Conditional full support for Lévy-driven moving averages

Mikko Pakkanen

Department of Mathematics, Imperial College London

Aarhus Conference on Probability, Statistics and Their Applications, 19 June 2015

Based on joint work with Tommi Sottinen and Adil Yazigi

Stochastic processes and their distributions

- Consider a stochastic process $X = (X_t)_{t\in[0,T]}$ on $(\Omega, \mathscr{F}, \mathbf{P})$.
- If the sample paths of X enjoy some regularity properties, for example they are càdlàg or continuous, then it is possible, and often also useful, to view X as a random variable

$$X : \Omega \to E,$$

with metric space $E = \big(D([0,T]), \rho_{\mathrm{Sko}}\big)$ or $E = \big(C([0,T]), \rho_{\mathrm{Sup}}\big)$ as its state space.
- The state space E is then infinite-dimensional.
- What can we learn about the distribution $\mathscr{L}(X) := \mathbf{P} \circ X^{-1}$ of X?

Difficulties in analysing infinite-dimensional probability distributions

- Interesting probability distributions on infinite-dimensional spaces are usually defined implicitly:
  1. via finite-dimensional projections (Kolmogorov's extension),
  2. as limits simpler probability distributions (Prohorov's theorem).
- Practical tools such as distribution functions or density functions are not available for distributions on infinite-dimensional spaces.
- Given a measurable transformation $T : E \to \mathbb{R}$, it is typically not easy to describe the distribution of the random variable $T(X)$, unless X is Gaussian and T linear, say.

Supports of probability distributions

- However, it is often possible to describe (practically) explicitly the support of an infinite-dimensional probability distribution.

Definition

1. The support of a Borel measure μ on a separable metric space E consists of all $x \in E$ such that

$$\mu\big(B(x,\varepsilon)\big) > 0 \quad \text{for all } \varepsilon > 0,$$

where $B(x,\varepsilon) := \{y \in E : d_E(x,y) < \varepsilon\}$. We denote the support by $\operatorname{supp} E$.

2. If μ is a Borel probability measure, then $\operatorname{supp} E$ is equivalently the smallest closed set with μ-probability one.

Example: the support of the Wiener measure

Question

Let $W = (W_t)_{t \in [0,T]}$ be a standard Brownian motion. With $E = C([0,T])$, what is $\operatorname{supp} \mathscr{L}(W)$?

Example: the support of the Wiener measure

Question

Let $W = (W_t)_{t \in [0,T]}$ be a standard Brownian motion. With $E = C([0,T])$, what is $\operatorname{supp} \mathscr{L}(W)$?

Two immediate observations:

1. Since $W_0 = 0$, we have
$$\operatorname{supp} \mathscr{L}(W) \subset C_0([0,T]) := \{f \in C([0,T]) : f(0) = 0\}.$$
2. The support is necessarily non-empty, so $g \in \operatorname{supp} \mathscr{L}(W)$ for some $g \in C_0([0,T])$.

Example: the support of the Wiener measure

Let now $p \in C_0([0, T])$ be a polynomial function and let $\varepsilon > 0$. Then, by Girsanov's theorem, there exists $\mathbf{Q}_p \sim \mathbf{P}$ such that

$$W - p \quad \text{is a standard Brownian motion under } \mathbf{Q}_p.$$

Thus, for $g \in \operatorname{supp} \mathscr{L}(W)$ it holds that

$$\begin{aligned} \mathbf{Q}_p[W \in B(p+g, \varepsilon)] &= \mathbf{Q}_p[W - p \in B(g, \varepsilon)] \\ &= \mathbf{P}[W \in B(g, \varepsilon)] > 0. \end{aligned}$$

Since $\mathbf{Q}_p \sim \mathbf{P}$, we find that

$$\mathbf{P}[W \in B(p+g, \varepsilon)] > 0,$$

so $p + g \in \operatorname{supp} \mathscr{L}(W)$, as $\varepsilon > 0$ was arbitrary.

Example: the support of the Wiener measure

Functions of the form $p+g$, where p is a polynomial functions with $p(0)=0$ are dense in $C_0([0,T])$ by Weierstrass's theorem. Since $\operatorname{supp}\mathscr{L}(W)$ is closed, we have

$$C_0([0,T]) \subset \operatorname{supp}\mathscr{L}(W).$$

But $\operatorname{supp}\mathscr{L}(W) \subset C_0([0,T])$, so

$$\operatorname{supp}\mathscr{L}(W) = C_0([0,T]).$$

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Conclusion

W has full support.

Full support

Definition

A continuous stochastic process $X = (X_t)_{t \in [0,T]}$ in $\mathbb{R}^d$ with $X_0 = x \in \mathbb{R}^d$ has full support if

$$\operatorname{supp} \mathscr{L}(X) = C_x([0,T], \mathbb{R}^d),$$

where $C_x([0,T]) := \{f \in C([0,T], \mathbb{R}^d) : f(0) = x\}$.

How to check? — Support theorems:

- diffusion processes (Stroock and Varadhan, 1972),
- Gaussian processes (Kallianpur, 1971).

Conditional full support

It is natural to formulate a similar property for conditional distributions:

Definition

A continuous stochastic process $X = (X_t)_{t\in[0,T]}$ in $\mathbb{R}^d$ has conditional full support (CFS) if for any $t \in [0, T)$,

$$\operatorname{supp} \mathscr{L}\Big((X_u)_{t\in[t,T]}\Big|\mathscr{F}_t\Big) = C_{X_t}([t, T], \mathbb{R}^d) \quad \text{almost surely,}$$

where $(\mathscr{F}_t)_{t\in[0,T]}$ is the natural augmented filtration of X and $C_x([t, T], \mathbb{R}^d) := \{f \in C([t, T], \mathbb{R}^d) : f(t) = x\}$ for any $x \in \mathbb{R}^d$.

Example: Brownian motion

Let $W = (W_t)_{t\in[0,T]}$ be a standard Brownian motion in $\mathbb{R}$. Fix $t \in [0, T)$. For $u \in [t, T]$, we have

$$W_u = W_u - W_t + W_t,$$

where $(W_u - W_t)_{u\in[t,T]}$ is a standard Brownian motion independent of $\mathscr{F}_t$. So, the conditional distribution of $(W_u)_{u\in[t,T]}$ given $\mathscr{F}_t$ is

$$\text{Wiener measure} + W_t.$$

As the support of this Wiener measure is $C_0([t, T], \mathbb{R})$, it follows that

$$\operatorname{supp} \mathscr{L}\big((W_u)_{t\in[t,T]}\big|\mathscr{F}_t\big) = C_{W_t}([t, T], \mathbb{R}),$$

that is, W has CFS.

Processes that have CFS

The following processes have CFS (possibly under some additional non-degeneracy conditions):

- Brownian motion,
- fractional Brownian motion (Guasoni, Rásonyi, and Schachermayer, 2008),
- univariate Brownian moving averages (Cherny, 2008),
- univariate stationary-increment Gaussian processes (Gasbarra, Sottinen, and van Zanten, 2011),
- diffusion processes (Guasoni, Rásonyi, and Schachermayer, 2008; Guasoni and Rásonyi, 2015),
- Brownian semimartingales (P., 2010; Herzegh, Prokaj, and Rásonyi, 2014),
- Brownian semistationary processes (P., 2011).

Applications of CFS in mathematical finance

The CFS property "unlocks" results in several areas:

- no-arbitrage, with and without transaction costs (Guasoni, Rásonyi, and Schachermayer, 2008; Bender, Sottinen, and Valkeila, 2008; Bender, 2011),
- super-hedging with transaction costs (Guasoni, Rásonyi, and Schachermayer, 2008; Blum, 2009; Dolinsky, 2012; Dolinsky and Soner, 2015),
- arbitrage and diverse markets (Herzegh, Prokaj, and Rásonyi, 2014),
- optimal arbitrage (Chau and Tankov, 2015),
- fragility of bubbles (Guasoni and Rásonyi, 2015),
- utility maximization under transaction costs, "shadow prices" (Czichowsky and Schachermayer, 2015).

Introduction

Multivariate Brownian moving averages

Multivariate Lévy-driven moving averages

Cherny's theorem

We endeavor to find a multivariate generalization of:

Theorem (Cherny, 2008)

Let $X = (X_t)_{t \in [0,T]}$ be a univariate, continuous Brownian moving average given by

$$X_t = \int_{-\infty}^{t} \big(g(t-s) - g(-s)\big)\mathrm{d}W_s,$$

where $g(t) = 0$ for any $t < 0$ and $g(t - \cdot\,) - g(-\cdot\,) \in L^2(\mathbb{R})$ for any $t \in \mathbb{R}$; and $(W_t)_{t \in \mathbb{R}}$ is a standard Brownian motion in $\mathbb{R}$. If

$$\int_0^{\varepsilon} |g(s)|\mathrm{d}s > 0 \quad \text{for any } \varepsilon > 0,$$

then X has CFS.

Multivariate Brownian moving averages

We consider a d-dimensional Brownian moving average

$$X_t := \int_{-\infty}^{t} \big(G(t-s) - H(-s)\big)\mathrm{d}W_s,$$

where

- G and H are matrix-valued functions $\mathbb{R} \to \mathbb{R}^{d\times d}$ such that $H(t) = 0 = G(t)$ for all $t < 0$ and that

$$G(t - \cdot\,) - H(-\cdot\,) \in L^2(\mathbb{R}, \mathbb{R}^{d\times d}),$$

- $(W_t)_{t\in\mathbb{R}}$ is a Brownian motion in $\mathbb{R}^d$.

Non-degeneracy condition

- In the 1-dimensional case, the kernel function g was assumed to satisfy
$$\int_0^{\varepsilon} |g(s)| \mathrm{d}s > 0 \quad \text{for any } \varepsilon > 0.$$
- We need a multivariate version of this non-degeneracy condition.
- A natural candidate would be
$$\int_0^{\varepsilon} \Big| \det\big(G(s)\big) \Big| \mathrm{d}s > 0 \quad \text{for any } \varepsilon > 0,$$
but this did not seem very fruitful...
- However, if we replace the classical determinant above with a convolution determinant, this idea will become successful.

Convolution determinant

- Let $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be the space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(s) = 0$ for all $s \leq 0$ and that

$$\int_0^t |f(s)|\mathrm{d}s < \infty \quad \text{for any } t > 0.$$

- Recall that the convolution of $f, g \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is given by

$$(f \star g)(t) := \int_0^t f(t-s)g(s)\mathrm{d}s, \quad t > 0,$$

and that $f \star g \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. (We set $(f \star g)(t) := 0$, $t \leq 0$.)

- We say that $G \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$ if the component functions $G_{i,j}$ $(i, j = 1, \ldots, d)$ of G are in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

Convolution determinant

Definition

The convolution determinant of $G \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$ is given by

$$\det{}^{\star}(G) := \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)\big(G_{1,\sigma(1)} \star \cdots \star G_{d,\sigma(d)}\big) \in L^1_{\mathrm{loc}}(\mathbb{R}_+),$$

where S_d is the group of permutations of $\{1, \dots, d\}$ and $\operatorname{sgn}(\sigma)$ is the signature of $\sigma \in S_d$.

- This is identical to the definition of the ordinary determinant, except that we replace products of real numbers with convolutions of functions.

CFS for multivariate Brownian moving averages

Theorem

Let $(W_t)_{t\in\mathbb{R}}$ be a Brownian motion in $\mathbb{R}^d$ with zero drift and a non-singular correlation matrix. Let $G \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$ and let H be such that $H(t) = 0$ for $t \leq 0$ and

$$G(t - \cdot\,) - H(-\,\cdot\,) \in L^2(\mathbb{R}, \mathbb{R}^{d\times d}), \quad \text{for any } t \in \mathbb{R}.$$

If

$$\int_0^{\varepsilon} |\mathrm{det}^{\star}(G)(s)|\mathrm{d}s > 0, \quad \text{for any } \varepsilon > 0,$$

then

$$X_t := \int_{-\infty}^{t} \big(G(t-s) - H(-s)\big)\mathrm{d}W_s, \quad t \in [0, T],$$

has CFS.

Regularly varying kernels

A wide class of kernel functions can characterized using the notion of regular variation. For such kernels, the "det-star" condition is fairly straightforward to check.

Definition

A measurable function $f : (0, \infty) \to (0, \infty)$ is said to be regularly varying at zero with index $\alpha \in \mathbb{R}$, and we write $f \in \mathrm{RV}_0(\alpha)$, if

$$\lim_{x \to 0} \frac{f(tx)}{f(x)} = t^{\alpha}, \quad \text{for any } t > 0.$$

If $f \in \mathrm{RV}_0(\alpha)$, then $f(t)$ behaves "almost" like $\mathrm{const} \cdot t^{\alpha}$ as $t \to 0$.

Regularly varying kernels

Regular variation at zero is preserved under convolutions:

Lemma

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \cap \mathrm{RV}_0(\alpha)$ *with* $\alpha > -1$ *and* $g \in L^1_{\mathrm{loc}}(\mathbb{R}_+) \cap \mathrm{RV}_0(\beta)$ *with* $\beta > -1$. *Then,*

$$\lim_{t \to 0} \frac{(f \star g)(t)}{t f(t) g(t)} = \mathrm{Beta}(\alpha + 1, \beta + 1),$$

which implies that $f \star g \in \mathrm{RV}_0(\alpha + \beta + 1)$.

Regularly varying kernels

We find the following sufficient condition for the "det-star" assumption in the 2-dimensional case:

Lemma (2-dimensional case)

Let $G \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{2\times 2})$ *be such that* $G_{i,j} \in \mathrm{RV}_0(\alpha_{i,j})$ *for some* $\alpha_{i,j} > -1$ *for any* $i, j = 1, 2$. *If*

$$\alpha_{1,1} + \alpha_{2,2} \neq \alpha_{2,1} + \alpha_{1,2},$$

then

$$\int_0^{\varepsilon} |\det^{\star}(G)(s)| \mathrm{d}s > 0, \quad \text{for any } \varepsilon > 0.$$

Regularly varying kernels

Proof.

By the definition of the convolution determinant,

$$\det{}^{\star}(G) = G_{1,1} \star G_{2,2} - G_{2,1} \star G_{1,2}$$

By the earlier Lemma, $G_{1,1} \star G_{2,2} \in \mathrm{RV}_0(\alpha_{1,1} + \alpha_{2,2} + 1)$ and $G_{2,1} \star G_{1,2} \in \mathrm{RV}_0(\alpha_{2,1} + \alpha_{1,2} + 1)$. When

$$\alpha_{1,1} + \alpha_{2,2} \neq \alpha_{2,1} + \alpha_{1,2},$$

we see that $\det^{\star}(G)$ is a difference of two regularly varying functions with different indices, so it cannot vanish in a neighborhood of zero. $\square$

Introduction

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Multivariate Lévy-driven moving averages

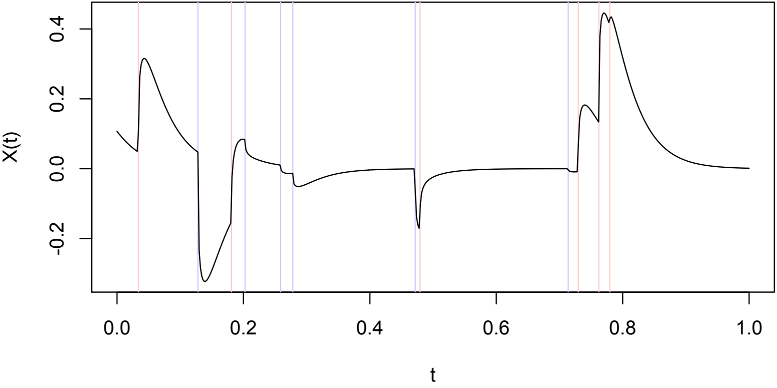
Moving averages driven by Lévy processes

- What if we replace the driving Brownian motion with a pure-jump Lévy process?
- CFS is formulated for continuous processes, but now the driving noise is discontinuous.
- However, a moving average driven by a pure-jump Lévy process can be continuous if the kernel function is "smooth".
- Let us look at an example:

$$X_t := \int_0^\infty g(t-s)\mathrm{d}L_s,$$

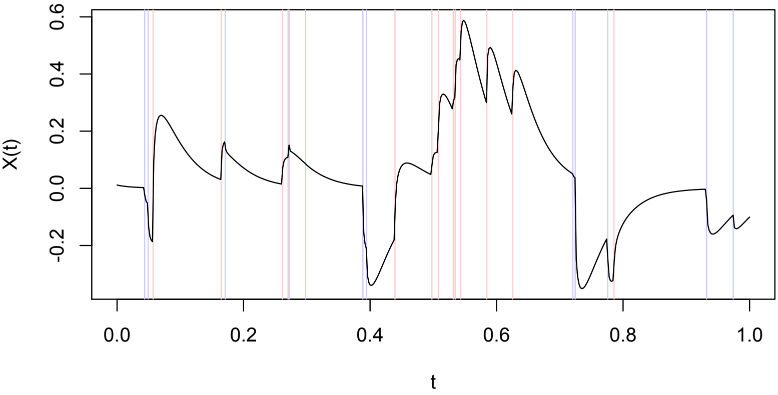
where $g(t) := t^\alpha \exp(-\rho t)$, where $\alpha > 0$ and $\rho > 0$; and $(L_t)_{t\in\mathbb{R}}$ is a two-sided compound Poisson process with standard Gaussian jumps arriving at rate $\lambda > 0$.

Moving averages driven by Lévy processes



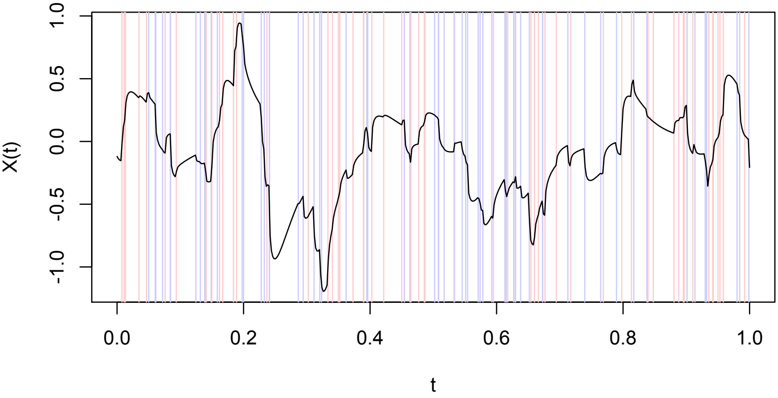
$\alpha = 0.3,\ \rho = 30,\ \lambda = 10$

Moving averages driven by Lévy processes



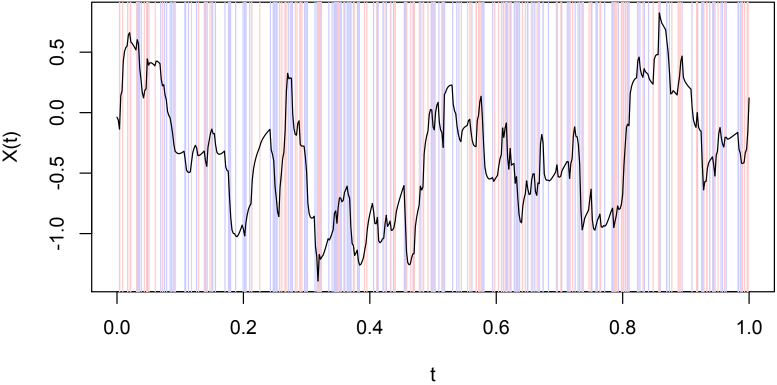
$\alpha = 0.3, \ \rho = 30, \ \lambda = 30$

Moving averages driven by Lévy processes



$\alpha = 0.3,\ \rho = 30,\ \lambda = 100$

Moving averages driven by Lévy processes



$\alpha = 0.3,\ \rho = 30,\ \lambda = 300$

Moving averages driven by Lévy processes

We consider the process

$$X_t := \int_{-\infty}^{t} \big(G(t-s) - H(-s)\big) \mathrm{d}L_s,$$

where

- $(L_t)_{t\in\mathbb{R}}$ is a two-sided d-dimensional Lévy process with triplet $(b, 0, \Lambda)$,
- G and H are measurable matrix-valued functions $\mathbb{R} \to \mathbb{R}^{d\times d}$ such that $H(t) = 0 = G(t)$ for all $t < 0$.

Moving averages driven by Lévy processes

Assumptions

1. The integral defining X_t exists for any $t \in \mathbb{R}$ in the sense of Rajput and Rosiński (1989),
2. X has a continuous modification,
3. The components of G are of finite variation.

Moving averages driven by Lévy processes

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2. X has a continuous modification,
3. The components of G are of finite variation.

Remark

- Condition 2 holds only if G is continuous and $G(0) = 0$ (Rosiński, 1989).
- If $\mathbf{E}[\|L_t\|^2] < \infty$, then Kolmogorov's criterion can be used to check condition 2.

CFS for Lévy-driven moving averages

Theorem

Let X be defined as before. If

$$\int_0^\varepsilon |\det{}^\star(G)(s)| \mathrm{d}s > 0, \quad \textit{for any } \varepsilon > 0,$$

and Λ is non-degenerate in the sense that

$$0 \in \operatorname{int}\operatorname{conv}\operatorname{supp}\Lambda\big(\,\cdot \cap B(0,\varepsilon)\big), \quad \textit{for any } \varepsilon > 0,$$

then X has CFS.

CFS for Lévy-driven moving averages

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Remark

In the special case $d = 1$, the non-degeneracy condition involving Λ reduces to $\Lambda\big((-\varepsilon, 0)\big) > 0$ and $\Lambda\big((0, \varepsilon)\big) > 0$ for any $\varepsilon > 0$.

Polar decomposition

When $d \geq 2$, the Lévy measure Λ has a polar decomposition

$$\Lambda(A) = \int_{\mathcal{S}^{d-1}} \int_{(0,\infty)} \mathbf{1}_A(ru)\rho_u(\mathrm{d}r)\lambda(\mathrm{d}u), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where

- λ is a finite measure on the unit sphere $\mathcal{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$,
- ρ_u is a Lévy measure on $\mathbb{R}_+$ that depends measurably on $u \in \mathcal{S}^{d-1}$.

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Example

With $\rho_u(\mathrm{d}r) = \frac{1}{r^{1+\alpha}}$ we recover a class of α-stable Lévy processes.

Polar decomposition

Using the polar decomposition, we can find sufficient conditions for the non-degeneracy of the Lévy measure Λ:

Lemma

Let Λ *have the polar decomposition* $\{\lambda, \rho_u : u \in \mathcal{S}^{d-1}\}$. *If*

1. $0 \in \operatorname{int} \operatorname{conv} \operatorname{supp} \lambda$,
2. $\rho_u\big((0,\varepsilon)\big) > 0$ *for any* $\varepsilon > 0$ *and* $u \in \operatorname{supp} \lambda$,

then

$$0 \in \operatorname{int} \operatorname{conv} \operatorname{supp} \Lambda\big(\,\cdot \cap B(0,\varepsilon)\big), \quad \textit{for any } \varepsilon > 0.$$

Other constructions multidimensional Lévy measures

The non-degeneracy condition can also be checked for Lévy measures constructed using

- Lévy copulas (Kallsen and Tankov, 2006),
- multivariate subordination (Barndorff-Nielsen, Pedersen, and Sato, 2001).

Finally. . .

Happy Birthday, Ole!