Zooming in on a Lévy process at its supremum

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A motivating result

- Let X be a linear Brownian motion with variance σ^2 and drift γ
- Define

$$M := \sup\{X_t : t \in [0, T]\}, \qquad M_\varepsilon := \max\{X_{i\varepsilon} : i = 0, \ldots, \lfloor T/\varepsilon \rfloor\}$$

- Asmussen, Glynn, and Pitman [1995] showed that

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- ≈120 citations

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What if X is an arbitrary non-monotone Lévy process with $(\gamma, \sigma^2, \Pi(\mathrm{d}x))$?

- Intuitive that (1) holds when $\sigma > 0, \Pi(\mathbb{R}) < \infty$ [Dia and Lamberton, 2011].
- Asymptotic expansions of the expected error $\mathbb{E}(M - M_\varepsilon)$: [Janssen and Van Leeuwaarden, 2009], [Dia, 2010], [Chen, 2011] ...

Further applications

Discretization error in simulation of a two-sided reflected Lévy process (with Søren Asmussen):

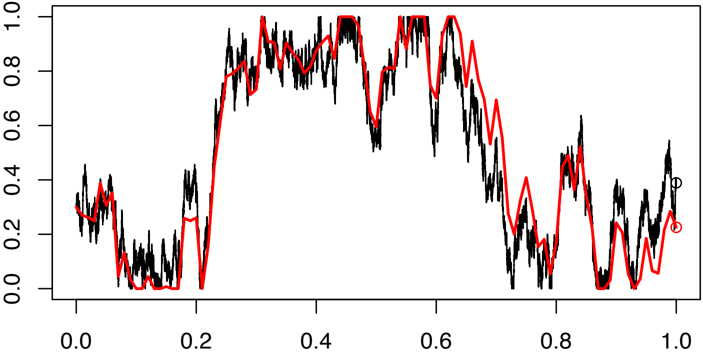

Figure: Example of a reflected sample path (black) and its discretized version (red) for $n = 100$.

Main ideas and results

Let τ be the time of the supremum of X on $[0, T]$.

- Study the weak (Renyi-mixing) limit

$$((X_{\tau+t\varepsilon} - M)/a_\varepsilon)_{t\in\mathbb{R}} \Rightarrow (\xi_t)_{t\in\mathbb{R}} \text{ as } \varepsilon \downarrow 0, \tag{2}$$

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Theorem

Under assumption (3) *and conditional on* $\tau \notin \{0, T\}$ *it holds that*

$$\left(\frac{M_\varepsilon - M}{a_\varepsilon}, \frac{\tau_\varepsilon - \tau}{\varepsilon}\right) \Rightarrow \left(\max_{i\in\mathbb{Z}} \xi_{U+i}, U + \mathrm{argmax}_{i\in\mathbb{Z}}\xi_{U+i}\right) \qquad \varepsilon \downarrow 0,$$

where U *and* ξ *are independent, and the convergence can be strengthened to Renyi-mixing.*

The underlying assumption

- We assume that
$$X_\varepsilon / a_\varepsilon \Rightarrow \widehat{X}_1 \qquad \text{as } \varepsilon \downarrow 0 \tag{3}$$
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- Zoming-in instead of the classical zooming-out of Lamperti [1962]:

Theorem

Assume that (4) *holds for a stochastically continuous, non-trivial process* $\widehat{X}$*. Then* $\widehat{X}$ *is self-similar with some index* $H > 0$*:*

$$(X_{ut})_{t\geq 0} \overset{\mathrm{d}}{=} (u^H X_t)_{t\geq 0} \qquad \text{for all } u > 0,$$

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Note: $(\xi_t)_{t\geq 0}, (\xi_{(-t)-})_{t\geq 0}$ must be self-similar as well

Self-similar Lévy processes

Let $\alpha = 1/H$ then it must be that $\alpha \in (0, 2]$.

The following is an exhaustive list of self-similar Lévy processes $\widehat{X}$:

(i) Brownian motion: $\widehat{\gamma} = 0, \widehat{\sigma} > 0, \widehat{\Pi} = 0$, in which case $\alpha = 2$;

(ii) Linear drift process: $\widehat{\gamma} \neq 0, \widehat{\sigma} = 0, \widehat{\Pi} = 0$, in which case $\alpha = 1$;

(iii) Strictly α-stable Lévy process for $\alpha \in (0, 2)$: $\widehat{\sigma} = 0$,

$$\widehat{\Pi}(\mathrm{d}x) = \mathbf{1}_{\{x>0\}}\widehat{c}_+ x^{-1-\alpha}\mathrm{d}x + \mathbf{1}_{\{x<0\}}\widehat{c}_-|x|^{-1-\alpha}\mathrm{d}x \tag{5}$$

for some $\widehat{c}_\pm \geq 0, \widehat{c}_+ + \widehat{c}_- > 0$, and, additionally,

$$\begin{aligned} \widehat{\gamma} = (\widehat{c}_+ - \widehat{c}_-)/(1-\alpha) \qquad & \text{if } \alpha \neq 1, \\ \widehat{c}_+ = \widehat{c}_-, \qquad & \text{if } \alpha = 1, \end{aligned}$$

Domains of attraction under zooming-in: literature

For each self-similar Lévy process $\widehat{X}$ specify the class of Lévy processes X with corresponding a_ε such that $X_\varepsilon/a_\varepsilon \Rightarrow \widehat{X}_1$.

- Rather similar to the classical zooming-out theory and the characterization of the strict domains of attraction for sums of i.i.d. random variables: Gnedenko and Kolmogorov [1954], Bingham, Goldie, and Teugels [1987] and Shimura [1990]
- Extensive literature on various aspects of small-time behavior of Lévy processes
- Doney and Maller [2002]: attraction to a Brownian motion and a linear drift process (simplification provided)
- Maller and Mason [2008]: non-strict attraction to stable processes

Domains of attraction under zooming-in

Define truncated mean and variance functions as well as the tails of Π:

$$m(x) = \gamma - \int_{x \le |y| < 1} y\Pi(\mathrm{d}y), \qquad v(x) = \sigma^2 + \int_{|y|<x} y^2 \Pi(\mathrm{d}y),$$

$$\overline{\Pi}_+(x) = \Pi(x, \infty), \qquad \overline{\Pi}_-(x) = \Pi(-\infty, -x), \qquad \overline{\Pi}(x) = \overline{\Pi}_+(x) + \overline{\Pi}_-(x).$$

Theorem

(i) *X is attracted to the Brownian motion with variance $\widehat{\sigma}$ if and only if*

$$v \in \mathrm{RV}_0 \qquad \textit{or equivalently} \qquad x^2\overline{\Pi}(x)/v(x) \to 0$$

as $x \downarrow 0$, and a_ε is chosen to satisfy $a_\varepsilon^2/v(a_\varepsilon) \sim \varepsilon/\widehat{\sigma}^2$.

(ii) *X is attracted to the non-zero linear drift $(\widehat{\gamma}t)_{t\ge 0}$ if and only if*

$$\sigma = 0, \quad m(x)/\widehat{\gamma} \textit{ is eventually positive}, \quad x\overline{\Pi}(x)/m(x) \to 0$$

as $x \downarrow 0$, and a_ε is chosen to satisfy $a_\varepsilon/m(a_\varepsilon) \sim \varepsilon/\widehat{\gamma}$.

(iii) *X is attracted to the strictly α-stable Lévy process with parameters $\widehat{c}_+, \widehat{c}_-, \widehat{\gamma}$ if and only if*

(a) *$\sigma = 0$, and $\gamma' = 0$ when X is b.v.,*

(b) *$\overline{\Pi}_\pm \in \mathrm{RV}_{-\alpha}$ if $\widehat{c}_\pm > 0$, and $\overline{\Pi}_+(x)/\overline{\Pi}_-(x) \to \widehat{c}_+/\widehat{c}_-$ as $x \downarrow 0$,*

(c) *for $\alpha = 1$ it is additionally required that*

$$\frac{m(x)}{x\overline{\Pi}_+(x)} \to \widehat{\gamma}/\widehat{c}_+ \qquad \textit{as } x \downarrow 0, \tag{6}$$

and a_ε is chosen to satisfy $\overline{\Pi}_\pm(a_\varepsilon) \sim \varepsilon^{-1}\widehat{c}_\pm/\alpha$ if $\widehat{c}_\pm > 0$.

Domains of attraction under zooming-in: comments

- If $\sigma \neq 0$ then take a standard Bm $\widehat{X}$ and $a_\varepsilon \sim \sigma\sqrt{\varepsilon}$
- If X is b.v. with linear drift $\gamma' \neq 0$ then take $\widehat{X}_t = \mathrm{sign}(\gamma')\mathrm{t}$ and $a_\varepsilon \sim |\gamma'|\varepsilon$
- There are Lévy processes not attracted to any $\widehat{X}$
- If not as above then $\alpha = \beta_{BG} := \inf\{\beta > 0 : \int_{|x|<1} |x|^\beta \Pi(\mathrm{d}x) < \infty\}$
- Possible to get limiting Bm, $\alpha = 2$, even when $\sigma = 0$
- $\alpha = 1$ corresponds to two different limits: linear drift and 1-strictly stable process
- ub.v process may have b.v. limit and vice versa, but only when $\alpha = 1$
- Non-strictly 1-stable process is attracted to linear drift

What is ξ appearing in the limit?

Completing the main result:

- $(\xi_t)_{t\geq 0}$ and $(-\xi_{(-t)-})_{t\geq 0}$ are independent and have the laws of $\widehat{X}$ conditioned to stay negative and positive, respectively
- $\widehat{X}$ conditioned to be negative can be seen as the limit law of the post-supremum process of $\widehat{X}$ on some finite time interval $[0, T]$ as $T \to \infty$
- Many other representations exist
- If $\widehat{X}$ is a standard Bm then $(-\xi_t)_{t\geq 0}$ and $(\xi_{(-t)-})_{t\geq 0}$ have the law of a 3-dimensional Bessel process

Proof ingredients: time-reversal and splitting of a killed Lévy process, invariance principle for conditioned processes (random walk case analyzed by Chaumont and Doney [2010]), existent theory on conditioned processes.

Conclusions

Assumption: $X_\varepsilon / a_\varepsilon \Rightarrow \widehat{X}_1$, see domains of attraction.
Result: $(M - M_\varepsilon)/a_\varepsilon \Rightarrow V_{\mathcal{L}(\widehat{X})}$, where the limit depends on the law of $\widehat{X}$ only.

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Result: $(M - M_\varepsilon)/a_\varepsilon \Rightarrow V_{\mathcal{L}(\widehat{X})}$, where the limit depends on the law of $\widehat{X}$ only.

- $V_{\mathcal{L}(-\widehat{X})} \stackrel{d}{=} V_{\mathcal{L}(\widehat{X})}$ and for any $c > 0$: $V_{\mathcal{L}(c\widehat{X})} \stackrel{d}{=} cV_{\mathcal{L}(\widehat{X})}$
- The result of Asmussen, Glynn, and Pitman [1995] holds true whenever $\sigma > 0$
- The same limit can be obtained when $\sigma = 0$, but then necessarily $\beta_{BG} = 2$
- If X is b.v. with linear drift $\gamma' \neq 0$ then

$$\frac{M - M_\varepsilon}{|\gamma'|\varepsilon} \Rightarrow U \qquad \text{on the event } \tau \notin \{0, T\}.$$

- Additionally, there is class (iii) of limits stemming from strictly-α-stable processes (single parameter class for a fixed α)

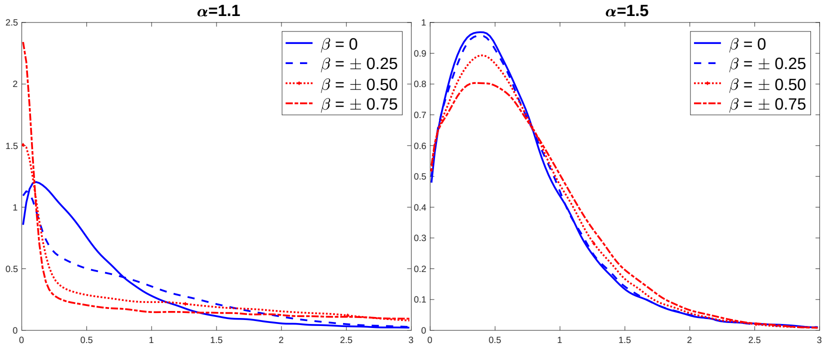

Figure: Estimated densities of V for strictly-α-stable process with skewness $\beta \in [-1, 1]$

Smaller α lead to a better rate but more disperse limit V.

Thank you!

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