Some noncommutative processes and related topics

RMT & Wireless systems, Dyson-Brownian motion, free Brownian motion and noncommutative fractional processes.

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Based on joint works with Dìaz, Nualart, Pardo, Pèrez G.

I. Large dimensional sample covariance matrices

The Marchenko-Pastur Distribution

I. Marchenko-Pastur law

Sample covariance matrix and its spectrum

- $H = H_{p \times n} = (Z_{j,k} : j = 1, .., p, k = 1, ..., n)$ i.i.d.r.v.

$$\mathbb{E}(Z_{1,1}) = 0,\ \mathbb{E}(|Z_{1,1}|^2) = 1,\ \mathbb{E}(|Z_{1,1}|^4) < \infty.$$

- Sample covariance matrix

$$S_n = \frac{1}{n} HH^*$$

- If $Z_{j,k}$ have $N(0,1)$ distribution, S_n is Wishart random matrix.
- Empirical Spectral Distribution (ESD)

$$\widehat{F}_p^{S_n} = \widehat{F}_p^{\frac{1}{n}HH^*} = \frac{1}{p} \sum_{j=1}^{p} \delta_{\lambda_j(S_n)}.$$

where $0 \leq \lambda_p(S_n) \leq \cdots \leq \lambda_1(S_n)$ are eigenvalues of S_n.

I. Marchenko-Pastur theorem

Mat. Sb. (1967)

Theorem

If $p/n \to c > 0$, $\widehat{F}_p^{S_n}$ converges weakly in probability to the **Marchenko-Pastur (MP) distribution***:*

$$\mu_c(\mathrm{d}x) = \begin{cases} f_c(x)\mathrm{d}x, & \text{if} \quad c \geq 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_c(x)\mathrm{d}x, & \text{if} \quad 0 < c < 1, \end{cases}$$

$$f_c(x) = \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\mathbf{1}_{[a,b]}(x)$$
$$a = (1-\sqrt{c})^2, \quad b = (1+\sqrt{c})^2.$$

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- MP distribution plays in free probability the role Poisson distribution does in classical probability.

II. RMT and Wireless Communications

Pioneering work of Emre Telatar

Telatar (1999), Capacity of multi-antenna Gaussian channels. *European Transactions on Telecommunications.*

- A $p \times 1$ complex Gaussian random vector $\mathbf{u} = (u_1 \ \cdots \ u_p)^\top$ has a *Q-circularly symmetric complex Gaussian distribution* if

$$\mathbb{E}[(\hat{\mathbf{u}} - \mathbb{E}[\hat{\mathbf{u}}])(\hat{\mathbf{u}} - \mathbb{E}[\hat{\mathbf{u}}])^*] = \frac{1}{2}\begin{bmatrix} \mathrm{Re}[Q] & -\mathrm{Im}[Q] \\ \mathrm{Im}[Q] & \mathrm{Re}[Q] \end{bmatrix},$$

  for some nonnegative definite Hermitian $p \times p$ matrix Q where

$$\hat{\mathbf{u}} = [\mathrm{Re}(u_1), \ldots, \mathrm{Re}(u_p), \mathrm{Im}(u_1), \ldots, \mathrm{Im}(u_p)]^\top .$$

II. Telatar: RMT and Channel Capacity

- n_T antennas at trasmitter and n_R antennas at receiver.
- Linear channel with Gaussian noise

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}.$$

- $\mathbf{x}$ is the n_T-dimensional input vector. $(n_T = n)$.
- $\mathbf{y}$ is the n_R-dimensional output vector. $(n_R = p)$.
- $\mathbf{n}$ is the receiver 0-mean Gaussian noise, $\mathbb{E}\left(\mathbf{n}\mathbf{n}^*\right) = \mathrm{I}_{n_T}$.
- The $n_R \times n_T$ random matrix $\mathbf{H}$ is the channel matrix.
- $\mathbf{H} = \{h_{jk}\}$ is a random matrix. It models the propagation coefficients between each pair of trasmitter-receiver antennas.
- $\mathbf{x}, \mathbf{H}$ and $\mathbf{n}$ are independent.

- h_{jk} are i.i.d. complex r.v. with 0-mean and variance one ($\mathrm{Re}(Z_{jk}) \sim N(0, \frac{1}{2})$ independent of $\mathrm{Im}(Z_{jk}) \sim N(0, \frac{1}{2})$).
- Total power constraint P: upper bound for variance $\mathbb{E}||\mathbf{x}||^2$ of the input signal amplitude.
- Signal to Noise Ratio (SNR)

$$SNR = \frac{\mathbb{E}||\mathbf{x}||^2 / n_T}{\mathbb{E}||\mathbf{n}||^2 / n_R} = \frac{P}{n_T}.$$

- Channel capacity is the maximum data rate which can be transmitted reliably over a channel (Shannon (1948)).
- The capacity of this MIMO system channel is

$$C(n_R, n_T) = \max_{Q} \mathbb{E}_{\mathbf{H}} \left[\log_2 \det \left(\mathrm{I}_{n_R} + \mathbf{H} Q \mathbf{H}^* \right) \right].$$

- Maximum capacity when $Q = SNR\mathrm{I}_{n_T}$

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- In terms of ESD $\widehat{F}_{n_T}^{\frac{1}{n_T}\mathbf{H}\mathbf{H}^*}$ of sample covariance $\frac{1}{n_T}\mathbf{H}\mathbf{H}^*$

$$C(n_R, n_T) = n_R \int_0^\infty \log_2(1 + Px)\,\mathrm{d}\widehat{F}_{n_T}^{\frac{1}{n_T}\mathbf{H}\mathbf{H}^*}.$$

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- By Marchenko-Pastur theorem, if $n_R / n_T \to c$,

$$\frac{C(n_R, n_T)}{n_R} \to \int_a^b \log_2(1 + Px)\, \mathrm{d}\mu_c(x) = K(c, P).$$

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- For fixed P

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- For fixed P

$$C(n_R, n_T) \sim n_R K(c, P).$$

- **Increase capacity with more transmitter and receiver antennas with same total power constraint** P.

II. RMT and Wireless Communication

Some further developments (NOT TODAY)

- Non Gaussian distribution for i.i.d. entries h_{ij} of the channel matrix $\mathbf{H}$: universality of the Marchenko-Pastor law.
  - Bai & Silverstein (2010). *Spectral Analysis of Large Dimensional Random Matrices.*
- Correlation models for $\mathbf{H}$, Kronecker correlation, etc..
  - Lozano, Tulino & Verdú. (2005). Impact of antenna correlation on the capacity of multiantenna channels. *IEEE Trans. Inform. Theor.*
  - Lozano, Tulino & Verdú (2006). Capacity-achieving input covariance for single-user multi-antenna channels. *IEEE Trans. Wireless Comm.*

II. RMT and Wireless Communication

Further developments (NOT TODAY)

- Books on RMT and Wireless Communications:
  - Tulino & Verdú (2004). *Random Matrix Theory and Wireless Communications.*
  - Couillet & Debbah (2011). *Random Matrix Methods for Wireless Communications.*
  - Bai, Fang & Ying-Chang (2014). *Spectral Theory of Large Dimensional Random Matrices and Its Applications to Wireless Communications and Finance Statistics*.
- Main problem is the computation of the asymptotic channel capacity, mainly done by a technique introduced by Girko (1990), solving a non-linear system of functional equations.
  - Couillet, R., Debbah, M., and Silverstein, J. (2011). A deterministic equivalent for the analysis of correlated MIMO multiple access channels. *IEEE Trans. Inform.Theor.*

II. RMT and Wireless Communication

Further developments (NOT TODAY)

- Recently, tools from Operator-valued free probability theory have been successful used as alternative to approximate the asymptotic capacity of new models:
- Ding (2014), Götze, Kösters & Tikhomirov (2015), Hachem, Loubaton & Najim (2007), Shlyakhtenko (1996), Helton, Far & Speicher (2007), Speicher, Vargas & Mai (2012), Belinschi, Speicher, Treilhard & Vargas (2014), Belinschi, Mai & Speicher, R. (2015),
- Diaz-Torres & PA (2017). On the capacity of block multiantena channels. *IEEE Trans. Inform. Theor.*

II. Time-varying random matrices: why?

Motivation for TODAY

Couillet & Debbah (2011), *Random Matrix Methods for Wireless Communications.* Chapter 19, Perspectives:

- Performance analysis of a typical network with users in motion according to some stochastic behavior, is not accessible to this date in the restrictive framework of random matrix theory.

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- It is to be believed that random matrix theory for wireless communications may move on a more or less long-term basis towards random matrix process theory for wireless communications. Nonetheless, these random matrix processes are nothing new and have been the interest of several generations of mathematicians.

Part III: Dyson-Brownian motion

III. Hermitian Brownian motion

- $\mathbf{B}(t) = (B_n(t))_{n\geq 1}$, $t \geq 0$.
- $B_n(t)$ is $n \times n$ Hermitian Brownian motion:

$$B_n(t) = (b_{ij}(t)), t \geq 0,$$

$$\mathrm{Re}(b_{ij}(t)) \sim \mathrm{Im}(b_{ij}(t)) \sim N(0, t(1+\delta_{ij})/2,$$

  where $\mathrm{Re}(b_{ij}(t)), \mathrm{Im}(b_{ij}(t)), 1 \leq i \leq j \leq n$ are independent one-dimensional Brownian motions.

- $(\lambda_1(t), ..., \lambda_n(t))_{t\geq 0}$ process of eigenvalues *of* $\{B_n(t)\}_{t\geq 0}$

$$\lambda_1(t) \geq \lambda_2(t) \geq ... \geq \lambda_n(t).$$

III. Dyson-Brownian motion

Time dynamics of the eigenvalues, dimension n fixed

Theorem

Dyson (1962):

1) If eigenvalues start at different positions, they never collide

$$\mathbb{P}\left(\lambda_1(t) > \lambda_2(t) > \dots > \lambda_n(t) \quad \forall t > 0\right) = 1.$$

2) They satisfy the Stochastic Differential Equation (SDE)

$$\lambda_i(t) = \lambda_i(0) + W_i(t) + \sum_{j \neq i} \int_0^t \frac{\mathrm{d}s}{\lambda_j(s) - \lambda_i(s)}, i = 1, \dots, n.$$

$\forall t > 0$, *where* $W_1, \dots, W_n$ *are 1-dimensional independent Bms.*

- Brownian part + repulsion force (at any time t).

III. Time-varying Wigner theorem and law of Free Bm

- Consider the Dyson spectral measure-valued processes

$$\mu_t^{(n)} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\{\lambda_j(t)/\sqrt{n}\}}, \quad t \geq 0, n \geq 1.$$

- Notation: For f μ-integrable function $\langle \mu, f \rangle = \int f(x)\mu(\mathrm{d}x)$.

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- Uniform Wigner theorem

$$\mathbb{P}\left(\lim_{n\to\infty}\sup_{0\leq t\leq T}\left|\left\langle \mu_t^{(n)}, f\right\rangle - \langle \mathrm{w}_t, f\rangle\right| = 0, \forall f \in C_b(\mathbb{R})\right) = 1.$$

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- The family of probability measures $\{\mathrm{w}_t\}_{t\geq 0}$ is the Law of the Free Brownian motion,

$$\mathrm{w}_t(\mathrm{d}x) = \frac{1}{2\pi t}\sqrt{4t - x^2}\,1_{[-2\sqrt{t},2\sqrt{t}]}(x)\mathrm{d}x.$$

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- Semicircle distribution plays in free probability the role Gaussian distribution does in classical probability.

III. Smooth vs non smooth SDE

A detour to understand why free Bm

- Interacting SDE with both **smooth drift & diffusion coefficients** β and α are of the form

$$X_{n,i}(t) = X_{n,i}(0) + \frac{1}{\sqrt{n}} \sum_{j \neq i} \int_0^t \beta(X_{n,j}(s), X_{n,i}(s)) \mathrm{d} W_i^{(n)}(t) + \frac{1}{n} \sum_{j \neq i} \int_0^t \alpha(X_{n,j}(s), X_{n,i}(s)) \mathrm{d}s.$$

- While Dyson-Brownian motion has **non smooth drift**

$$X_{n,i}(t) = X_{n,i}(0) + \frac{1}{\sqrt{n}} W_i^{(n)}(t) + \frac{1}{n} \sum_{j \neq i} \int_0^t \frac{1}{X_{n,i}(s) - X_{n,j}(s)} \mathrm{d}s.$$

- Empirical measure valued process

$$\mu_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{X_{n,j}(t)}, \quad t \geq 0, n \geq 1.$$

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For Interacting SDE with both **smooth drift & diffusion coefficients:**

- McKean (1967): $\left\{\mu_t^{(n)}\right\}_{t\geq 0}$ converges weakly in probability to $\{\mu_t\}_{t\geq 0}$, which is the law of a stochastic differential equation.

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- Interacting SDE with **non smooth drift coefficient** arise from eigenvalue processes of matricial processes [Bru (1989), Rogers & Shi (1993), Konig & O´Connell (2001), Cabanal-Duvillard & Guionnet (2001), Katori & Tanemura (2004)].

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- The family of probabilities $\{w_t, t \geq 0\}$ is not the law of a SDE equation, but the law of a noncommutative process: Free Brownian motion.

Part IV: Free Brownian motion

IV. Noncommutative probability spaces

- A *noncommutative probability space* $(\mathcal{A}, \varphi)$ is a unital algebra $\mathcal{A}$ over $\mathbb{C}$ with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi(1_{\mathcal{A}}) = 1$. Elements of $\mathcal{A}$ are called *noncommutative random variables*.
- We should think of φ as playing the role of the expectation in classical probability theory.
- Distribution μ on $\mathbb{R}$ (bounded support), of a self-adjoint $\mathbf{a} \in \mathcal{A}$ in a C^*-probability space $(\mathcal{A}, \varphi)$

$$\varphi(f(\mathbf{a})) = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x), \quad \forall f \in C_b(\mathbb{R}).$$

- A family of subalgebras $\{\mathcal{A}_i\}_{i\in I} \subset \mathcal{A}$ in a noncommutative probability space is *free* (**freely independent**) if

$$\varphi(\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_n) = 0$$

whenever $\varphi(\mathbf{a}_j) = 0$, $\mathbf{a}_j \in \mathcal{A}_{i_j}$, and $i_1 \neq i_2$, $i_2 \neq i_3$, ..., $i_{n-1} \neq i_n$.

IV. Free independence allows to compute joint moments

Example

Computation of $\varphi(\mathbf{abab})$ when $\mathbf{a}$ & $\mathbf{b}$ are freely independent:
Suppose $\{\mathbf{a}_1, \mathbf{a}_3\}$ and $\{\mathbf{a}_2, \mathbf{a}_4\}$ are freely independent. Since

$$\varphi(\mathbf{a}_i - \varphi(\mathbf{a}_i)1_{\mathcal{A}}) = 0,$$

$$\varphi(\mathbf{a}_1 - \varphi(\mathbf{a}_1)1_{\mathcal{A}})\varphi(\mathbf{a}_2 - \varphi(\mathbf{a}_2)1_{\mathcal{A}})\varphi(\mathbf{a}_3 - \varphi(\mathbf{a}_3)1_{\mathcal{A}})\varphi(\mathbf{a}_4 - \varphi(\mathbf{a}_4)1_{\mathcal{A}}) = 0$$

Computations yield

$$\begin{aligned}\varphi(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4) = \varphi(\mathbf{a}_1\mathbf{a}_3)\varphi(\mathbf{a}_2)\varphi(\mathbf{a}_4) + \varphi(\mathbf{a}_1)\varphi(\mathbf{a}_3)\varphi(\mathbf{a}_2\mathbf{a}_4) \\ - \varphi(\mathbf{a}_1)\varphi(\mathbf{a}_2)\varphi(\mathbf{a}_3)\varphi(\mathbf{a}_4).\end{aligned}$$

In particular if $\mathbf{a}_1 = \mathbf{a}_3 = \mathbf{a}$ and $\mathbf{a}_2 = \mathbf{a}_4 = \mathbf{b}$

$$\varphi(\mathbf{abab}) = \varphi(\mathbf{a})^2\varphi(\mathbf{b}^2) + \varphi(\mathbf{a}^2)\varphi(\mathbf{b})^2 - \varphi(\mathbf{a})^2\varphi(\mathbf{b})^2 \neq \varphi(\mathbf{a}^2)\varphi(\mathbf{b}^2).$$

IV. Free Brownian motion

A noncommutative process

A Free Brownian motion is a family $S = \{S_t\}_{t \geq 0}$ of self-adjoint random variables in a noncommutative probability space $(\mathcal{A}, \varphi)$ such that:

1. $S_0 = 0$.
2. For $t_2 \geq t_1 \geq 0$, $S_{t_2} - S_{t_1}$ has law $\mathrm{w}_{t_2 - t_1}$.
3. For all $n \geq 1$ and $t_n > \cdots > t_1 > 0$, the increments $S_{t_n} - S_{t_{n-1}}, ..., S_{t_2} - S_{t_1}, S_{t_1}$ are freely independent with respect to φ.

- For every $t \geq 0$, S_t has semicircle law w_t of zero mean and variance one.

Part V: From Fractional Wishart process to Noncommutative Wishart process

V. Fractional Wishart process

- $m, n \geq 1$, $m \times n$ matrix process

$$\{B_{m,n}(t)\}_{t\geq 0} = \left\{ \left(b_{m,n}^{j,k}(t) \right)_{1\leq j\leq m, 1\leq k\leq n} \right\}_{t\geq 0},$$

$\left\{ \mathrm{Re}\left(b_{m,n}^{j,k}(t) \right) \right\}_{t\geq 0}$ & $\left\{ \mathrm{Im}\left(b_{m,n}^{j,k}(t) \right) \right\}_{t\geq 0}$ independent 1-dimensional fractional Bm of parameter $H \in [1/2, 1)$.

- Fractional Laguerre, fractional Wishart process: $n \times n$ matrix-valued process

$$L_{m,n}(t) = B_{m,n}^{*}(t) B_{m,n}(t), t \geq 0.$$

- $0 \leq \lambda_n(t) \leq \cdots \leq \lambda_1(t)$ eigenvalues of $L_{m,n}(t)/n$.
- For $H \in [1/2, 1)$ the noncoliding property holds

$$\mathbb{P}\left(\lambda_1(t) > \lambda_2(t) > ... > \lambda_n(t) > 0 \quad \forall t > 0\right) = 1.$$

V. Fractional Wishart process

- $H = 1/2$:
  - Bru (1989): noncoliding property and stochastic dynamics

$$
\mathrm{d}\lambda_i(t) = \lambda_i(0) + \frac{1}{\sqrt{n}}\sqrt{2\lambda_i(t)}W_i(t)
$$

$$
+\frac{1}{n}\int_0^t \left(m + \sum_{j\neq i}\frac{\lambda_i(s)+\lambda_j(s)}{\lambda_i(s)-\lambda_j(s)}\right)\mathrm{d}s,\ 1 \leq i \leq n.
$$

  - Cabanal-Duvillard & Guionnet (2001), PA & Tudor (2009): limiting measure-valued process, when $n/m \to c > 0$, is dilation of free Poisson law.
- $H \in (1/2, 1)$: Pardo, Pérez G., PA (2017):
  - Noncoliding, stochastic dynamics of eigenvalues.
  - Limiting measure valued process is fractional dilation of MP law.

V. Dilation of MP law

Law of noncommutative fractional Wishart process

The limit, when $n/m \to c > 0$, of $\mu_t^{(n)} = \frac{1}{n}\sum_{j=1}^n \delta_{\lambda_j(t)}$, $t \geq 0$,

- *is not the law* $\{\mathrm{m}_{ct}\}_{t\geq 0}$,

$$\mathrm{m}_{ct}(\mathrm{d}x) = \begin{cases} f_{ct}(x)\mathrm{d}x, & ct \geq 1 \\ (1-ct)\delta_0(\mathrm{d}x) + f_{ct}(x)\mathrm{d}x, & 0 \leq ct < 1, \end{cases}$$

$$f_{ct}(x) = \frac{1}{2\pi x}\sqrt{4ct - (x-(1+ct))^2}\,\mathbf{1}_{[(1-\sqrt{ct})^2,(1+\sqrt{ct})^2]}(x)$$

V. Dilation of MP law

Law of noncommutative fractional Wishart process

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- *rather fractional dilations* of m_c: $\mu_c^H(t) = \mathrm{m}_{c,} \circ h_t^{-1}$, for $h_t(x) = t^{2H}x$, i.e.

$$\mu_c^H(t)(\mathrm{d}x) = \begin{cases} \widetilde{f}_c^t(x)\mathrm{d}x, & c \geq 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_{a,b}(x)\mathrm{d}x, & 0 \leq c < 1, \end{cases}$$

$$\widetilde{f}_c^t(x) = \frac{1}{2\pi t^{2H} x}\sqrt{4ct^{2H} - (x-t^H(1+c))^2}\mathbf{1}_{[t^{2H}(1-\sqrt{c})^2,t^{2H}(1+\sqrt{c})^2]}$$

V. Characterization of the law

Cabanal-Duvillard & Guionnet (2001): $H = 1/2$.

Pardo, Pérez G, PA (2017, *JFA*): $H \in (1/2, 1)$.

Theorem

The family $(\mu_c^H(t), t \geq 0)$ is characterized by the property that its Cauchy transform $G_{c,H}$ is the unique solution to

$$\frac{\partial G_{c,H}}{\partial t}(t,z) = 2Ht^{2H-1}\left[\begin{array}{c} G_{c,H}^2(t,z) + \\ (1 - c + 2zG_{c,1/2}(t,z))\frac{\partial G_{c,H}}{\partial z}(t,z) \end{array}\right], t > 0$$

$$G_{c,H}(0,z) = \int_{\mathbb{R}} \frac{\mu_{c,H}(0)(dx)}{x - z}.$$

V. Identification H=1/2, general c

Free Wishart process of Capitanie and Donati-Martin (2005)

- If $S = (S_t)_{t\geq 0}$ is a free (complex) Brownian motion $(H = 1/2)$, $\mathrm{W}_t = S_t^* S_t$ is a **free Wishart process**.
- It is a **free diffusion**: $c > 1, 0 < x \in \mathcal{A}$:

$$dW_t = c1_{\mathcal{A}} dt + \sqrt{\mathrm{W}_t} dS_t + dS_t^* \sqrt{\mathrm{W}_t}, \quad \mathrm{W}_0 = x.$$

- $(\mathrm{W}_t)_{t\geq 0}$ does not have free increments, **but** If $(S_t)_{t\geq 0}$, $\left(\widetilde{S}_t\right)_{t\geq 0}$ are free as well as $x, \widetilde{x}$, with parameters c_1, c_2 then $\left(\mathrm{W}_t + \widetilde{\mathrm{W}}_t\right)_{t\geq 0}$ is a free Wishart proces with parameter $c_1 + c_2$ and initial condition $x + \widetilde{x}$.
- **Open problems:**
  - $H \in (1/2, 2)$, description of the noncommutative fractional Wishart process?
  - How is related to the noncommutative fractional Brownian motion of Nourdin and Taqqu?

Some references

Dyson-Brownian motion and other noncoliding processes

- Anderson, Guionnet & Zeitouni (2010). *An Introduction to Random Matrices.* Cambridge University Press..
- Bru (1989). Diffusions of perturbed principal component analysis. *J. Multivariat. Anal.*
- Cabanal-Duvillard & Guionett (2001). Large deviations upper bounds for the laws of matrix-valued processes and non-communicative entropies. *Ann. Probab.*
- Dyson (1962). A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.*
- König & O 'Connel (2001). Eigenvalues of the Laguerre process as Non-colliding squared Bessel processes. *Electron. Commun. Probab.*
- Tao (2012). *Topics on Random Matrix Theory.* American Mathematical Society.

Some references

Dyson-Brownian motion and other noncoliding processes, more recently

- Holcomb & Paquette (2017). Tridiagonal models for Dyson Brownian motion. Arxive 1707.02700.
- Katori & Tanemura (2013). Complex Brownian motion representation of the Dyson model. *Electron. Commun. Probab.*
- Pérez-Abreu & Nualart (2014). *Stoch. Proces. & Appl.*
- Pérez-Abreu & Rocha-Arteaga (2016). On the process of the eigenvalues of Hermitian Lévy processes. In: *The Fascination of Probability, Statistics and their Applications: Festschrift in Honour of Ole E. Barndorff-Nielsen.*

Some references

Free stochastic calculus and infinite divisibility

- Anshelevich (2002). Itô formula for free stochastic integrals. *J. Funct. Anal.*
- Biane (1995). Brownian motion, free stochastic calculus and random matrices. *Free Probability Theory. Fields Inst. Commun. Amer. Math. Soc.*
- Biane & Speicher (1998). Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. *Probab. Theory. Relat. Fields.*
- Bercovici & Pata with an appendix by Biane (1999). Stable laws and domains of attraction in free probability theory. *Ann. Math.*
- Benaych-Georges (2005). Classical and free i.d. distributions and random matrices. *Ann. Probab.*
- Cabanal-Duvillard (2005): A matrix representation of the Bercovici-Pata bijection. *Electron. J. Probab.*

Some references

Noncommutative processes: free and fractional

- An & Gao (2015). Poisson processes in free probability. Arxive 1506.03130.
- Barndorff-Nielsen and Thorjorsen (2006). Classical and free infinite divisibility and Lévy processes. *Lecture Note Math.*
- Capitanie & Donata-Martin (2005). Free Wishart processes. *J. Theoret. Probab.*
- Nica and Speicher (2006). *Lectures on Combinatorics for Free Probability.* CUP.
- Voiculescu, Dykema and Nica (1992). *Free Random Variables.* Amer. Math. Soc.
- Nourdin & Taqqu (2014). Central and non-central limit theorems in a free probability setting. *J. Theoret. Probab.*
- Pardo, Pérez G. & Pérez-Abreu (2016). A random matrix approximation to the noncommutative fractional Brownian motion. *J. Theoret. Probab.*
- Pardo, Pérez G. & Pérez-Abreu (2017). On the free fractional Wishart process. *J. Funct. Anal.*

Thanks

Part V:

Free Brownian motion

and

Noncommutative fractional Brownian motion

V. Noncommutative probability spaces

- A *noncommutative probability space* $(\mathcal{A}, \varphi)$ is a unital algebra $\mathcal{A}$ over $\mathbb{C}$ with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with $\varphi(1_{\mathcal{A}}) = 1$. Elements of $\mathcal{A}$ are called *noncommutative random variables*.

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- $\mathcal{A} = \mathbb{M}_d(\mathbb{C})$ $d \times d$ matrices with complex entries

$$\varphi(\cdot) = \mathrm{tr}_d(\cdot) = \frac{1}{d}\mathrm{tr}(\cdot).$$

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$$\varphi(\cdot) = \mathbb{E}\mathrm{tr}_d(\cdot).$$

- $\mathcal{A} = L(\mathcal{H})$ algebra of linear operators on a Hilbert space, $u \in H$, $\|u\| = 1$

$$\varphi(\cdot) = \langle \cdot u, u \rangle$$

- We should think of φ as playing the role of the expectation in classical probability theory.
- We talk about the moments of a, referring to the values of $\varphi\left(a^k\right), \quad k \geq 0$.
- More generally, for a tuple $a_1, \ldots, a_n \in \mathcal{A}$, the values

$$\varphi(a_{i_1}^{m_1} \ldots a_{i_k}^{m_k})$$

  for $k \geq 0$, $1 \leq i_1, \ldots, i_k \leq n$, $m_1, \ldots m_k \geq 0$, are known as the joint moments of $a_1, \ldots, a_n$.
- Let $\mathbb{C}\langle X_1, \ldots X_n\rangle$ the algebra of polynomials in n noncommutative indeterminates with coefficients in $\mathbb{C}$.
- Let $a_1, \ldots, a_n$ be elements in a noncommutative probability space $(\mathcal{A}, \varphi)$. The (algebraic) distribution of $a_1, \ldots, a_n$ is the $\mu_{a_1,\ldots,a_n} : \mathbb{C}\langle X_1, \ldots X_n\rangle \to \mathbb{C}$ determined by

$$\mu_{a_1,\ldots,a_n}(X_{i_1}^{m_1} \ldots X_{i_k}^{m_k}) = \varphi(a_{i_1}^{m_1} \ldots a_{i_k}^{m_k})$$

  for each $k \geq 0$, $1 \leq i_1, \ldots, i_k \leq n$, $m_1, \ldots m_k \geq 0$.
- When an algebraic distribution is given by an analytic distribution?

V. Noncommutative probability spaces

Generality needed to deal with free probability

Remember classical case: A real random variable R has distribution μ on $\mathbb{R}$ iff

$$\mathbb{E}f(R) = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x), \quad \forall f \in B_b(\mathbb{R}).$$

Noncommutative case needs:

(i) **Given a p.m.** μ on $\mathbb{R}$ with **bounded support**, there exist a C^*-probability space $(\mathcal{A}, \varphi)$ and a self-adjoint $\mathbf{a} \in \mathcal{A}$ with

$$\varphi(f(\mathbf{a})) = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x), \quad \forall f \in C_b(\mathbb{R}).$$

(ii) **Given a p.m.** μ on $\mathbb{R}$, there exists a W^*-probability space $(\mathcal{A}, \varphi)$ and self-adjoint operator $\mathbf{a}$ on a Hilbert space H such that

$$f(\mathbf{a}) \in \mathcal{A} \quad \forall f \in B_b(\mathbb{R}), \tag{1}$$

$$\varphi(f(\mathbf{a})) = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x), \quad \forall f \in B_b(\mathbb{R}).$$

If (1) holds, it is said that $\mathbf{a}$ *is affiliated with* $\mathcal{A}$.

V. Free Random Variables

Definition

(i) A family of subalgebras $\{\mathcal{A}_i\}_{i\in I} \subset \mathcal{A}$ in a noncommutative probability space is *free* (**freely independent**) if

$$\varphi(\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n) = 0$$

whenever $\varphi(\mathbf{a}_j) = 0$, $\mathbf{a}_j \in \mathcal{A}_{i_j}$, and $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$.

Definition

If $\mathbf{a}_1, \mathbf{a}_2$ are freely independent, with distributions μ_{a_1} and μ_{a_2}, the distribution of $\mathbf{a}_1 + \mathbf{a}_2$ is the free convolution $\mu_{a_1} \boxplus \mu_{a_2}$.

V. Free independence allows to compute joint moments

Example

Computation of $\varphi(\mathbf{abab})$ when $\mathbf{a}$ & $\mathbf{b}$ are freely independent:
Suppose $\{\mathbf{a}_1, \mathbf{a}_3\}$ and $\{\mathbf{a}_2, \mathbf{a}_4\}$ are freely independent. Since

$$\varphi(\mathbf{a}_i - \varphi(\mathbf{a}_i)1_{\mathcal{A}}) = 0,$$

$$\varphi(\mathbf{a}_1 - \varphi(\mathbf{a}_1)1_{\mathcal{A}})\varphi(\mathbf{a}_2 - \varphi(\mathbf{a}_2)1_{\mathcal{A}})\varphi(\mathbf{a}_3 - \varphi(\mathbf{a}_3)1_{\mathcal{A}})\varphi(\mathbf{a}_4 - \varphi(\mathbf{a}_4)1_{\mathcal{A}}) = 0$$

Computations yield

$$\varphi(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4) = \varphi(\mathbf{a}_1\mathbf{a}_3)\varphi(\mathbf{a}_2)\varphi(\mathbf{a}_4) + \varphi(\mathbf{a}_1)\varphi(\mathbf{a}_3)\varphi(\mathbf{a}_2\mathbf{a}_4) - \varphi(\mathbf{a}_1)\varphi(\mathbf{a}_2)\varphi(\mathbf{a}_3)\varphi(\mathbf{a}_4).$$

In particular if $\mathbf{a}_1 = \mathbf{a}_3 = \mathbf{a}$ and $\mathbf{a}_2 = \mathbf{a}_4 = \mathbf{b}$

$$\varphi(\mathbf{abab}) = \varphi(\mathbf{a})^2\varphi(\mathbf{b}^2) + \varphi(\mathbf{a}^2)\varphi(\mathbf{b})^2 - \varphi(\mathbf{a})^2\varphi(\mathbf{b})^2 \neq \varphi(\mathbf{a}^2)\varphi(\mathbf{b}^2).$$

V. Application: Free Central Limit Theorem

Theorem

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots$ *be a sequence of independent free random variables with the same distribution with all moments. Assume that* $\varphi(\mathbf{a}_1) = 0$ *and* $\varphi(\mathbf{a}_1^2) = t$. *Then the distribution of*

$$\mathbf{Z}_m = \frac{1}{\sqrt{m}}(\mathbf{a}_1 + \ldots + \mathbf{a}_m)$$

converges, as $m \to \infty$, *to the semicircle distribution*

$$\mathrm{w}_t(x) = \frac{1}{2\pi}\sqrt{4t - x^2}, \quad |x| \leq 2\sqrt{t}$$

with moments $m_{2k+1} = 0$ *and* $m_{2k} = t^{2k}\binom{2k}{k}/(k+1)$.

- **Semicircle or Wigner distribution plays the role of classical Gaussian in free probability.**

V. Free Brownian motion

A noncommutative process

A Free Brownian motion is a family $S = \{S_t\}_{t \geq 0}$ of self-adjoint random variables in a noncommutative probability space $(\mathcal{A}, \varphi)$ such that:

1. $S_0 = 0$.
2. For $t_2 \geq t_1 \geq 0$, $S_{t_2} - S_{t_1}$ has law $\mathrm{w}_{t_2 - t_1}$.
3. For all $n \geq 1$ and $t_n > \cdots > t_1 > 0$, the increments $S_{t_n} - S_{t_{n-1}}, \ldots, S_{t_2} - S_{t_1}, S_{t_1}$ are freely independent with respect to φ.

- For every $t \geq 0$, S_t has semicircle law w_t of zero mean and variance one.

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- For every $t \geq 0$, S_t has semicircle law w_t of zero mean and variance one.
- One has Stochastic calculus for the free Brownian motion (Anshelevich, 2002, Biane, 1997, Biane & Speicher, 1998).

V. Semicircular process

- Free Brownian motion is an example of a Semicircular process $X = \{X_t\}_{t\geq 0} \subset \mathcal{A}$, self-adjoint random variables: For every $k \geq 1$, $t_1, ..., t_k \in [0, \infty)$ and $\theta_1, .., \theta_k \in \mathbb{R}$, the noncommutative random variable $\theta_1 X_{t_1} + \cdots + \theta_k X_{t_k}$ has Semicircle law

$$\mathrm{w}_{m,\sigma^2}(\mathrm{d}x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-m)^2}\mathbf{1}_{[m-2\sigma, m+2\sigma]}(x)\mathrm{d}x.$$

for some $m \in \mathbb{R}$, $\sigma^2 > 0$.

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- The law of a centered semicircular processes ($\varphi(X_t) = 0$ for every $t > 0$) is uniquely determined by its covariance function

$$\Gamma(s, t) = \varphi(X_t X_s).$$

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$$\Gamma(s,t) = \varphi(X_t X_s).$$

- Centered semicircular $X = \{X_t\}_{t\geq 0}$ has stationary increments

$$\Gamma(s,t) = \Gamma(|t-s|) = \varphi(X_{|t-s|}).$$

V. Noncommutative Fractional Brownian Motion

Nourdin and Taqqu (2104)

- Let $H \in (0,1)$. A noncommutative fractional Brownian motion (ncfBm) of Hurst parameter H is a centered semicircular process $S^H = \{S_t^H\}_{t \geq 0}$ in a noncommutative probability space $(\mathcal{A}, \varphi)$ with covariance function

$$\varphi(S_t^H S_s^H) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

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- ncfBm has stationary increments: For every $t, s > 0$

$$\varphi(\left(S_t^H - S_s^H\right)^2) = |t-s|^{2H}.$$

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- For $H > 1/2$ the process has long-range dependence

$$\sum_{n=1}^{\infty} \varphi(S_1^H(S_{n+1}^H - S_n^H)) = \infty$$

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- For $H > 1/2$ the process has long-range dependence

$$\sum_{n=1}^{\infty} \varphi(S_1^H (S_{n+1}^H - S_n^H)) = \infty$$

- and the increments are positively correlated: For $s_1 < t_1 < s_2 < t_2$

$$\varphi((S_{t_2}^H - S_{s_2}^H)(S_{t_1}^H - S_{s_1}^H)) > 0.$$

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- For $H < 1/2$ the increments are negatively correlated.

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Nourdin and Taqqu (2104)

- ncfBm is self-similar: For all $a > 0$, $(a^{-H}S^H_{at})_{t\geq 0} \overset{\text{law}}{=} (S^H_t)_{t\geq 0}$

$$\varphi(a^{-H}S^H_{at}a^{-H}S^H_{as}) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

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- Existence: Wigner integral representation of S^H with respect to free Brownian motion S

$$S^H_t = \frac{1}{c_H}\left(\int_0^\infty \left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\right) \mathrm{d}S_u\right).$$

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- Similar to the Wiener integral representation of 1-dimensional fractional Brownian motion $b^H = \left(b^H(t)\right)_{t\geq 0}$ with respect to 1-dimensional Brownian motion $b = \left(b(t)\right)_{t\geq 0}$

$$b^H(t) = \frac{1}{c_H}\left(\int_0^\infty \left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}\right) \mathrm{d}b(u)\right).$$

Part VI: From matrix fractional Bm to noncommutative fractional Bm

(Time-varying random matrix models for the noncommutative fractional Bm)

VI. One-dimensional fractional Brownian motion

A one-dimensional fractional Brownian motion $b^H = \left\{ b^H(t) \right\}_{t \geq 0}$ is a zero-mean classical Gaussian process with covariance

$$\mathbb{E} b^H(t) b^H(s)) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t-s|^{2H} \right).$$

- Stationary increments: For $s, t > 0$

$$\mathbb{E} \left| b^H(t) - b^H(s)) \right|^2 = |t-s|^{2H}.$$

- Self-similarity: $(a^{-H} b^H(at))_{t \geq 0} \overset{\text{law}}{=} (b^H(t))_{t \geq 0}$.

- $H = 1/2$ is 1-dimensional Bm (independent increments).

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- $H = 1/2$ is 1-dimensional Bm (independent increments).
- Itô stochastic calculus cannot be used for $H \neq 1/2$.

VI. One-dimensional fractional Brownian motion

A one-dimensional fractional Brownian motion $b^H = \left\{b^H(t)\right\}_{t\geq 0}$ is a zero-mean classical Gaussian process with covariance

$$\mathbb{E} b^H(t) b^H(s)) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- Stationary increments: For $s, t > 0$

$$\mathbb{E}\left|b^H(t) - b^H(s))\right|^2 = |t-s|^{2H}.$$

- Self-similarity: $(a^{-H} b^H(at))_{t\geq 0} \overset{\text{law}}{=} (b^H(t))_{t\geq 0}$.
- $H = 1/2$ is 1-dimensional Bm (independent increments).
- Itô stochastic calculus cannot be used for $H \neq 1/2$.
- Need *classical* fractional stochastic calculus: Skorohod, Young.

VI. Motivation for ncfBm of Nourdin and Taqqu

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$$\sum_{k,l}^{n} \rho(k-l) \sim K n^{2H} L(n) \text{ as } n \to \infty$$

with $0 < H < 1$, $K > 0$ and $L : (0,\infty) \to (0,\infty)$ a slowly varying function at infinity $(\forall a > 0, \lim_{x\to\infty} L(ax)/L(x) = 1)$.

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- Take the sequence of noncommutative stochastic processes

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- Then the finite dimensional distributions (f.d.d.) of Z_n converge in law to those of $\sqrt{K} S^H$ where S^H is a noncommutative fractional Brownian motion.

VI. Matrix fractional Brownian motion

Consider $n(n+1)/2$ independent 1-dimensional fractional Brownian motions with $H \in (1/2, 1)$.

$$\{\{b_{i,j}^H(t), t \geq 0\}, 1 \leq i, j \leq n\}.$$

- $n \times n$ symmetric matrix fractional Brownian motion:

$$\mathbf{B}_n^H(t) = (B_{ij}^H(t))_{i,j=1}^n$$
$$B_{ij}^H(t) = b_{i,j}^H \text{ if } i < j$$
$$B_{ii}^H(t) = \sqrt{2} b_{i,i}^H(t).$$

- For $0 < t_1 < \cdots < t_p$, the increments $(\mathbf{B}_n^H(t_k - t_{k-1}))_{n \geq 1}$, $k = 1, ..., p$ are not independent nor asymptotically free.
- Let $\lambda_1(t) \geq \lambda_2(t) \geq ... \geq \lambda_n(t)$ be the eigenvalues of $\mathbf{B}_n^H(t)$.

VI. Matrix fractional Brownian motion

Nualart and PA (2014)

1. If $\lambda_1(0) > \lambda_2(0) > ... > \lambda_n(0))$ the eigenvalues never collide:

$$\mathbb{P}\left(\lambda_1(t) > \lambda_2(t) > ... > \lambda_n(t) \quad \forall t > 0\right) = 1. \qquad (*)$$

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$$\lambda_i(t) = \lambda_i(0) + Y_i(t) + 2H \sum_{j \neq i} \int_0^t \frac{1}{\lambda_i(s) - \lambda_j(s)} \mathrm{d}s$$

$$Y_i(t) = \sum_{k \leq h} \int_0^t \frac{\partial \lambda_i(s)}{\partial b_{kh}^H(s)} \delta b_{kh}^H(s). \qquad (**)$$

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- Stochastic integral in (**) is in the sense of Skorohod. Classical Itô stochastic calculus cannot be used for $H \neq 1/2$.
- Proof of (*) uses the Young stochastic integral.
- $Y_i(t)$ is not a fractional Brownian motion, but it is a self-similar process: $\forall\ a > 0$, $(a^{-H} Y_i(at))_{t \geq 0} \stackrel{\text{law}}{=} (Y_i(t))_{t \geq 0}$.

VI. Time-varying Wigner theorem

Pardo, Pérez G, PA (2016)

Consider the empirical spectral measure-valued processes of the re-scaled matrix fractional Bm $\mathbf{B}_n^H(t)/\sqrt{n}$

$$\mu_t^{(n)} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{\lambda_j(t)/\sqrt{n}\}}, \; t \geq 0, n \geq 1.$$

1. Fix $T > 0$. For all continuous bounded function f and $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leq t\leq T} \left| \int f(x)\mathrm{d}\mu_t^{(n)}(x) - \int f(x)\mathrm{w}_t^H(x)\mathrm{d}x \right| > \varepsilon \right) = 0$$

   where w_t^H is the semicircle distribution on $(-2t^H, 2t^H)$.

2. The family of measure-valued processes $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ converges to $(\mathrm{w}_t^H)_{t\geq 0}$, the law of a noncommutative fractional Bm of Hurst parameter $H \in (1/2, 1)$.

VI. Precise statement

Pardo, Pérez G, PA (2016)

1. The family of measure-valued empirical spectral processes $\{(\mu_t^{(n)})_{t\geq 0} : n \geq 1\}$ converges weakly in $C(\mathbb{R}_+, \mathcal{P}(\mathbb{R}))$ to the unique continuous probability-measure valued function $(\mu_t)_{t\geq 0}$ satisfying, for each $t \geq 0$, $f \in C_b^2(\mathbb{R})$,

$$\langle \mu_t, f \rangle = \langle \mu_0, f \rangle + H \int_0^t ds \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} s^{2H-1} \mu_s(dx)\mu_s(dy).$$

   Moreover $\mu_t = \mathrm{w}_t^H$.

2. The Cauchy transform $G_t(z) = \int_{\mathbb{R}} \frac{\mu_t(dx)}{z-x}$ of μ_t is the unique solution to the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} G_t(z) = Ht^{2H-1} G_t(z) \frac{\partial}{\partial z} G_t(z), & t > 0, \\ G_0(z) = \int_{\mathbb{R}} \frac{\mu_0(dx)}{z-x}, & z \in \mathbb{C}^+. \end{cases}$$