SKEW DISPERSION & LOCAL TIME CONTINUITY

ED WAYMIRE

(with J. Ramirez*, E. Thomann)

OREGON STATE UNIVERSITY

Celebrating the scientific achievements of Ole E. Barndorff-Nielsen

Aarhus, Denmak

June 2015

*Universidad Nacional de Colombia

Partially Supported by a grant DMS-1408947 from the National Science Foundation

``Random change of time is key to understanding the nature of various stochastic processes and gives rise to interesting mathematical results and insights of importance for the modeling and interpretation of empirically observed dynamic processes.''

- O. E. Barndorff-Nielsen & A. N. Shiraev (2010)

OUTLINE OF TALK

- Some Motivating Examples
- An Applied Probability Perspective
- Statement of Main Results
- Sketch for Piecewise Constant Coefficients

Example 1: Dispersion in Heterogeneous Media

B. Wood, CBEE LAB, OSU

$$D(y) = \begin{cases} D^+ & \text{if } y \geq 0 \\ D^- & \text{if } y < 0. \end{cases}$$

c = concentration of injected dye

Fickean Dispersion Model:

$$\frac{\partial c}{\partial t} = \frac{1}{2}\nabla(D(y)\nabla c)$$

$$[D(y)\nabla c]_0 = 0$$

Experimentally Observed Asymmetry

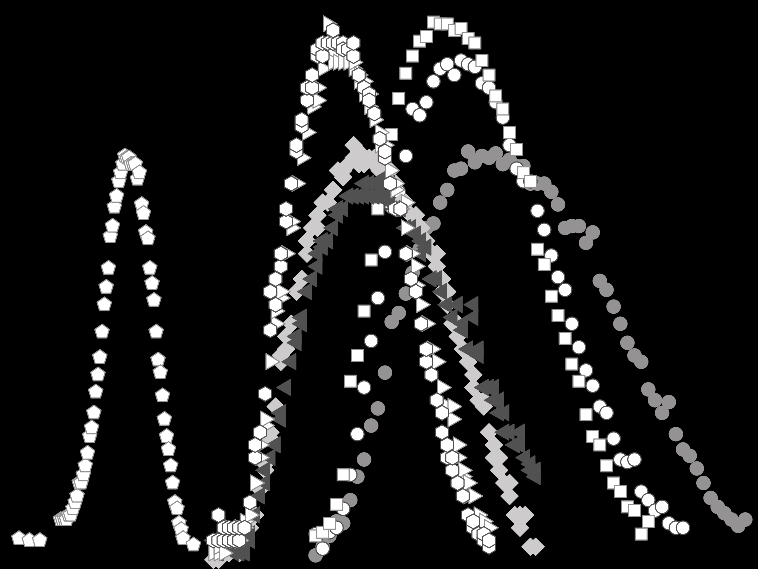
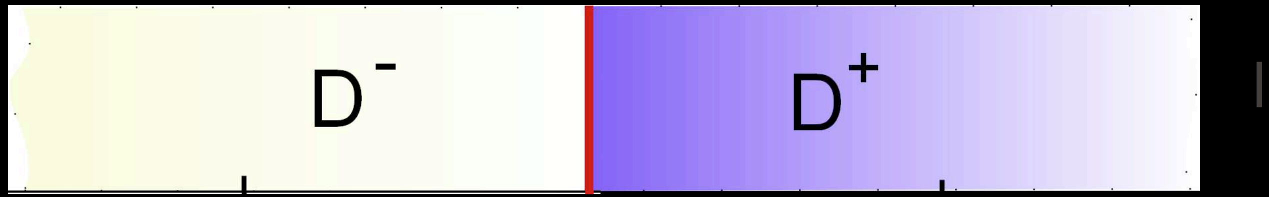


Figure 2. Experiments showing differences in breakthrough behavior for coarse to fine (C-F) and fine-to-coarse (F-C) directions of flow at flow rates of 0.3, 0.4, and 1 mL/min. The tracer input pulse times are 5, 3.75, and 1.5 min, and the sample collection intervals are 5, 3.75, and 1.5 min, respectively. Solid symbols represent C-F direction, and open symbols represent F-C direction. Two experiments in each direction were measured for the 0.3 and 0.4 mL/min flow rates. Each point represents an average of three measurements. The vertical axis shows electrical conductivity (EC), which is directly proportional to concentration.

B. Berkowitz, A. Cortis, I Dror and H Scher - 2009., B. Wood -2010.

PROBLEM: Explain the asymmetry from the Fickian Dispersion Model ?

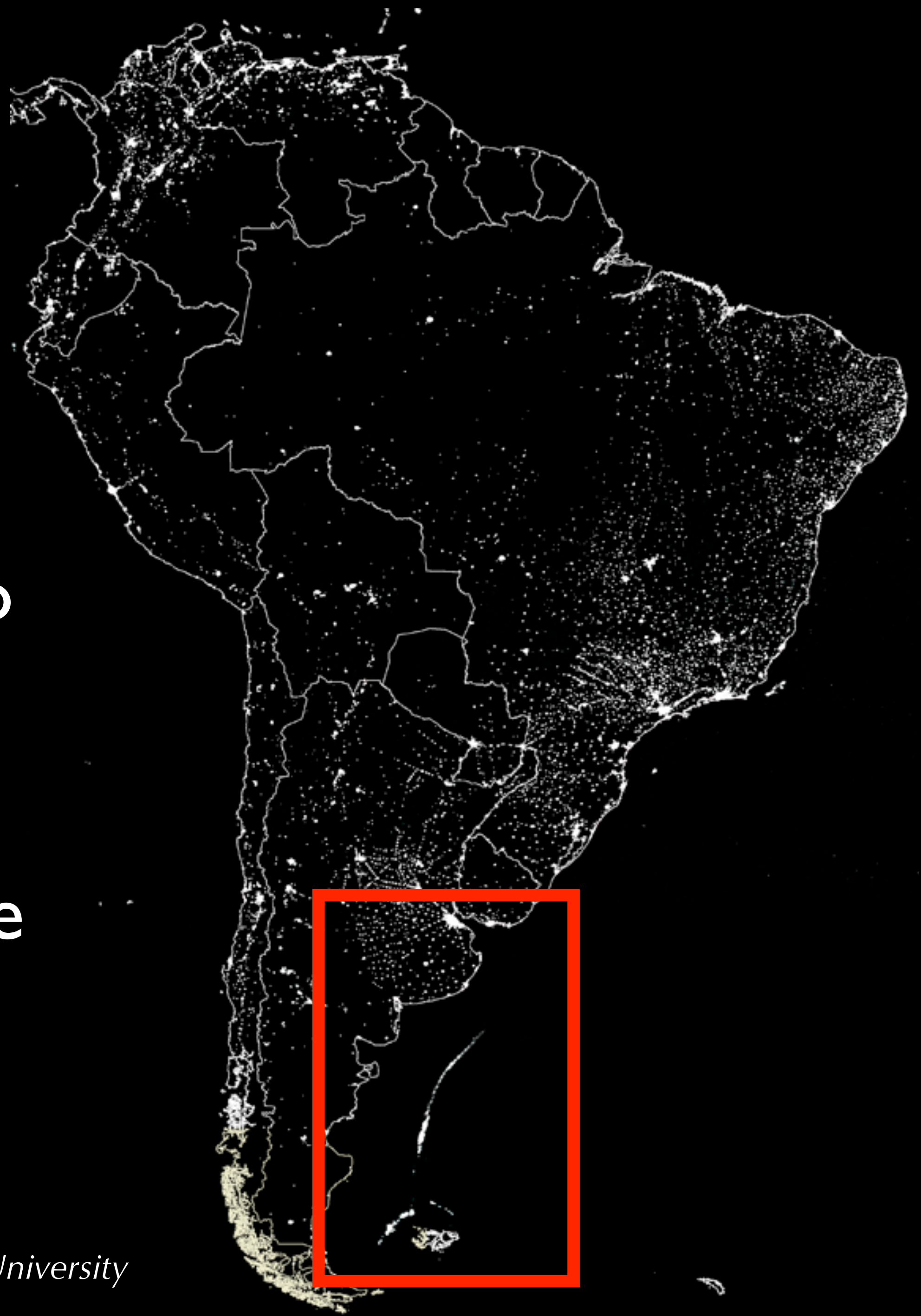


Q: Assume that D- < D+. Which is more likely removed first, a particle injected at -l and removed at l, or a particle injected at l and removed at -l ?

Example 2: Physical Oceanography - Upwelling off the coast of Argentina

Highlighted region points to flotilla of ‘fishing factories’

Concentration of ships on the continental break where upwelling occurs.



Acknowledgment: Ricardo Matano - COAS - Oregon State University

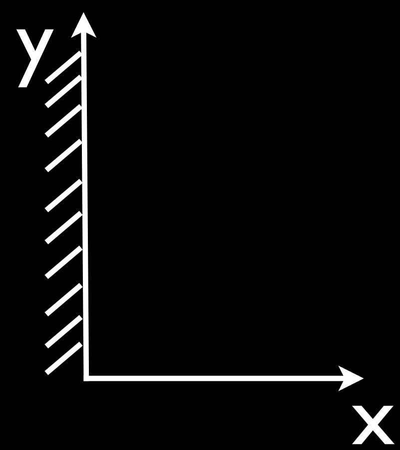
Arrested Topographic Wave Model

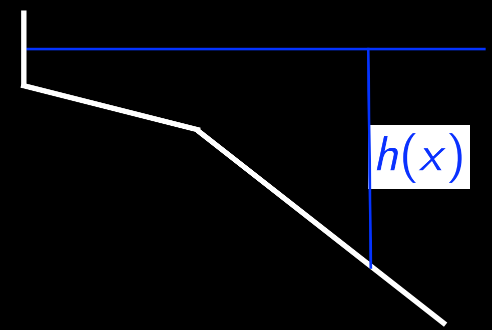
Quasi geostrophic balance - Hydrostatic Approximation.

$$-fv = -g\frac{\partial \eta}{\partial x}$$

$$-fu = -g\frac{\partial \eta}{\partial y} - \frac{rv}{h}$$

$$0 = \frac{\partial uh}{\partial x} + \frac{\partial vh}{\partial y}$$





$\eta(x, y)$ = free surface, *h(x)* = ocean depth

Eliminate velocity in terms of free surface

$$\frac{\partial \eta}{\partial y} + \frac{r}{f}\left(\frac{\partial h}{\partial x}\right)^{-1}\frac{\partial^2 \eta}{\partial x^2} = 0$$

f < 0 in Southern hemisphere

Arrested Topographic Wave Model

Transport = velocity times height of water column

Height of water column = $h(x) + \eta(x, y)$

$s^{\pm}$ = slope of continental shelf (-)and slope (+)

$$D^{+} = \frac{2r}{|f|s^{+}}, \quad D^{-} = \frac{2r}{|f|s^{-}}$$ (Shelf Break Interface)

Arrested Topographic Wave Model:

$$\frac{\partial \eta}{\partial y} = \frac{1}{2} D^{\pm} \frac{\partial \eta^2}{\partial x^2} \qquad \frac{\partial \eta}{\partial x}(y, I^{+}) = \frac{\partial \eta}{\partial x}(y, I^{-})$$

Example 3: FENDERS BLUE BUTTERFLY

Fender's Blue	Kincaid's Lupin	Patch Distribution
		

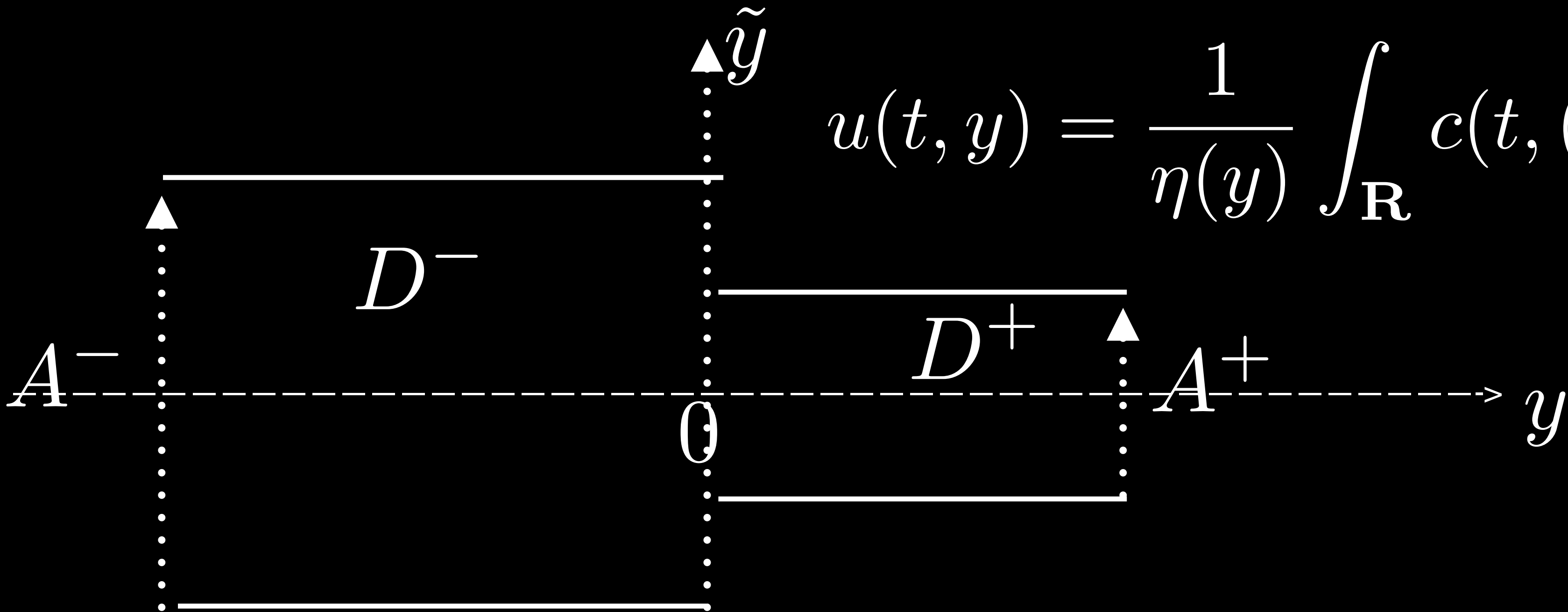
Ecology, 82(7), 2001, pp. 1879–1892
© 2001 by the Ecological Society of America

EDGE-MEDIATED DISPERSAL BEHAVIOR IN A PRAIRIE BUTTERFLY

CHERYL B. SCHULTZ[1] AND ELIZABETH E. CRONE[2]

Given past research on the Fender's blue and the potential to investigate response to patch boundaries in this system, we ask two central questions. First, how do organisms respond to habitat edges? Second, what are the implications of this behavior for residence time?”

Example 4: **(MARINE PROTECTED AREAS)**



$$u(t,y) = \frac{1}{\eta(y)} \int_{\mathbf{R}} c(t,(y,\tilde{y}))d\tilde{y}$$

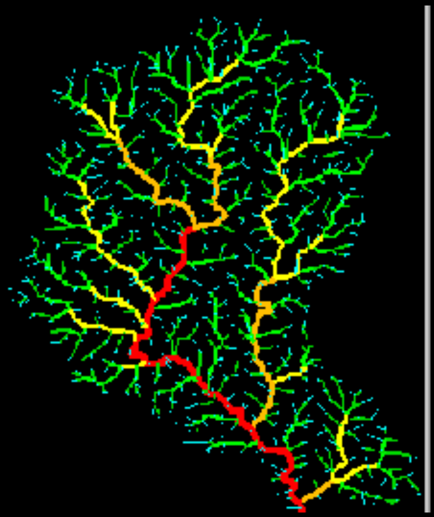
$\frac{1}{\eta(y)}$ = biomass of individuals at cross-sectional location y

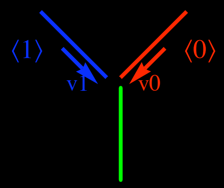
$c(t,(y,\tilde{y}))$ = biomass concentration at location $(y,\tilde{y})$ at time t

$$\frac{\partial u}{\partial t} = \frac{1}{\eta}\frac{\partial}{\partial y}(\frac{1}{2}D\frac{\partial u}{\partial y}) \qquad [D\frac{\partial u}{\partial y}]_0 = 0, \quad [\frac{u}{A}]_0 = 0$$

Rekow, J. (2015); Impact of biased movement on marine protected area effectiveness, OSU

EXAMPLE 5: DRIFT PARADOX IN RIVER NETWORKS





$$\frac{\partial c}{\partial t} = rc(t,y) - \mu c(t,y) + \mu \int_T p(x,y)c(t,x)dx$$

DRIFT PARADOX PROBLEM: IDENTIFY SUBNETWORKS FOR WHICH C = 0 IS UNSTABLE EQUILIBRIA?

Reference:

Ramirez, J.M. (2012) Population persistence under advection-dispersion in river networks, **J. Math. Bio.**

Example 6: **Modeling Bounces and Sinks of Financial Firms**

Financial firms in distress `bounce or sink' around some distress level. How can the (stock price) probabilities be coded into a model for distressed firms ?

Reference:

Nilsen, W., and H. Sayit (2011): No arbitrage in markets with bounces and sinks, *Intl. Rev. Appld. Financial Issues and Economics*, v3(4), 696-699.

A. Fokker-Planck Conservation Equation:

$$\frac{\partial u}{\partial t} = \frac{1}{\eta(y)}\frac{\partial}{\partial y}\left(D(y)\frac{\partial u}{\partial y}\right)$$

(Continuity of Flux) $[D(y)\frac{\partial u}{\partial y}]_I = 0$

$$[\beta(y)u]_I = 0$$

A. Fokker-Planck Conservation Equation:

$$\frac{\partial u}{\partial t} = \frac{1}{\eta(y)} \frac{\partial}{\partial y} (D(y) \frac{\partial u}{\partial y})$$

(Continuity of Flux) $[D(y) \frac{\partial u}{\partial y}]_I = 0$

$$[\beta(y) u]_I = 0$$

NOTATION: $[g]_I = g(y_j^+) - g(y_j^-)$

$I = \{\cdots < y_{-1} < y_0 = 0 < y_1 < \cdots\}$

$D(y)$ piecewise continuous with jump discontinuities in I

$\eta(y)$ piecewise continuous with jump discontinuities in I

$\beta(y)$ piecewise constant with jump discontinuities in I

A. Fokker-Planck Conservation Equation:

$$\frac{\partial u}{\partial t} = \frac{1}{\eta(y)} \frac{\partial}{\partial y} (D(y) \frac{\partial u}{\partial y})$$

(Continuity of Flux) $[D(y) \frac{\partial u}{\partial y}]_I = 0$

$$[\beta(y) u]_I = 0$$

$$I = \{\cdots < y_{-1} < y_0 = 0 < y_1 < \cdots\}$$

$D(y)$ piecewise continuous with jump discontinuities in I

$\eta(y)$ piecewise continuous with jump discontinuities in I

$\beta(y)$ piecewise constant with jump discontinuities in I

B. Adjoint Backward Equation:

$$\frac{\partial(\frac{v}{\eta})}{\partial t} = \frac{1}{\eta(x)}\frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}(\frac{v}{\eta})\right)$$

$$[\frac{D(x)}{\beta(x)}\frac{\partial}{\partial x}(\frac{v}{\eta})]_I = 0$$

$$\left[\ \frac{v}{\eta}\ \right]_I = 0$$

B. Adjoint Backward Equation:

$$\frac{\partial(\frac{v}{\eta})}{\partial t} = \frac{1}{\eta(x)}\frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}(\frac{v}{\eta})\right) \qquad (= L(\frac{v}{\eta}))$$

$$[\frac{D(x)}{\beta(x)}\frac{\partial}{\partial x}(\frac{v}{\eta})]_I = 0$$

$$\left[\ \frac{v}{\eta}\ \right]_I = 0$$

$$w = \frac{v}{\eta} \in C_0(\mathbb{R}) \qquad A = L(\eta v)$$

(Fokker-Planck) (Adjoint)

$$q(t,x,y) = \frac{\eta(x)}{\eta(y)} p(t,x,y)$$

C. Generic Infinitesimal Generator:

$$\mathcal{D}_A \subseteq C_0(\mathbb{R}) \quad A = \frac{1}{2\rho(x)}\frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}\right) \quad [\Lambda\frac{\partial}{\partial x}]_I = 0$$

$X = \{X(t) : t \geq 0\}$ Associated Feller Diffusion

Three-fold Perspective:

A. Forward Equation as Backward Equation:

$$\beta = 1, \rho = \eta, \Lambda = D$$

C. Generic Infinitesimal Generator:

$$\mathcal{D}_A \subseteq C_0(\mathbb{R}) \quad A = \frac{1}{2\rho(x)} \frac{\partial}{\partial x}\left(D(x) \frac{\partial}{\partial x}\right) \quad [\Lambda \frac{\partial}{\partial x}]_I = 0$$

$$X = \{X(t) : t \geq 0\} \text{ Associated Feller Diffusion}$$

Three-fold Perspective:

A. Forward Equation as Backward Equation:

$$\beta = 1, \rho = \eta, \Lambda = D$$

B. Adjoint Equation as Backward Equation:

$$\rho = \eta, \Lambda = \frac{D(x)}{\beta(x)}$$

C. Generic Infinitesimal Generator:

$$\mathcal{D}_A \subseteq C_0(\mathbb{R}) \quad A = \frac{1}{2\rho(x)}\frac{\partial}{\partial x}\Big(D(x)\frac{\partial}{\partial x}\Big) \quad [\Lambda\frac{\partial}{\partial x}]_I = 0$$

$$X = \{X(t) : t \geq 0\} \text{ Associated Feller Diffusion}$$

Three-fold Perspective:

A. Forward Equation as Backward Equation:

$$\beta = 1, \rho = \eta, \Lambda = D$$

B. Adjoint Equation as Backward Equation:

$$\rho = \eta, \Lambda = \frac{D(x)}{\beta(x)}$$

B'. Prescribed Backward Equation:

$$\rho = 1, D, \Lambda \text{ Given parameters}$$

C. Generic Infinitesimal Generator:

$$\mathcal{D}_A \subseteq C_0(\mathbb{R}) \quad A = \frac{1}{2\rho(x)}\frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}\right) \quad [\Lambda\frac{\partial}{\partial x}]_I = 0$$

$X = \{X(t) : t \geq 0\}$ Associated Feller Diffusion

C. Generic Infinitesimal Generator:

$$\mathcal{D}_A \subseteq C_0(\mathbb{R}) \quad A = \frac{1}{2\rho(x)}\frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}\right) \quad [\Lambda\frac{\partial}{\partial x}]_I = 0$$

$X = \{X(t) : t \geq 0\}$ Associated Feller Diffusion

Speed and Scale Measures: $\frac{D(x_j^-)}{\varphi_{j-1}} f'(x_j^-) = \frac{D(x_j^+)}{\varphi_j} f'(x_j^+)$

$$\frac{\varphi_j}{\varphi_{j-1}} = \frac{D(x_j^+)(1-\lambda_j)}{D(x_j^-)\lambda_j} = \frac{\beta_j^+}{\beta_j^-}, \ \varphi_0 = 1.$$

$$Af(x) = \frac{\varphi_j}{\eta(x)}\frac{d}{dx}\left(\frac{D(x)}{2\varphi_j}\frac{df}{dx}\right), \ x \in (x_j, x_{j+1})$$

$$s'(x) = \frac{2\varphi_j}{D(x)}, \ m'(x) = \frac{\eta(x)}{\varphi_j}, \ x \in (x_j, x_{j+1})$$

(D, I, ρ, Λ) – Assumptions:

D piecewise differentiable and bounded variation
ρ piecewise continuous

I has no accumulation points.

$g = D, \rho \in$ LeGall Class: $\exists$ strictly increasing f

$$|g(y) - g(x)|^2 \leq |f(y) - f(x)|$$

g is bounded between two positive constants.

$$\int_0^\infty \int_0^x \frac{\rho(y)}{D(y)} dy dx = \int_{-\infty}^0 \int_x^0 \frac{\rho(y)}{D(y)} dy dx = \infty$$

C. Generic Infinitesimal Generator:

$$\mathcal{D}_A \subseteq C_0(\mathbb{R}) \;\; A = \frac{1}{2\rho(x)} \frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}\right) \;\; [\Lambda \frac{\partial}{\partial x}]_I = 0$$

$X = \{X(t) : t \geq 0\}$ Associated Feller Diffusion

Speed and Scale Measures:

$$s'(x) = \frac{2\varphi_j}{D(x)}, \;\; m'(x) = \frac{\eta(x)}{\varphi_j}, \;\; x \in (x_j, x_{j+1})$$

Theorem X is the unique strong solution to

$$X(t) = \int_0^t \sqrt{\frac{D(X(s))}{\eta(X(s))}}\, dB(s) - \int_0^t \frac{D'(X(s))}{2\eta(X(s))}\, ds + \sum_j \frac{2\lambda_j - 1}{2\lambda_j} L_+^X(t, x_j)$$

$L^+(t, x)$ is right continuous semimartingale local time of X.

Proof. Use BN-Shiraev time change principle !

Interfaces Abound !

Examples: Hydrology, Biology/Ecology, Oceanography, Astrophysics, Finance

c.f. Ramirez, Thomann, W. (2013): Statistical Science, IMS

Basic Question: How do the sample path properties of X exhibit the parameters D, I, Λ?

$e.g.,$ i. First Passage Times
ii. Occupation Times

$\Big\{$ Stochastic order problems

iii. Local Times ✓

On Brownian Motion Observations

``The trajectories are confused and complicated so often and so rapidly that it is impossible to follow them; the trajectory actually measured is very much simpler and shorter than the real one. Similarly, the apparent mean speed of a grain during a given time varies in a wildest way in magnitude and direction, and does not tend to a limit as the time taken for an observation decreases, as may be easily shown by noting, in the camera lucida, the positions occupied by a grain from minute to minute, and then every five seconds, or, better still, by photographing them every twentieth of a second, as has been done by Victor Henri Comandon, and de Broglie when kinematographing the movement. It is impossible to fix a tangent, even approximately, at any point on a trajectory, and we are thus reminded of the continuous underived functions of the mathematicians."

- Jean Baptiste Perrin, Atoms 1913

Q: What might Perrin report on paths of X ?

Ans: Characterize continuity/discontinuities in local time.

On Brownian Motion Observations

``The trajectories are confused and complicated so often and so rapidly that it is impossible to follow them; the trajectory actually measured is very much simpler and shorter than the real one. Similarly, the apparent mean speed of a grain during a given time varies in a wildest way in magnitude and direction, and does not tend to a limit as the time taken for an observation decreases, as may be easily shown by noting, in the camera lucida, the positions occupied by a grain from minute to minute, and then every five seconds, or, better still, by photographing them every twentieth of a second, as has been done by Victor Henri Comandon, and de Broglie when kinematographing the movement. It is impossible to fix a tangent, even approximately, at any point on a trajectory, and we are thus reminded of the continuous underived functions of the mathematicians."

- Jean Baptiste Perrin, Atoms 1913

Q: What might Perrin report on paths of X ?

Ans: Characterize continuity/discontinuities in local time.

New Technology: Newburgh et al (2006): Einstein, Perrin, and the reality of atoms: 1905 revisited, Am. J. Physics

LOCAL TIMES:

1. SEMIMARTINGALE LOCAL TIME:

$$\int_0^t \varphi(X(s))d\langle X\rangle_s = \int_{\mathbb{R}} \varphi(x)L^X(x,t)dx$$

LOCAL TIMES:

1. SEMIMARTINGALE LOCAL TIME:

$$\int_0^t \varphi(X(s))d\langle X\rangle_s = \int_{\mathbb{R}} \varphi(x)L^X(x,t)dx$$

2. DIFFUSION LOCAL TIME:

$$\int_0^t \varphi(X(s))ds = \int_{\mathbb{R}} \varphi(x)\tilde{L}^X(x,t)m(dx)$$

LOCAL TIMES:

1. SEMIMARTINGALE LOCAL TIME:

$$\int_0^t \varphi(X(s))d\langle X\rangle_s = \int_{\mathbb{R}} \varphi(x)L^X(x,t)dx$$

2. DIFFUSION LOCAL TIME:

$$\int_0^t \varphi(X(s))ds = \int_{\mathbb{R}} \varphi(x)\tilde{L}^X(x,t)m(dx)$$

3. NATURAL LOCAL TIME:

$$\int_0^t \varphi(X(s))ds = \int_{\mathbb{R}} \varphi(x)\ell^X(x,t)dx$$

LOCAL TIMES: (+Right, Left, Symmetric Flavors)

1. SEMIMARTINGALE LOCAL TIME:

$$\int_0^t \varphi(X(s))d\langle X\rangle_s = \int_{\mathbb{R}} \varphi(x)L^X(x,t)dx$$

2. DIFFUSION LOCAL TIME:

$$\int_0^t \varphi(X(s))ds = \int_{\mathbb{R}} \varphi(x)\tilde{L}^X(x,t)m(dx)$$

3. NATURAL LOCAL TIME:

$$\int_0^t \varphi(X(s))ds = \int_{\mathbb{R}} \varphi(x)\ell^X(x,t)dx$$

LOCAL TIMES: (+Right, Left, Symmetric Flavors)

[UNITS]

1. SEMIMARTINGALE LOCAL TIME: [LENGTH]

$$\int_0^t \varphi(X(s))d\langle X\rangle_s = \int_{\mathbb{R}} \varphi(x)L^X(x,t)dx$$

2. DIFFUSION LOCAL TIME: [DIMENSIONLESS]

$$\int_0^t \varphi(X(s))ds = \int_{\mathbb{R}} \varphi(x)\tilde{L}^X(x,t)m(dx)$$

3. NATURAL LOCAL TIME: [TIME/LENGTH]

$$\int_0^t \varphi(X(s))ds = \int_{\mathbb{R}} \varphi(x)\ell^X(x,t)dx$$

Theorem

Consider the Feller diffusion $X\ for\ \eta, D, \Lambda$.

$$\frac{\ell^X(t, x_j^+)}{\ell^X(t, x_j^-)} = \frac{\eta(x_j^+)}{\eta(x_j^-)} \frac{D(x_j^-)}{D(x_j^+)} \frac{\lambda_j}{1-\lambda_j}.$$

Corollary

Let u denote solution to FP eqn for $D, \eta, \beta_j^{\pm}, j \in \mathbb{Z}$.

$$\frac{\ell^X(t, x_j^+)}{\ell^X(t, x_j^-)} = \frac{\eta(x_j^+)}{\eta(x_j^-)} \frac{\beta_j^-}{\beta_j^+}$$

Special Case: Appuhamillage, Bokil, Thomann, W., Wood (2014), Jour. Stat.Phys.

Assume piecewise constant D

Single interface at 0 $\quad (\eta \equiv 1)$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial y}(D(y)\frac{\partial c}{\partial y})$$

$$\lambda \frac{\partial c}{\partial y}\Big|_{y=0^+} - (1-\lambda)\frac{\partial c}{\partial y}_{y=0^-} = 0$$

Theorem : X is a Feller diffusion such that

$$X(t) = \sqrt{D}(B^{(\alpha(\lambda))}) \qquad \alpha(\lambda) = \frac{\lambda\sqrt{D^-}}{\lambda\sqrt{D^-} + (1-\lambda)\sqrt{D^+}}$$

$$X(t) = X(0) + \int_0^t \sqrt{D}(X(s))dB(s) + \frac{1}{2}[\Lambda]_0 \frac{\ell_+(0,t)}{m'_+(0)}$$

where $x \to \dfrac{\ell_+(x,t)}{m'_+(x)} = \tilde{L}_+(x,t)$ is continuous.

$$m'_+(x) = \begin{cases} \frac{\lambda}{D^+} & \text{if } x \geq 0 \\ \frac{1-\lambda}{D^-} & \text{if } x < 0 \end{cases}$$

$$[\Lambda]_0 = \lambda - (1-\lambda) = 2\lambda - 1.$$

PROOF OF THEOREM :

$$\ell_+^X(0,t) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t 1[0 \le X(s) < \epsilon)] ds$$

$$= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t 1[0 \le \sqrt{D^+} B^{(\alpha)}(s) < \epsilon)] ds \quad = \frac{1}{\sqrt{D^+}} \ell_+^{B^{(\alpha)}}(0,t)$$

Similarly, $$\ell_-^X(0,t) = \frac{1}{\sqrt{D^-}} \ell_-^{B^{(\alpha)}}(0,t)$$

Thus, $$\frac{\ell_+^X(0,t)}{\ell_-^X(0,t)} = \frac{\alpha}{1-\alpha} \frac{\sqrt{D^-}}{\sqrt{D^+}} \quad = \frac{\lambda D^-}{(1-\lambda) D^+}$$

$$\frac{D^+}{\lambda} \ell_+^X(0,t) = \frac{D^-}{1-\lambda} \ell_-^X(0,t)$$

SOME IMPLICATIONS FOR RESIDENCE TIMES (especially Examples 1, 3)

Definition: For a continuous semimartingale Y we refer to

$$\Gamma_G^Y(t) = \int_0^t 1(Y(s) \in G)ds \qquad \text{(Units of TIME)}$$

as **natural occupation time.**
(in place of integration w.r. to quadratic variation units L^2).

Corollary. Let Y be natural diffusion for the parameters $D^{\pm}, \lambda$. Then,

$$E\Gamma^Y_{[0,\infty)}(t) \geq E\Gamma^Y_{(-\infty,0]}(t)$$

if and only if

$$\lambda \geq \frac{\sqrt{D^+}}{\sqrt{D^+} + \sqrt{D^-}}$$

PROOF.

$$E\Gamma^{+}_{[0,\infty)} = E\int_0^t 1[Y^{(\alpha(\lambda))}(s) > 0]ds$$

$$= E\int_0^t 1[\sqrt{D^+}B^{(\alpha(\lambda))}(s) > 0]ds$$

$$= \int_0^t P(B^{(\alpha(\lambda))}(s) > 0)ds$$

$$= t\alpha(\lambda)$$

$E\Gamma^{-}_{[0,\infty)} = t(1-\alpha(\lambda))$ Compute when ratio > 1. QED

Recall: Conservative case

$$\lambda = \frac{D^+}{D^+ + D^-} \quad > \frac{\sqrt{D^+}}{\sqrt{D^+} + \sqrt{D^-}}$$

$$D^+ > D^-$$

IMPLICATION: The physical diffusion spends a longer time in the region with larger diffusion. Mass conservation principles (models) may not be appropriate to ecological examples, e.g., individual animal dispersion.

THANK YOU

(More References Follow)

REFERENCES

Appuhamillage, T., V. Bokil, E. Thomann, E. Waymire, and B. Wood (2014): Skew dispersion and continuity of local time, J. Stat. Phys., 156(2), 384-394.

Bass, R.F., Z-Q Chen (2005): One-dimensional stochastic differential equations with singular and degenerate coefficients,Sankhya: 67(1), 19-45.

Le Gall, J.-F. (1984) One-dimensional stochastic differential equations involving the local times of the unknown process. 51--82, Lecture Notes in Math., 1095, Springer, Berlin

Lejay, Antoine (2006): On the constructions of the skew Brownian motion, Probability Surveys, 3(2) 413-466 (electronic).

Martinez, M., D. Talay (2012): One dimensional parabolic diffraction equations, EJP, 27, 1-32.

Ouknine, Y. (1991): Skew-Brownian motion and associated processes Theory Probab. Appl. 35, no. 1, 163--169.

Ramirez, J.~M. (2011), Multi-skewedBrownian motion and diffusion in layered media, Proc AMS, 139, 3739-3752.

Ramirez, J., E. Thomann and E. Waymire (2013): Advection-dispersion across interfaces, Statistical Sciences, vol 28 (4), IMS/Bernoulli Society, 487--509.

Ramirez, J. E. Thomann, E. Waymire (2015), Continuity of local time: An applied view

Walsh, J. (1978): A diffusion with a discontinuous local time, Asterisque ,,37–45.

THEOREMS

Theorem (D, I, ρ, Λ) defines a Feller diffusion X on $\mathbb{R}$. X is the path-wise unique strong solution of

$$X(t) = X(0) + \int_0^t \sqrt{D}(X(s))dB(s) + \frac{1}{2}\int_0^t D'_-(X(s))ds + \frac{1}{2}\int_0^t \sum_j \int_{\mathbb{R}} [\Lambda]_{x_u} \frac{\ell_+(x_j, ds)}{m'_+(x_j)}$$

cf.LeGall(1984), Martinez-Talay(2012), Bass-Chen(2005)