ds + \int_0^t \sigma(s, x(s, Y), u(s, Y), Y) dB(s \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, Y), u(s, Y), Y, \zeta) \tilde{N}(ds, d\zeta) \\
&= x + \int_0^t b(s, x(s, y), u(s, y), y)_{y=Y} ds + \int_0^t \sigma(s, x(s, y), u(s, y), y)_{y=Y}
\end{aligned}
$$

(4.6)

$$
+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, y), u(s, y), y, \zeta)_{y=Y} \tilde{N}(ds, d\zeta)
$$

$$= x + \int_0^t \int_{\mathbb{R}} b(s, x(s, y), u(s, y), y)\delta_Y(y)dyds + \int_0^t \int_{\mathbb{R}} \sigma(s, x(s, y), u(s,$$

$$+ \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \gamma(s, x(s, y), u(s, y), y, \zeta)\delta_Y(y)\tilde{N}(ds, d\zeta)$$

$$= x + \int_{\mathbb{R}} [\int_0^t b(s, x(s, y), u(s, y), y)ds$$

$$+ \int_0^t \sigma(s, x(s, y), u(s, y), y)dB(s)$$

(4.7)

$$+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, y), u(s, y), y, \zeta)\tilde{N}(ds, d\zeta)]\delta_Y(y)dy.$$

Comparing (4.5) and (4.6) we see that (4.5) holds if we choose $x(t, y)$ for each y as the solution of the classical SDE

(4.8)

$$\begin{cases} dx(t,y) = b(t,x(t,y),u(t,y),y)dt + \sigma(t,x(t,y),u(t,y),y)dB(t) \\ + \int_{\mathbb{R}} \gamma(t,x(t,y),u(t,y),y,\zeta)\tilde{N}(dt,d\zeta); \quad t \geq 0 \\ x(0,y) = x, \quad x \in \mathbb{R}, \end{cases}$$

Let $\mathcal{A}$ be a given family of admissible $\mathbb{H}-$adapted control processes u. The *performance functional* $J(u)$ of $u \in \mathcal{A}$ is defined by

$$(4.9) \qquad \begin{aligned} J(u) &= \mathbb{E}[\int_0^T f(t,X(t),u(t),Y)dt + g(X(T),Y)] \\ &= \mathbb{E}[\int_{\mathbb{R}}\{\int_0^T f(t,x(t,y),u(t,y),y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt \\ &+ g(x(T,y),y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]\}dy] \end{aligned}$$

We consider the problem to find $u^\star \in \mathcal{A}$ such that

$$(4.10) \qquad \sup_{u\in\mathcal{A}} J(u) = J(u^\star).$$

5. A sufficient maximum principle

The problem (4.10) is a stochastic control problem with a standard (albeit parametrized) stochastic differential equation (4.8) for the state process $x(t, y)$, but with a non-standard performance functional given by (4.9). We can solve this problem by a modified maximum principle approach, as follows:

Define the *Hamiltonian*
$H : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \to \mathbb{R}$ by

$$\begin{aligned} H(t, x, y, u, p, q, r) &= H(t, x, y, u, p, q, r, \omega) \\ &= \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] f(t, x, u, y) + b(t, x, u, y)p + \sigma(t, x, u, y)q \\ &\quad + \int_{\mathbb{R}} \gamma(t, x, u, y) r(t, y, \zeta) \nu(d\zeta). \end{aligned} \tag{5.1}$$

Here $\mathcal{R}$ denotes the set of all functions $r(., y) : \mathbb{R} \to \mathbb{R}$ such that the last integral above converges.

We define the *adjoint* processes $p(t,y), q(t,y)$ as the solution of the y-parametrized BSDE

(5.2)

$$\begin{cases} dp(t,y) = -\frac{\partial H}{\partial x}(t,y)dt + q(t,y)dB(t) + \int_{\mathbb{R}} r(t,y,\zeta)\tilde{N}(dt,d\zeta); \\ p(T,y) = g'(x(T,y),y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \end{cases}$$

Let $J(u(.,y))$ be defined by

$$J(u(.,y)) = \mathbb{E}[\int_0^T f(t,x(t,y),u(t,y),y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt$$

$$+ g(x(T,y),y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]] \tag{5.3}$$

Comparing with (4.9) we see that

$$J(u) = \int_{\mathbb{R}} J(u(.,y))dy. \tag{5.4}$$

Therefore, it suffices to maximise $J(u(.,y))$ over u for each $y \in \mathbb{R}$:

Problem

Given $y \in \mathbb{R}$, find $u^\star(.,y) \in \mathcal{A}$ such that

$$\sup_{u(.,y)\in\mathcal{A}} J(u(.,y)) = J(u^\star(.,y)). \tag{5.5}$$

To study this problem we present two maximum principles. The first is the following:

Theorem

[Sufficient maximum principle]
Let $\hat{u} \in \mathcal{A}$ *with associated solution* $\hat{x}(t,y), \hat{p}(t,y), \hat{q}(t,y), \hat{r}(t,y,\zeta)$ *of* (4.8) *and* (5.2). *Assume that the following hold:*

1. $x \to g(x)$ *is concave*
2. $(x,u) \to H(t,x,y,u,\widehat{p}(t,y),\widehat{q}(t,y),\hat{r}(t,y,\zeta))$ *is concave for all* t,y,ζ
3. $\sup_{w \in \mathbb{U}} H\big(t,\widehat{x}(t,y),w,\widehat{p}(t,y),\widehat{q}(t,y),\hat{r}(t,y,\zeta)\big) = H\big(t,\widehat{x}(t,y),\widehat{u}(t,y),\widehat{p}(t,y),\widehat{q}(t,y),\hat{r}(t,y,\zeta)\big)$ *for all* t,y,ζ.

Then $\widehat{u}(.,y)$ *is an optimal insider control for problem (5.5).*

6. A necessary maximum principle

We proceed to establish a corresponding necessary maximum principle. For this, we do not need concavity conditions, but in stead we need the following assumptions about the set of admissible control values:

- A_1. For all $t_0 \in [0, T]$ and all bounded $\mathcal{H}_{t_0}$-measurable random variables $\alpha(y, \omega)$, the control $\theta(t, y, \omega) := \mathbf{1}_{[t_0, T]}(t)\alpha(y, \omega)$ belongs to $\mathcal{A}$.
- A_2. For all $u; \beta_0 \in \mathcal{A}$ with $\beta_0(t, y) \le K < \infty$ for all t, y define

  $$\delta(t, y) = \frac{1}{2K} dist((u(t, y), \partial\mathbb{U}) \wedge 1 > 0 \tag{6.1}$$

  and put

  $$\beta(t, y) = \delta(t, y)\beta_0(t, y). \tag{6.2}$$

  Then the control

  $$\widetilde{u}(t, y) = u(t, y) + a\beta(t, y); \quad t \in [0, T]$$

  belongs to $\mathcal{A}$ for all $a \in (-1, 1)$.

- $A3$. For all β as in (6.2) the derivative process

$$\chi(t,y) := \frac{d}{da} x^{u+a\beta}(t,y)|_{a=0}$$

exists, and belong to $\mathbf{L}^2(\lambda \times \mathbf{P})$ and

(6.3)

$$\begin{cases} d\chi(t,y) = [\frac{\partial b}{\partial x}(t,y)\chi(t,y) + \frac{\partial b}{\partial u}(t,y)\beta(t,y)]dt + [\frac{\partial \sigma}{\partial x}(t,y)\chi(t,y \\ + \int_{\mathbb{R}} [\frac{\partial \gamma}{\partial x}(t,y,\zeta)\chi(t,y) + \frac{\partial \gamma}{\partial u}(t,y,\zeta)\beta(t,y)]\tilde{N}(dt,d\zeta) \\ \chi(0,y) = \frac{d}{da} x^{u+a\beta}(0,y)|_{a=0} = 0. \end{cases}$$

Theorem

[Necessary maximum principle]
Let $\hat{u} \in \mathcal{A}$. Then the following are equivalent:

1. $\frac{d}{da} J(\hat{u} + a\beta, y)|_{a=0} = 0$ *for all bounded* $\beta \in \mathcal{A}$ *of the form (6.2).*
2. $\frac{\partial}{\partial u} H(t, \widehat{x}(t, y), y, u, \widehat{p}(t, y), \widehat{q}(t, y), \hat{r}(t, y, \zeta))_{u=\hat{u}} = 0$ *for all* $t \in [0, T]$.

7. Applications

We now apply the general theory of the previous sections to study the problem of optimal portfolio for an insider who knows the value of a given $\mathcal{F}_{T_0}$ - measurable random variable Y, where $T_0 > T$.

In the following we assume that

(7.1)
$$\mathbb{E}[\int_0^T \{\mathbb{E}[D_t\delta_Y(y)|\mathcal{F}_t]^2 + \int_{\mathbb{R}} \mathbb{E}[D_{t,z}\delta_Y(y)|\mathcal{F}_t]^2\nu(dz)\}dt] < \infty.$$

7.1. Utility maximization for an insider, part 1 (N=0)

Consider a financial market where the unit price $S_0(t)$ of the risk free asset is

$$S_0(t) = 1, \quad t \in [0, T] \tag{7.2}$$

and the unit price process $S(t)$ of the risky asset has no jumps and is given by (assuming $\sigma_0 > 0$)

$$\begin{cases} dS(t) & = S(t)[b_0(t, Y)dt + \sigma_0(t, Y)dB(t)]; \quad t \in [0, T] \\ S(0) & > 0. \end{cases} \tag{7.3}$$

Then the wealth process $X(t) = X^{\Pi}(t)$ associated to a portfolio $u(t) = \Pi(t) = \Pi(t, Y)$, interpreted as the fraction of the wealth invested in the risky asset at time t, is given by

(7.4)

$$\begin{cases} dX(t) & = \Pi(t)X(t)[b_0(t, Y)dt + \sigma_0(t, Y)dB(t)]; \quad t \in [0, T] \\ X(0) & = x_0 > 0. \end{cases}$$

Let U be a given utility function. We want to find $\Pi^* \in \mathcal{A}$ such that

$$J(\Pi^*) = \sup_{\Pi \in \mathcal{A}} J(\Pi), \tag{7.5}$$

where

$$J(u) := \mathbb{E}[U(X^{\Pi}(T))]. \tag{7.6}$$

Note that, in terms of our process $x(t, y)$ we have

(7.7)

$$\begin{cases} dx(t,y) = \pi(t,y)x(t,y)[b_0(t,y)dt + \sigma_0(t,y)dB(t)]; \quad t \in [0,T] \\ x(0,y) = x_0 > 0 \end{cases}$$

where

(7.8) $$\Pi(t, Y) = \pi(t, y)_{y=Y}$$

and the performance functional gets the form

$$J(\pi) = U(x(T, y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T].$$

This is a problem of the type investigated in the previous sections (in the special case with no jumps) and we can apply the results there to solve it, as follows:

The Hamiltonian gets the form, with $u = \pi$,

$$H(t, x, y, \pi, p, q) = \pi x[b_0(t, y)p + \sigma_0(t, y)q] \tag{7.9}$$

while the BSDE for the adjoint processes becomes

(7.10)

$$\begin{cases} dp(t, y) & = -\pi(t, y)[b_0(t, y)p(t, y) \\ & \quad +\sigma_0(t, y)q(t, y)]dt + q(t, y)dB(t); \quad t \in [0, T] \\ p(T, y) & = U'(x(T, y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \end{cases}$$

Since the Hamiltonian H is a linear function of π, it can have a finite maximum over all π only if

$$x(t,y)[b_0(t,y)p(t,y) + \sigma_0(t,y)q(t,y)] = 0 \tag{7.11}$$

Substituted into (7.10) this gives

$$\begin{cases} dp(t,y) & = q(t,y)dB(t) + \int_{\mathbb{R}} r(t,y,\zeta)\tilde{N}(dt,d\zeta) \\ p(T,y) & = U'(x(T))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \end{cases} \tag{7.12}$$

If we assume that, for all t, y,

(7.13) $$x(t,y) > 0,$$

then we get from (7.11) that

(7.14) $$q(t,y) = -\frac{b_0(t,y)}{\sigma_0(t,y)} p(t,y).$$

Substituting this into (7.12), we get the equation

(7.15) $$\begin{cases} dp(t,y) &= -\frac{b_0(t,y)}{\sigma_0(t,y)} p(t,y) dB(t) \\ p(T,y) &= U'(x(T,y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \end{cases}$$

Thus we obtain that

(7.16)

$$p(t,y) = p(0,y)\exp(-\int_0^t \frac{b_0(s,y)}{\sigma_0(s,y)}dB(s) - \frac{1}{2}\int_0^t (\frac{b_0(s,y)}{\sigma_0(s,y)})^2 ds),$$

for some, not yet determined, constant $p(0,y)$. In particular, if we put $t = T$ and use (7.15) we get

$$U'(x(T,y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]$$

(7.17)

$$= p(0,y)\exp(-\int_0^T \frac{b_0(s,y)}{\sigma_0(s,y)}dB(s) - \frac{1}{2}\int_0^T (\frac{b_0(s,y)}{\sigma_0(s,y)})^2 ds).$$

To make this more explicit, we proceed as follows:
Define

$$M(t, y) := \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] \tag{7.18}$$

Then by the generalised Clark-Ocone theorem ([AaØPU])
(7.19)

$$\begin{cases} dM(t, y) = \mathbb{E}[D_t\delta_Y(y)|\mathcal{F}_t]dB(t) = \Phi(t, y)M(t, y)dB(t) \\ M(0, y) = 1 \end{cases}$$

where

$$\Phi(t, y) = \frac{\mathbb{E}[D_t\delta_Y(y)|\mathcal{F}_t]}{\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]} \tag{7.20}$$

Solving this SDE for $M(t)$ we get

$$M(t) = \exp(\int_0^t \Phi(s, y)dB(s) - \frac{1}{2}\int_0^t \Phi^2(s, y)ds). \tag{7.21}$$

Substituting this into (7.17) we get

$$
\begin{aligned}
&U'(x(T,y)) \\
&= p(0,y)\exp\Big(-\int_0^T \{\Phi(s,y)+\frac{b_0(s,y)}{\sigma_0(s,y)}\}dB(s) \\
&+\frac{1}{2}\int_0^T \{\Phi^2(s,y)-\frac{b_0^2(s,y)}{\sigma_0^2(s,y)}\}ds\Big).
\end{aligned}
\tag{7.22}
$$

It remains to determine the constant $p(0,y)$:
From (7.22) we get

$$
x(T,y) = I(p(0,y)\Gamma(T))
\tag{7.23}
$$

where

$$
I = (U')^{-1} \text{ and}
\tag{7.24}
$$

$$
\begin{aligned}
\Gamma(T) = exp\Big(&-\int_0^T \{\Phi(s,y)+\frac{b_0(s,y)}{\sigma_0(s,y)}\}dB(s) \\
&+\frac{1}{2}\int_0^T \{\Phi^2(s,y)-\frac{b_0^2(s,y)}{\sigma_0^2(s,y)}\}ds\Big).
\end{aligned}
\tag{7.25}
$$

We can write the differential stochastic equation of $x(t, y)$ as

$$(7.26) \quad \begin{cases} dx(t, y) = \pi(t, y)x(t, y)[b_0(t, y)dt + \sigma_0(t, y)dB(t)] \\ x(T, y) = I(p(0, y)\Gamma(T, y)) \end{cases}$$

If we define

$$(7.27) \qquad z(t, y) = \pi(t, y)x(t, y)\sigma_0(t, y)$$

then equation (7.26) becomes the linear BSDE

$$(7.28) \qquad \begin{cases} dx(t, y) = \frac{z(t,y)b_0(t,y)}{\sigma_0(t,y)}dt + z(t, y)dB(t)] \\ x(T, y) = I(p(0, y)\Gamma(T, y)) \end{cases}$$

in the unknown $(x(t, y), z(t, y))$.

The solution of this BSDE is

$$(7.29) \qquad x(t,y) = \frac{1}{\Gamma_1(t,y)} \mathbb{E}[I(p(0,y)\Gamma(T,y))\Gamma_1(T,y)|\mathcal{F}_t],$$

where

$$(7.30) \quad \Gamma_1(t,y) = \exp\{\int_0^t -\frac{b_0(s,y)}{\sigma_0(s,y)} dB(s) - \frac{1}{2}\int_0^t (\frac{b_0(s,y)}{\sigma_0(s,y)})^2 ds\}.$$

In particular,

$$(7.31) \qquad x_0 = x(0,y) = \mathbb{E}[I(p(0,y)\Gamma(T,y))\Gamma_1(T,y)].$$

This is an equation which (implicitly) determines the value of $p(0, y)$. When $p(0, y)$ is found, we have the optimal terminal wealth $x(T, y)$ given by (7.28). Solving the resulting BSDE for $z(t, y)$, we get the corresponding optimal portfolio $\pi(t, y)$ by (7.27). We summarize what we have proved in the following theorem:

Theorem

The optimal portfolio $\Pi^*(t)$ *for the insider portfolio problem* (7.5) *is given by*

$$\Pi^*(t) = \int_{\mathbb{R}} \pi^*(t, y)\delta_Y(y)dy = \pi^*(t, Y), \tag{7.32}$$

where

$$\pi^*(t, y) = \frac{z(t, y)}{x(t, y)\sigma_0(t, y)} \tag{7.33}$$

with $x(t, y), z(t, y)$ *given as the solution of the BSDE* (7.28) *and* $p(0, y)$ *given by* (7.31).

7.2. The logarithmic utility case (N=0)

We now look at the special case when U is the *logarithmic utility*, i.e.,

$$U(x) = \ln x; \;\; x > 0. \tag{7.34}$$

Recall the equation for $x(t, y)$:

$$\begin{cases} dx(t,y) = \pi(t,y)x(t,y)[b_0(t,y)dt + \sigma_0(t,y)dB(t)] \\ x(0,y) = x_0 > 0 \end{cases} \tag{7.35}$$

By the Itô formula for forward integrals, we get that the solution of this equation is

$$\begin{aligned} & x(t,y) \\ & = x_0 \exp\{\int_0^t [\pi(s,y)b_0(s,y) - \frac{1}{2}\pi^2(s,y)\sigma_0^2(s,y)]ds \\ & + \int_0^t \pi(s,y)\sigma_0(s,y)dB(s)\}. \end{aligned} \tag{7.36}$$

Therefore,

$$
\begin{aligned}
U'(x(T,y)) &= \frac{1}{x(T,y)} \\
&= \frac{1}{x}\exp\{-\int_0^T [\pi(s,y)b_0(s,y) - \frac{1}{2}\pi^2(s,y)\sigma_0^2(s,y)]ds \\
&- \int_0^T \pi(s,y)\sigma_0(s,y)dB(s)\}
\end{aligned}
$$

(7.37)

Comparing with (7.22) we try to choose $\pi(s, y)$ such that

$$
\begin{aligned}
& \frac{1}{x} \exp\{-\int_0^T \pi(s, y)\sigma_0(s, y)dB(s) \\
& - \int_0^T [\pi(s, y)b_0(s, y) - \frac{1}{2}\pi^2(s, y)\sigma_0^2(s, y)]ds\} \\
& = p(0, y) \exp\{-\int_0^T \{\Phi(s, y) + \frac{b_0(s, y}{\sigma_0(s, y)}\}dB(s) \\
& + \frac{1}{2}\int_0^T \int_0^t \{\Phi^2(s, y) - \frac{b_0^2(s, y)}{\sigma_0^2(s, y)}\}ds\} \qquad (7.38)
\end{aligned}
$$

Thus we try to put

$$p(0,y) = \frac{1}{x} \tag{7.39}$$

and choose $\pi(s,y)$ such that, using (7.20),

(7.40)

$$\pi(s,y)\sigma_0(s,y) = \Phi(s,y) + \frac{b_0(s,y)}{\sigma_0(s,y)} = \frac{\mathbb{E}[D_s\delta_Y(y)|\mathcal{F}_s]}{\mathbb{E}[\delta_Y(y)|\mathcal{F}_s]} + \frac{b_0(s,y)}{\sigma_0(s,y)}$$

This gives

$$\pi(s,y) = \frac{b_0(s,y)}{\sigma_0^2(s,y)} + \frac{\mathbb{E}[D_s\delta_Y(y)|\mathcal{F}_s]}{\sigma_0(s,y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_s]} \tag{7.41}$$

We now verify that with this choice of $\pi(s,y)$, also the other two terms on the exponents on each side of (7.38) coincide, i.e. that

$$\pi(s,y)b_0(s,y) - \frac{1}{2}\pi^2(s,y)\sigma_0^2(s,y) = \Phi^2(s,y) - \frac{b_0^2(s,y)}{\sigma_0^2(s,y)} \tag{7.42}$$

Hence we have proved the following result, which has been obtained earlier in [ØR1] by a different method:

Theorem

The optimal portfolio $\Pi = \Pi^*$ with respect to logarithmic utility for an insider in the market (7.2)-(7.3) *and with the inside information (4.1) is given by*

(7.43)

$$\Pi^*(s) = \frac{b_0(s,Y)}{\sigma_0^2(s,Y)} + \frac{\mathbb{E}[D_s\delta_Y(y)|\mathcal{F}_s]_{y=Y}}{\sigma_0(s,Y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_s]_{y=Y}}, \quad 0 \le s \le T < T_0.$$

Substituting (3.15) and (3.16) in (7.43) we obtain:

Corollary

Suppose that Y is Gaussian of the form (3.12). Then the optimal insider portfolio is given by

(7.44)

$$\Pi^*(s) = \frac{b_0(s, Y(T_0))}{\sigma_0^2(s, Y(T_0))} + \frac{(Y(T_0) - Y(s))\psi(s)}{\sigma_0(s, Y(T_0))\|\psi\|^2_{[s,T_0]}}, \quad 0 \leq s \leq T < T_0.$$

In particular, if $Y = B(T_0)$ we get the following result, which was also proved in [PK], in the case when the coefficients do not depend on Y:

Corollary

Suppose that $Y = B(T_0)$. Then the optimal insider portfolio is given by

(7.45)

$$\Pi^*(s) = \frac{b_0(s, B(T_0))}{\sigma_0^2(s, B(T_0))} + \frac{B(T_0) - B(s)}{\sigma_0(t, Y(T_0))(T_0 - s)}, \quad 0 \leq s \leq T < T_0.$$

8. The general Itô-Lévy process case

We now extend the financial applications in the previous sections to the general case with both a Brownian motion component $B(t)$ and a compensated Poisson random measure component $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$, as in Section 2. Thus we consider a financial market where the unit price $S_0(t)$ of the risk free asset is

$$S_0(t) = 1, \quad t \in [0, T] \tag{7.46}$$

and the unit price process $S(t)$ of the risky asset is given by

(7.47)

$$\begin{cases} dS(t) & = S(t)[b_0(t, Y)dt + \sigma_0(t, Y)dB(t) + \int_{\mathbb{R}} \gamma_0(t, Y, \zeta)\tilde{N}(dt, d\zeta)]; \\ S(0) & > 0. \end{cases}$$

Then the wealth process $X(t) = X^{\Pi}(t)$ associated to a portfolio $u(t) = \Pi(t)$, interpreted as the fraction of the wealth invested in the risky asset at time t, is given by

$$\begin{cases} dX(t) & = \Pi(t)X(t)[b_0(t,Y)dt + \sigma_0(t,Y)dB(t) \\ & \quad + \int_{\mathbb{R}} \gamma_0(t,Y,\zeta)\tilde{N}(dt,d\zeta)]; \quad t \in [0,T] \\ X(0) & = x_0 > 0. \end{cases}$$

Let U be a given utility function. We want to find $\Pi^* \in \mathcal{A}$ such that

$$J(\Pi^*) = \sup_{\Pi \in \mathcal{A}} J(\Pi), \tag{7.48}$$

where

$$J(\Pi) := \mathbb{E}[U(X^{\Pi}(T))]. \tag{7.49}$$

Note that, in terms of our process $x(t, y)$ we have

$$\begin{cases} dx(t,y) & = \pi(t,y)x(t,y)[b_0(t,y)dt + \sigma_0(t,y)dB(t) \\ & \quad + \int_{\mathbb{R}} \gamma_0(t,y,\zeta)\tilde{N}(dt,d\zeta)]; \quad t \in [0,T] \\ x(0,y) & = x_0(y) > 0, \end{cases}$$

which has the solution

$$\begin{aligned} x(t,y) = x_0(y) \exp \Big(& \int_0^t \{\pi(s,y)b_0(s,y) - \frac{1}{2}\pi^2(s,y)\sigma_0^2(s) \\ & + \int_0^t \int_{\mathbb{R}} [\ln(1 + \pi(s,y)\gamma_0(s,y,\zeta)) - \pi(s,y)\gamma_0(s,y,\zeta)]\nu(d\zeta)\} ds \\ & + \int_0^t \pi(s,y)\sigma_0(s,y)dB(s) \end{aligned}$$

(7.50)

$$+ \int_0^t \int_{\mathbb{R}} \ln(1 + \pi(s,y)\gamma_0(s,y,\zeta))\tilde{N}(ds,d\zeta)\Big).$$

The performance functional gets the form

$$J(\pi) = \mathbb{E}[U(x(T,y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]], \tag{7.51}$$

where

$$\Pi(t) = \pi(t, Y). \tag{7.52}$$

In this case the Hamiltonian gets the form

$$\begin{aligned} &H(t,x,y,\pi,p,q,r) \\ &= \pi x[b_0(t,y)p + \sigma_0(t,y)q + \int_{\mathbb{R}} \gamma_0(t,y,\zeta)r(t,y,\zeta)\nu(d\zeta)], \end{aligned} \tag{7.53}$$

while the BSDE for the adjoint processes becomes

$$\begin{cases} dp(t,y) = -\pi(t,y)[b_0(t,y)p(t,y) + \sigma_0(t,y)q(t,y) \\ + \int_{\mathbb{R}} \gamma_0(t,y,\zeta)r(t,y,\zeta)\nu(d\zeta)]dt \\ +q(t,y)dB(t) + \int_{\mathbb{R}} r(t,y,\zeta)\tilde{N}(dt,d\zeta); \quad t \geq 0 \\ p(T,y) = U'(x(T,y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \end{cases}$$

For simplicity, let us from now on consider the case with logarithmic utility, i.e.

$$U(x) = \ln x; x > 0.$$

Then

$$\mathbb{E}[U(x(T,y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]] = \mathbb{E}[\ln(x(T,y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]]$$

$$= \mathbb{E}\big[\{\int_0^T \{\pi(t,y)b_0(t,y) - \frac{1}{2}\pi^2(t,y)\sigma_0^2(t)$$

$$+ \int_0^T \int_{\mathbb{R}} [\ln(1 + \pi(t,y)\gamma_0(t,y,\zeta)) - \pi(t,y)\gamma_0(t,y,\zeta)]\nu(d\zeta)\}dt$$

$$+ \int_0^T \pi(t,y)\sigma_0(t,y)dB(t)$$

(7.54)

$$+ \int_0^T \int_{\mathbb{R}} \ln(1 + \pi(t,y)\gamma_0(t,y,\zeta))\tilde{N}(dt,d\zeta)\}\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]\big]$$

We now use the following duality formulas:

- (Generalized duality formula (1))
  Let $F \in \mathbf{L}^2(\mathcal{F}_T, \mathbf{P})$ and let $\phi(t) \in \mathbf{L}^2(\lambda \times \mathbf{P})$ be adapted. Then

$$\mathbb{E}[F \int_0^T \phi(t) dB(t)] = \mathbb{E}[\int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] \phi(t) dt]$$

- (Generalized duality formula (2))
  Let $F \in \mathbf{L}^2(\mathcal{F}_T, \mathbf{P})$ and $\psi(t, z) \in \mathbf{L}^2(\lambda \times \nu \times \mathbf{P})$ be adapted. Then

$$\begin{aligned} &\mathbb{E}[F \int_0^T \int_{\mathbb{R}_0} \psi(t, z) \tilde{N}(ds, dz)] \\ &= \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{t,z} F | \mathcal{F}_t] \psi(t, z) \nu(dz) dt] \end{aligned}$$

This enables us to write (7.54) as the expectation of a *dt*-integral and we get:

$$
\begin{aligned}
&\mathbb{E}[U(x(T,y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]] = \mathbb{E}[\ln(x(T,y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]] \\
&= \mathbb{E}\Big[\int_0^T \{\pi(t,y)b_0(t,y) - \frac{1}{2}\pi^2(t,y)\sigma_0^2(t)\}dt\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \\
&+ \int_0^T \int_{\mathbb{R}} [\ln(1+\pi(t,y)\gamma_0(t,y,\zeta)) \\
&- \pi(t,y)\gamma_0(t,y,\zeta)]\nu(d\zeta)dt\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \\
&+ \int_0^T \mathbb{E}[D_t\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]\pi(t,y)\sigma_0(t,y)dt
\end{aligned}
\tag{7.55}
$$

$$
+ \int_0^T \int_{\mathbb{R}} \mathbb{E}[D_{t,\zeta}\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]]\ln(1+\pi(t,y)\gamma_0(t,y,\zeta))\nu(d\zeta)dt\Big]
$$

Note that

$$D_t \mathbb{E}[\delta_Y(y) | \mathcal{F}_T] = \mathbb{E}[D_t \delta_Y(y) | \mathcal{F}_T] \tag{7.56}$$

and

$$D_{t,\zeta} \mathbb{E}[\delta_Y(y) | \mathcal{F}_T] = \mathbb{E}[D_{t,\zeta} \delta_Y(y) | \mathcal{F}_T]. \tag{7.57}$$

Therefore, if we substitute this in (7.55) and take for each t the conditional expectation with respect to $\mathcal{F}_t$ of the integrand, we get

$$
\begin{aligned}
&\mathbb{E}[\ln(x(T,y))\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]] \\
&= \mathbb{E}[\int_0^T \{\pi(t,y)b_0(t,y) - \frac{1}{2}\pi^2(t,y)\sigma_0^2(t)\}dt\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \\
&+ \int_0^T \int_{\mathbb{R}} [\ln(1+\pi(t,y)\gamma_0(t,y,\zeta)) \\
&- \pi(t,y)\gamma_0(t,y,\zeta)]\nu(d\zeta)dt\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \\
&+ \int_0^T \mathbb{E}[D_t\delta_Y(y)|\mathcal{F}_t]\pi(t,y)\sigma_0(t,y)dt \\
&+ \int_0^T \int_{\mathbb{R}} \mathbb{E}[D_{t,\zeta}\delta_Y(y)|\mathcal{F}_t]\ln(1+\pi(t,y)\gamma_0(t,y,\zeta))\nu(d\zeta)dt\Big]
\end{aligned}
\tag{7.58}
$$

We can maximize this by maximizing the integrand with respect to $\pi(t, y)$ for each t and y. Doing this we obtain that the optimal portfolio $\pi(t, y)$ for Problem (7.48) is given implicitly as the solution $\pi(t, y)$ of the first order condition

$$[b_0(t, y) - \pi(t, y)\sigma_0^2(t, y)]\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + \sigma_0(t, y)\mathbb{E}[D_t\delta_Y(y)|\mathcal{F}_t]$$
$$- \int_{\mathbb{R}} \frac{\pi(t, y)\gamma_0^2(t, y, \zeta)}{1 + \pi(t, y)\gamma_0(t, y, \zeta)}[\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]$$
(7.59)
$$+ \int_{\mathbb{R}} \frac{\gamma_0(t, y, \zeta)}{1 + \pi(t, y)\gamma_0(t, y, \zeta)}\mathbb{E}[D_{t,\zeta}\delta_Y(y)|\mathcal{F}_t] = 0.$$

If we define

$$\Phi(t,y) := \frac{\mathbb{E}[D_t \delta_Y(y) | \mathcal{F}_t]}{\mathbb{E}[\delta_Y(y) | \mathcal{F}_t]} \tag{7.60}$$

and

$$\Psi(t,\zeta,y) := \frac{\mathbb{E}[D_{t,\zeta} \delta_Y(y) | \mathcal{F}_t]}{\mathbb{E}[\delta_Y(y) | \mathcal{F}_t]} \tag{7.61}$$

then (7.59) can be written

$$\begin{aligned} & b_0(t,y) - \pi(t,y)\sigma_0^2(t,y) - \int_{\mathbb{R}} \frac{\pi(t,y)\gamma_0^2(t,y,\zeta)}{1+\pi(t,y)\gamma_0(t,y,\zeta)} \nu(d\zeta) \\ & + \sigma_0(t,y)\Phi(t,y) + \int_{\mathbb{R}} \frac{\gamma_0(t,y,\zeta)}{1+\pi(t,y)\gamma_0(t,y,\zeta)} \Psi(t,y,\zeta)\nu(d\zeta) = 0. \end{aligned} \tag{7.62}$$

Thus we have proved the following theorem:

Theorem

The optimal portfolio with respect to logarithmic utility for an insider in the market (7.46)-(7.47)*and with the inside information (4.1) is given implicitly as the solution* $\Pi(t) = \Pi^*(t)$ *of the equation*

$$b_0(t, Y) - \Pi(t)\sigma_0^2(t, Y) - \int_{\mathbb{R}} \frac{\Pi(t)\gamma_0^2(t, Y, \zeta)}{1 + \Pi(t)\gamma_0(t, Y, \zeta)}\nu(d\zeta)$$

(7.63)

$$+ \sigma_0(t, y)\Phi(t, Y) + \int_{\mathbb{R}} \frac{\gamma_0(t, Y, \zeta)}{1 + \Pi(t)\gamma_0(t, Y, \zeta)}\Psi(t, Y, \zeta)\nu(d\zeta) = 0,$$

provided that a solution exists.

The equation (7.63) for the optimal portfolio $\Pi(t)$ holds for a general insider random variable Y. In the case when Y is of the first order chaos form (3.5), then we can substitute the expressions found in Section 3 into (7.60) and (7.61), and get a more explicit equation as follows:

Theorem

Suppose Y is as in (3.5). Then in the equation (7.63) for the optimal portfolio $\Pi(t) = \Pi(t, Y)$ we have

$$\Phi(t,y) = \frac{\beta(t)\int_{\mathbb{R}} F(t,x,y)x dx}{\int_{\mathbb{R}} F(t,x,y) dx} \tag{7.64}$$

$$\Psi(t,y,z) = \frac{\int_{\mathbb{R}} F(t,x,y)(e^{ix\psi(t,z)} - 1) dx}{\int_{\mathbb{R}} F(t,x,y) dx} \tag{7.65}$$

(7.66)

where

$$\begin{aligned} F(t,x,y) = \int_{\mathbb{R}} \exp\Big[\int_0^t \int_{\mathbb{R}} ix\psi(s,\zeta)\tilde{N}(ds,d\zeta) + \int_0^t x\beta(s)dB(s) \\ + \int_t^{T_0} \int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta))\nu(d\zeta)ds \\ + \int_t^{T_0} \frac{1}{2}x^2\beta^2(s)ds - ixy\Big] dx. \end{aligned}$$

- N. Agram and B. Øksendal: Malliavin calculus and optimal control of stochastic Volterra equations. J. Optim. Theory Appl. DOI: 10.1007/s10957-015-0753-5 (May 2015).
- K. Aase, B. Øksendal, N. Privault and J. Ubøe: White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. Finance Stoch. 4 (2000), 465-496.
- K. Aase, B. Øksendal and J. Ubøe: Using the Donsker delta function to compute hedging strategies. Potential Analysis 14 (2001), 351-374.
- L. Breiman: Probability. Addison-Wesley 1968.
- O. E. Barndorff-Nielsen, F.E. Benth and B. Szozda: On stochastic integration for volatility modulated Brownian-driven Volterra processes via white noise analysis. arXiv:1303.4625v1, 19 March 2013.

- F. Biagini and B. Øksendal: A general stochastic calculus approach to insider trading.Appl. Math. & Optim. 52 (2005), 167-181.
- K.R. Dahl, S.-E. A. Mohammed, B. Øksendal and E. R. Røse; Optimal control with noisy memory and BSDEs with Malliavin derivatives. arXiv: 1403.4034 (2014).
- G. Di Nunno, T. Meyer-Brandis, B. Øksendal and F. Proske: Malliavin calculus and anticipative Itô formulae for Lévy processes. Inf. Dim. Anal. Quantum Prob. Rel. Topics 8 (2005), 235-258.
- G. Di Nunno, T. Meyer-Brandis, B. Øksendal and F. Proske: Optimal portfolio for an insider in a market driven by Lévy processes. Quant. Finance 6 (2006), 83-94.
- G. Di Nunno and B. Øksendal: The Donsker delta function, a representation formula for functionals of a Lévy process and application to hedging in incomplete markets. Séminaires et

Congrèes, Societé Mathématique de France, Vol. 16 (2007), 71-82.

O. Draouil and B. Øksendal: A Donsker delta functional approach to optimal insider control and applications to finance. http://arxiv.org/abs/1504.02581 (April 2015) To appear in Communication in Mathematics and Statistics.

G. Di Nunno and B. Øksendal: A representation theorem and a sensitivity result for functionals of jump diffusions. In A.B. Cruzeiro, H. Ouerdiane and N. Obata (editors): Mathematical Analysis and Random Phenomena. World Scientific 2007, pp. 177 - 190.

G. Di Nunno, B. Øksendal and F. Proske: Malliavin Calculus for Lévy Processes with Applications to Finance. Universitext, Springer 2009.

H. Holden, B. Øksendal, J. Ubøe and T. Zhang: Stochastic Partial Differential Equations. Universitext, Springer, Second Edition 2010.

- A. Lanconelli and F. Proske: On explicit strong solution of Itô-SDEs and the Donsker delta function of a diffusion. Inf. Dim. Anal. Quatum Prob Rel. Topics 7 (2004),437-447.
- S. Mataramvura, B. Øksendal and F. Proske: The Donsker delta function of a Lévy process with application to chaos expansion of local time. Ann. Inst H. Poincaré Prob. Statist. 40 (2004), 553-567.
- T. Meyer-Brandis and F. Proske: On the existence and explicit representability of strong solutions of Lévy noise driven SDEs with irregular coefficients. Commun. Math. Sci. 4 (2006), 129-154.
- B. Øksendal and E. Røse: A white noise approach to insider trading. Manuscript University of Oslo, 20 November 2014
- B. Øksendal and E. Røse: Applications of white noise to mathematical finance. Manuscript University of Oslo, 5 February 2015

B. Øksendal and A. Sulem: Applied Stochastic Control of Jump Diffusions. Second Edition. Springer 2007

B. Øksendal and A. Sulem: Risk minimization in financial markets modeled by Itô-Lévy processes. Afrika Matematika (2014), DOI: 10.1007/s13370-014-02489-9.

B. Øksendal, A. Sulem and T. Zhang : A stochastic HJB equation for optimal control of forward-backward SDEs (13 pages). *arXiv* 1312.1472 (December 2013).To appear in F.E.Benth and G. Di Nunno (editors): Stochastics of Environmental and Financial Economics, Springer.

P. Protter: Stochastic Integration and Differential Equations. Second Edition. Springer 2005

I. Pikovsky and I. Karatzas: Anticipative portfolio optimization. Adv. Appl. Probab. 28 (1996), 1095-1122.

F. Russo and P. Vallois: Forward, backward and symmetric stochastic integration. Probab. Theor. Rel. Fields 93 (1993), 403-421.

F. Russo and P. Vallois. The generalized covariation process and Itô formula. Stoch. Proc. Appl., 59(4):81-104, 1995.

F. Russo and P. Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. Stoch. Stoch. Rep., 70(4):1-40, 2000.