Asymptotic indifference pricing in exponential Lévy models

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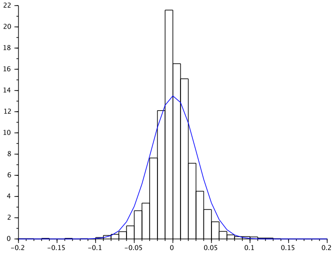
Exponential Lévy models

In the Black-Scholes-Samuelson model, the distribution of price returns (log-increments) is Gaussian, and the probability of extreme price moves is under-estimated.



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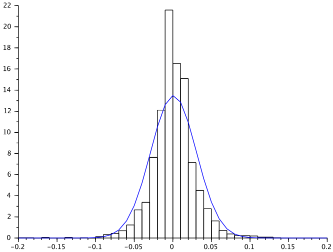

For this reason, it has been suggested to model prices with non-Gaussian processes with stationary and independent increments (Lévy processes).

We assume that the price process S of the stock satisfies

$$\frac{dS_t}{S_{t-}} = dX_t,$$

where X is a Lévy process.

Utility indifference pricing in exponential Lévy models

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Fair price: price at which seller is indifferent between selling and not selling (Hodges and Neuberger '89):

$$\max_{\varphi} \mathbb{E}\left[U\left(V_0 + \int_0^T \varphi_t dS_t\right)\right] = \max_{\varphi} \mathbb{E}\left[U\left(V_0 + p + \int_0^T \varphi_t dS_t - H_T\right)\right]$$

(the maximum is taken over a suitably defined set of admissible strategies)

Utility indifference price with exponential utility

We are interested in the exponential utility function $U(x) = -e^{-\alpha x}$, and define the set of admissible strategies

$$\Theta = \{\varphi \in L(S) \mid \exists L^* \text{ with } \mathbb{E}[e^{-\alpha L^*}] < \infty \text{ s.t. } (\varphi \cdot S)_t \geq L^* \, \forall t \in [0,T] \text{ a.s.}\}$$

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We make the standing assumption that the Lévy process X is not a.s. monotone and has bounded jumps: $|\Delta X_t| \leq \delta < 1$ a.s. for all $t \in [0, T]$.

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Then, for a bounded contingent claim H_T, the seller's indifference price satisfies

$$p = \frac{1}{\alpha} \log \frac{\min_{\varphi \in \Theta} \mathbb{E}\left[\exp\left(-\alpha \int_0^T \varphi_t dS_t + \alpha H_T\right)\right]}{\min_{\varphi \in \Theta} \mathbb{E}\left[\exp\left(-\alpha \int_0^T \varphi_t dS_t\right)\right]}$$

Pure investment problem

The pure investment problem

$$\min_{\varphi \in \Theta} \mathbb{E}\left[\exp\left(-\alpha \int_0^T \varphi_t dS_t\right)\right]$$

admits an explicit solution:

$\varphi_t^* = \frac{\vartheta^*}{S_{t-}}$ where ϑ^* is such that $\ell(-\alpha\vartheta^*) = \inf_u \ell(u)$ with

$$\ell(u) = \gamma u + \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{ux} - 1 - ux)\nu(dx).$$

Utility indifference price with exponential utility

Define a measure $\mathbb{Q}^*$ via

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{\exp\left(-\alpha \int_0^T \varphi_t^* dS_t\right)}{\mathbb{E}\left[\exp\left(-\alpha \int_0^T \varphi_t^* dS_t\right)\right]} = \frac{e^{-\alpha\vartheta^* X_T}}{\mathbb{E}[e^{-\alpha\vartheta^* X_T}]}.$$

Then, S is a martingale under $\mathbb{Q}^*$ and

$$p = \frac{1}{\alpha} \log \min_{\varphi} \mathbb{E}^{\mathbb{Q}^*}\left[\exp\left(-\alpha \int_0^T \varphi_t dS_t + \alpha H_T\right)\right]$$

$\mathbb{Q}^*$ is called minimal entropy martignale measure.

Dual representation of the indifference price

From Bellini and Frittelli (2002) (see also Delbaen et al., 2002):

$$p = \sup_{\mathbb{Q} \in \mathrm{EMM}(\mathbb{Q}^*)} \left\{ \mathbb{E}^{\mathbb{Q}}[H] - \frac{1}{\alpha} H(\mathbb{Q}|\mathbb{Q}^*) \right\},$$

where EMM $(\mathbb{Q}^*)$ denotes the set of martingale measures, equivalent to $\mathbb{Q}^*$ and $H(\mathbb{Q}|\mathbb{Q}^*)$ is defined by

$$H(\mathbb{Q}|\mathbb{Q}^*) = \mathbb{E}^* \left[\frac{d\mathbb{Q}}{d\mathbb{Q}^*} \ln \frac{d\mathbb{Q}}{d\mathbb{Q}^*} \right]$$

whenever this quantity is finite and equals $+\infty$ otherwise.

Approximating the indifference price: idea

Computation of the indifference price is a stochastic control problem leading to a nonlinear partial integro-differential equation difficult to solve in real-time

In the Black-Scholes model, the price is unique: the indifference price does not depend on the utility function and is given by $p = \mathbb{E}^{\mathbb{Q}^*}[H_T]$.

If a Lévy model is close to Black-Scholes, in the sense that H_T is almost replicated, can we find an approximation of the indifference price?

Reminder on quadratic hedging and martingale representation

- Quadratic hedging (Föllmer and Sondermann '85):
  Let $\tilde{H}_T = H_T - \mathbb{E}^{\mathbb{Q}^*}[H_T]$, $\tilde{H}_t = \mathbb{E}^{\mathbb{Q}^*}[\tilde{H}_T|\mathcal{F}_t]$ and let $\bar{\varphi}$ be the minimizer of

$$\mathbb{E}^{\mathbb{Q}^*}\left[\left(\int_0^T \varphi_s dS_s - \tilde{H}_T\right)^2\right] \quad \Rightarrow \quad \bar{\varphi}_t = \frac{d\langle \tilde{H}, S\rangle_t}{d\langle S, S\rangle_t}$$

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- The process $(\tilde{H}_t)_{0\leq t\leq T}$ admits the representation

$$\tilde{H}_t = \int_0^t \sigma_s dX_s^c + \int_0^t \int_{\mathbb{R}} \gamma_s(z)\tilde{J}_X(ds \times dz),$$

  where X^c is the continuous martingale part of the process X under $\mathbb{Q}^*$ and $\tilde{J}_X$ is the compensated jump measure of the process X under $\mathbb{Q}^*$.

Non-asymptotic approximation of the indifference price

Assume that there exists a constant $L < \infty$ with $2\delta L\alpha < 1$ such that

$$|H - \mathbb{E}^*[H]| \leq L \quad \text{a.s.},$$

$$|\sigma_t| \leq L \quad \text{a.s. for all } t \in [0, T],$$

$$|\gamma_t(z)| \leq L|z| \quad \text{a.s. for all } t \in [0, T] \text{ and all } z \in \operatorname{supp} \nu.$$

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Then there exists a constant $C < \infty$ such that for every $\varepsilon \in (0, 1]$ the seller's indifference price of the claim H satisfies

$$\left| p - \mathbb{E}^*[H] - \frac{\alpha}{2}\mathbb{E}^*\left[\left(\int_0^T \bar{\varphi}_s dS_s - \tilde{H}_T\right)^2\right]\right| \leq \alpha^{1+\varepsilon} C \mathbb{E}^*\left[\sup_{0\leq t\leq T}\left|\int_0^t \bar{\varphi}_s dS_s - \tilde{H}_t\right|^{2+\varepsilon}\right].$$

Structure of the indifference price

Seller's price:

$$p^{\text{sell}} \approx \mathbb{E}^{\mathbb{Q}^*}[H_T] + \frac{\alpha}{2}\mathbb{E}^{\mathbb{Q}^*}\left[\left(\int_0^T \bar{\varphi}_s dS_s - \tilde{H}_T\right)^2\right]$$

Buyer's price:

$$p^{\text{buy}} \approx \mathbb{E}^{\mathbb{Q}^*}[H_T] - \frac{\alpha}{2}\mathbb{E}^{\mathbb{Q}^*}\left[\left(\int_0^T \bar{\varphi}_s dS_s - \tilde{H}_T\right)^2\right]$$

Linear part (the same for all agents): $\mathbb{E}^{\mathbb{Q}^*}[H_T]$

Spread between seller's price and buyer's price: proportional to the risk aversion and the unhedgeable part of the risk.

$$\alpha\mathbb{E}^{\mathbb{Q}^*}\left[\left(\int_0^T \bar{\varphi}_s dS_s - \tilde{H}_T\right)^2\right] = \alpha \min_{\varphi} \mathbb{E}^{\mathbb{Q}^*}\left[\left(\int_0^T \varphi_s dS_s - \tilde{H}_T\right)^2\right]$$

Relationship to the literature

- Several authors (Kallsen and Rheinländer ’11; Kramkov and Sirbu ’07; Mania and Schweizer ’05; Becherer ’06, Delbaen et al., ’02) study the asymptotics of the indifference price as $\alpha \to 0$ in various settings.

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- By contrast, our approximation is non-asymptotic, and provides an error bound for finite α, as soon as $2\delta L\alpha < 1$.
- It allows to recover a variety of asymptotic approximations, for example, as $\alpha \downarrow 0$,

$$p = \mathbb{E}^*[H] + \frac{\alpha}{2}\mathbb{E}^*\left[\left(\int_0^T \bar{\varphi}_s dS_s - \tilde{H}_T\right)^2\right] + O(\alpha^2),$$

extending Kallsen and Rheinländer (11).

Non-asymptotic approximation: idea of the proof

Under the assumptions of the Theorem,

$$\bar{\varphi}_t = \frac{1}{S_{t-}} \frac{\sigma\sigma_t + \int_{\mathbb{R}} z\gamma_t(z)\nu(dz)}{\sigma^2 + \int_{\mathbb{R}} z^2\nu(dz)}$$

and therefore $|S_{t-}\bar{\varphi}_t| \leq L$ a.s. for all $t \in [0, T]$.

Upper bound: use a suboptimal strategy:

Define

$$\tau = \inf\{t \geq 0 : \left|\int_0^t \bar{\varphi}_s dS_s - \tilde{H}_t\right| \geq 1\} \wedge T.$$

Then,

$$\left|\int_0^\tau \bar{\varphi}_s dS_s - \tilde{H}\right| \leq 3L + 1$$

and we can use the suboptimal strategy $\varphi_t = \bar{\varphi}_t \mathbf{1}_{t\leq\tau}$ and perform the Taylor expansion of the exponential.

Non-asymptotic approximation: idea of the proof

Lower bound: use the duality formula:

$$p = \sup_{\mathbb{Q}\in \mathrm{EMM}(\mathbb{Q}^*)} \left\{ \mathbb{E}^{\mathbb{Q}}[H] - \frac{1}{\alpha} H(\mathbb{Q}|\mathbb{Q}^*) \right\}, \qquad H(\mathbb{Q}|\mathbb{Q}^*) = \mathbb{E}^{\mathbb{Q}^*}\left[\frac{d\mathbb{Q}}{d\mathbb{Q}^*} \ln \frac{d\mathbb{Q}}{d\mathbb{Q}^*}\right],$$

and take $\mathbb{Q} = D\mathbb{Q}^*$ with

$$D = 1 + \alpha \int_0^\tau \bar{\varphi}_t dS_t - \alpha \tilde{H}_\tau$$

and

$$\tau = \inf\left\{ t \geq 0 : \left| \int_0^t \bar{\varphi}_s dS_s - \tilde{H}_t \right| \geq \frac{1}{2} - \alpha L \delta \right\} \wedge T.$$

How to measure 'closedness to Black-Scholes'

An idea proposed for smooth linear functionals by Cerny, Denkl and Kallsen (2013).

Lévy model: $\frac{dS_t}{S_{t-}} = dX_t$ Black-Scholes model: $\frac{dS_t}{S_{t-}} = \sigma dW_t$

Recall that X is a martingale Lévy process with diffusion coefficient A and Lévy measure ν.

Let $X^\lambda_t = \lambda X_{t/\lambda^2}$. If $\int x^2 \nu(dx) < \infty$ then

$$(X^\lambda_t)_{t\geq 0} \xrightarrow[\lambda\to 0]{d} (\bar\sigma W_t)_{t\geq 0}, \quad \bar\sigma^2 = A + \int x^2 \nu(dx).$$

λ is an artificial small parameter allowing to expand a Lévy model around Black-Scholes.

A Taylor expansion of the indifference price

Let p_λ be the indifference price evaluated for X^λ. Then, $p_1 = p$ is the price of interest and p_0 is the Black-Scholes price.

If we can find a representation

$$p_\lambda = p_0 + \lambda p_0' + \frac{\lambda^2}{2} p_0'' + o(\lambda^2),$$

then p can be approximated by

$$p_0 + p_0' + \frac{1}{2} p_0''.$$

Expansion for the indifference price

Assume that

- The pay-off satisfies $H = h(S_T)$ where h is bounded, satisfies $|xh(x)| \leq L$ for some constant L, is a.e. differentiable and h' has finite variation.
- Either $\sigma > 0$ or there exists $\beta > 0$ such that $\liminf_{r\downarrow 0} \frac{\int_{[-r,r]} x^2\nu(dx)}{r^{2-\beta}} > 0$.

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Then, as $\lambda \to 0$,

$$\begin{aligned}
p^\lambda =& P_{BS}(S_0) + \frac{\lambda m_3 T}{6} S_0^3 P_{BS}^{(3)}(S_0) + \frac{\lambda^2 m_4 T}{24} S_0^4 P_{BS}^{(4)}(S_0) \\
&+ \frac{\lambda^2 m_3^2 T^2}{72}\left\{6 S_0^3 P_{BS}^{(3)}(S_0) + 18 S_0^4 P_{BS}^{(4)}(S_0) + 9 S_0^5 P_{BS}^{(5)}(S_0) + S_0^6 P_{BS}^{(6)}(S_0)\right\} \\
&+ \frac{\alpha\lambda^2}{8}\left(m_4 - \frac{m_3^2}{\bar\sigma^2}\right)\mathbb{E}^{BS}\left[\int_0^T \left(S_t^2 \frac{\partial^2 P_{BS}(t,S_t)}{\partial S^2}\right)^2 dt\right] + o(\lambda^2)
\end{aligned}$$

where $m_3 = \int_{\mathbb{R}} x^3\nu(dx)$, $m_4 = \int_{\mathbb{R}} x^4\nu(dx)$ and $\mathbb{E}^{BS}$ / P^{BS} denote the expectation / option price in the Black-Scholes model with volatility $\bar\sigma$.

Expansion for the linear part

No pay-off regularity is needed for this part due to the smoothing effect of the Lévy density

Assume that

- The function h is bounded measurable with polynomial growth.
- Either $\sigma > 0$ or there exists $\beta \in (0, 2)$ with $\liminf_{r \downarrow 0} \frac{\int_{[-r,r]} x^2 \nu(dx)}{r^{2-\beta}} > 0$.

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Then, as $\lambda \to 0$,

$$\mathbb{E}^{\mathbb{Q}^*}[H_T^\lambda] = P_{BS}(S_0) + \frac{\lambda m_3 T}{6} S_0^3 P_{BS}^{(3)}(S_0) + \frac{\lambda^2 m_4 T}{24} S_0^4 P_{BS}^{(4)}(S_0)$$
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See also Cerny, Denkl and Kallsen (2013) for the case of C^∞ pay-offs.

Expansion for the quadratic part

Put-style regularity needed, otherwise convergence in λ is slower

Under the assumptions of the Theorem,

$$\mathbb{E}^{\mathbb{Q}^*}\left[\left(\int_0^T \bar\varphi_s^\lambda dS_s^\lambda - \tilde H_T^\lambda\right)^2\right]$$

$$= \frac{\lambda^2}{4}\left(m_4 - \frac{m_3^2}{\bar\sigma^2}\right)\mathbb{E}^{BS}\left[\int_0^T \left(S_t^2 \frac{\partial^2 P_{BS}(t,S_t)}{\partial S^2}\right)^2 dt\right] + o(\lambda^2).$$

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In addition, for the put option,

$$\mathbb{E}^{BS}\left[\int_0^T \left(S_t^2 \frac{\partial^2 P_{BS}(t,S_t)}{\partial S^2}\right)^2 dt\right] = \frac{K^2}{2\pi\bar{\sigma}^2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} e^{-\frac{d^2}{1+t}}$$

where $d = \frac{1}{\bar{\sigma}\sqrt{T}}\log\frac{S_0}{K} - \frac{\bar{\sigma}\sqrt{T}}{2}$.

See also Cerny, Denkl and Kallsen (2013) for the case of C^∞ pay-offs.

Estimation of the residual term

Put-style regularity needed

Under the assumptions of the theorem, let $M_t^\lambda = \int_0^t \bar{\varphi}_s^\lambda dS_s^\lambda - H_t^\lambda$ and define $\bar{M}_T^\lambda = \sup_{0\leq t\leq T} |M_t^\lambda|$. Then $\forall q > 2$, as $\lambda \to 0$

$$\mathbb{E}^* \left[(\bar{M}_T^\lambda)^q\right] = O(\lambda^q (\ln \frac{1}{\lambda})^{\frac{q}{2}})$$

Numerical illustration

In the numerical illustration, we let $\lambda = 1$ and approximate the indifference price by

$$
\begin{aligned}
p^1 =& P_{BS}(S_0) + \frac{m_3 T}{6} S_0^3 P_{BS}^{(3)}(S_0) + \frac{m_4 T}{24} S_0^4 P_{BS}^{(4)}(S_0) \\
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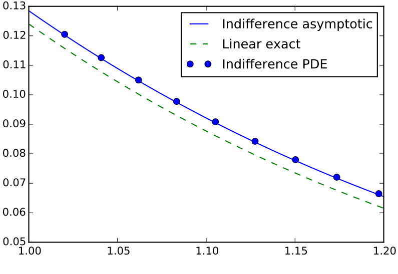
&+ \frac{\alpha}{8} \left(m_4 - \frac{m_3^2}{\bar{\sigma}^2} \right) \mathbb{E}^{BS} \left[\int_0^T \left(S_t^2 \frac{\partial^2 P_{BS}(t, S_t)}{\partial S^2} \right)^2 dt \right].
\end{aligned}
$$

Numerical illustration



Left: Indifference price with PIDE / asymptotic method, together with the linear part of the price $\mathbb{E}^{\mathbb{Q}^*}[(K - S_T)_+]$, in Merton model as function of S_0 for $\alpha = 10$.

Parameters: strike $K = 1$, maturity $T = 1$, diffusion volatility $\sigma = 0.2$, jump intensity $\lambda = 5$, average log jump -5%, log jump size std. dev. 10%.

Right: indifference price for ATM put as function of α.

Spread between buyer's and seller's price

The (half)-spread between the buyer's and the seller's indifference price may be seen as a valuation adjustment reflecting the difference between the model value of the option and its potential market price.

In the neighborhood of the Black-Scholes model, this spread is approximately

$$p^{\text{sell}} - p^{\text{buy}} \approx \underbrace{\alpha}_{\text{Risk aversion}} \times \underbrace{\frac{1}{4}\left(m_4 - \frac{m_3^2}{\bar{\sigma}^2}\right)}_{\text{Lévy model}} \times \underbrace{\mathbb{E}^{BS}\left[\int_0^T \left(S_t^2 \frac{\partial^2 P_{BS}(t, S_t)}{\partial S^2}\right)^2 dt\right]}_{\text{Jump risk sensitivity of the option}}$$

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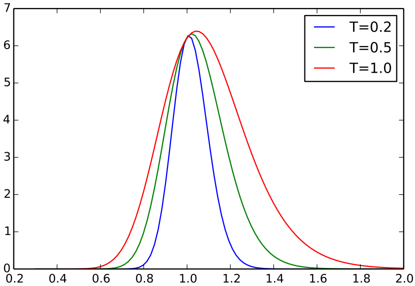
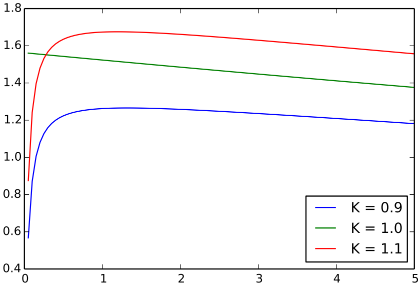
The factor

$$\mathbb{E}^{BS}\left[\int_0^T \left(S_t^2 \frac{\partial^2 P_{BS}}{\partial S^2}(t, S_t)\right)^2 dt\right]$$

can therefore be seen as a model-independent adjustment for mark to market risk for a European option in a Lévy model in the limit of small jumps.

Jump risk sensitivity





Left: jump risk sensitivity as function of K, $S_0 = 1$, $\bar{\sigma} = 0.2$.

Right: jump risk sensitivity as function of T, $S_0 = 1$, $\bar{\sigma} = 0.4$.

Extension: Indifference price and bid-ask spread under calibration constraints

- Assume that in the market, options with pay-offs $\boldsymbol{B} = (B_1, \dots, B_n)$ are liquidly traded, and (WLOG) their prices at time zero are equal to zero.
- For liquid options bid and ask prices coincide: $\mathbb{E}^*[\boldsymbol{B}] = 0$.
- In practice, MEMM can be found by *calibration* to market prices.
- Allowing (static) investment into the liquid options, the seller's indifference price becomes

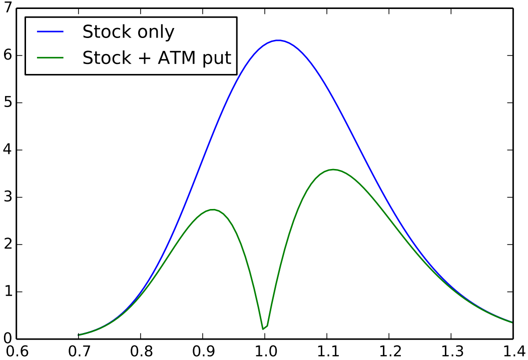
$$\bar{p}^s = \frac{1}{\alpha} \log \min_{\varphi \in \Theta, \boldsymbol{\beta} \in \mathbb{R}^n} \mathbb{E}^* \left[\exp\left(-\alpha \int_0^T \varphi_t dS_t + \alpha H - \alpha \boldsymbol{\beta}^T \boldsymbol{B} \right) \right]$$

Asymptotic spread of a European option under calibration constraints

$$\bar{p}^{\text{sell}} - \bar{p}^{\text{buy}} \approx \underbrace{\alpha}_{\text{Risk aversion}} \times \underbrace{\frac{1}{4}\left(m_4 - \frac{m_3^2}{\bar{\sigma}^2}\right)}_{\text{Lévy model}}$$
$$\times \underbrace{\min_{\boldsymbol{\beta}\in\mathbb{R}^n} \mathbb{E}^{BS}\left[\int_0^T S_t^4 \left(\frac{\partial^2 P_{BS}(t,S_t)}{\partial S^2} - \sum_{i=1}^n \beta_i \frac{\partial^2 P^i_{BS}(t,S_t)}{\partial S^2}\right)^2 dt\right]}_{\text{Jump risk valuation adjustment under calibration constraints}}$$

$\Rightarrow$ the jump risk valuation adjustment and the hedge ratios β_i are model independent.

Jump risk sensitivity reduction: hedging with options



Jump risk sensitivity of a European put as function of strike, hedged by an ATM put. Parameters: $S_0 = 1$, $T = 0.5$, $\bar{\sigma} = 20\%$.

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