Imperial College London

HYBRID MARKED POINT PROCESSES

CHARACTERISATION, EXISTENCE AND UNIQUENESS

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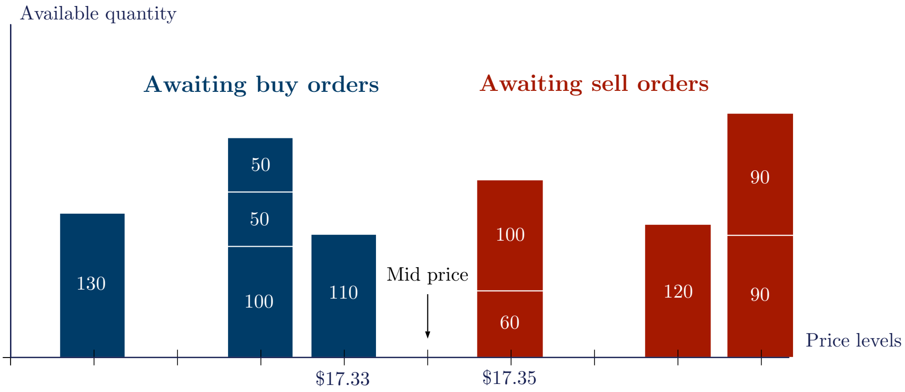
Motivation: limit order book modelling



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- It evolves with the submission of each order.

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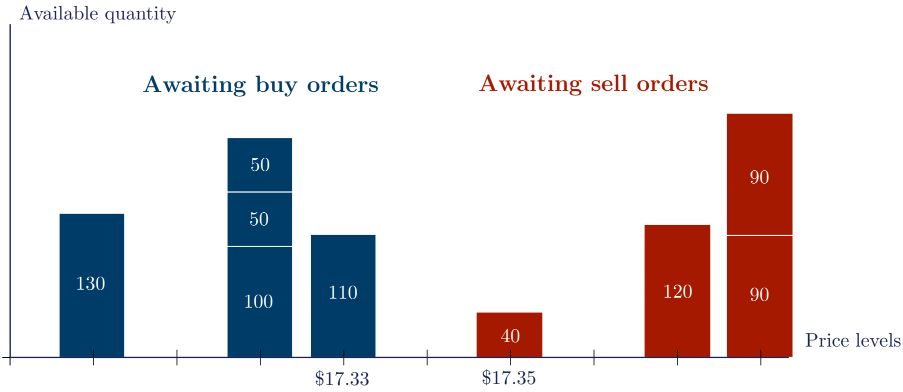

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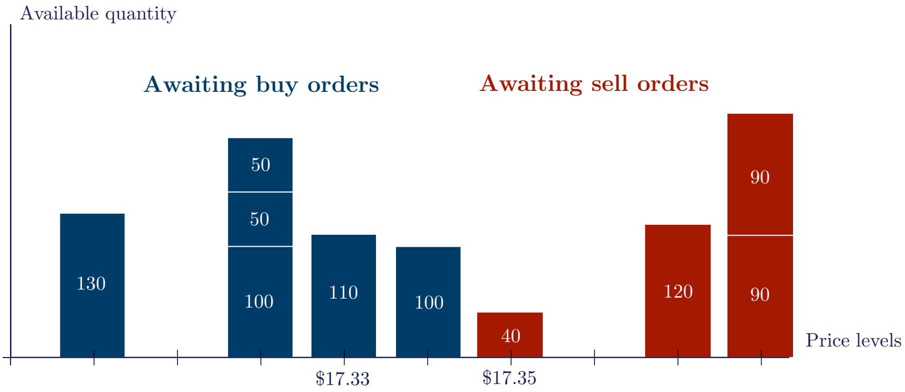

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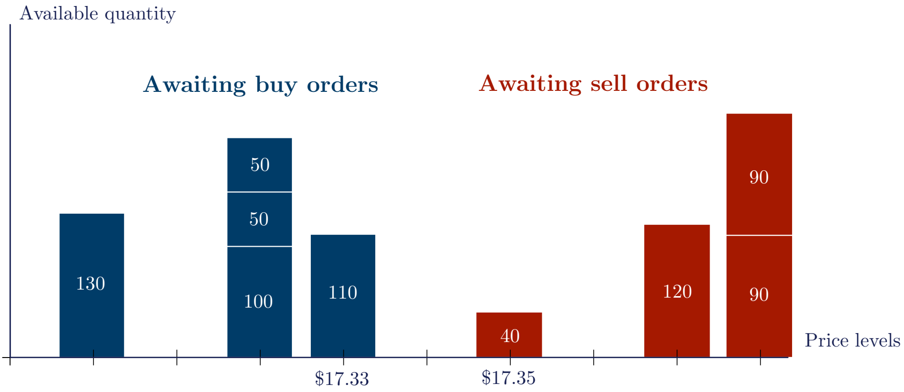

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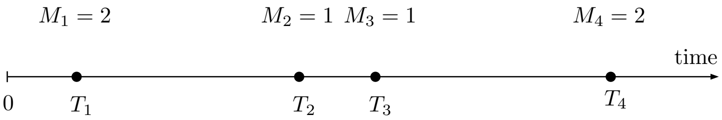
Motivation: marked point processes

- A **marked point process** (MPP) consists of
  - an increasing sequence of random times $(T_n)_{n\in\mathbb{N}}$ in $(0,\infty]$;
  - a sequence of random marks $(M_n)_{n\in\mathbb{N}}$, where $M_n \in \{1,\dots,m\}$, say.



- Define the counting processes

$$N_i(t) := \sum_{n\in\mathbb{N}} \mathbb{1}(T_n \le t, M_n = i), \quad t \ge 0, i \in \{1,\dots,m\}.$$

- Approximate definition: the **intensity process** $\lambda_i(t)$ of N_i at time t is such that

$$\mathbb{E}\left[N_i(t+dt) - N_i(t) \,|\, \mathcal{F}_t^N\right] \approx \lambda_i(t)dt,$$

where $\mathcal{F}_t^N := \sigma(N_i(s), s \le t, i = 1,\dots,m)$.

Motivation: Hawkes processes

- A **Hawkes process** is an MPP that admits intensities such that

$$\lambda_i(t) = \nu_i + \sum_{j=1}^{m} \int_{[0,t)} k_{ji}(t-s)dN_j(s), \quad t \geq 0,\ i = 1, \ldots, m,$$

  where $\nu_i > 0$ and $k_{ji} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.
- The kernels k_{ij} allow for self- and cross-excitation effects.
  - The arrival at time t of an event of type j increases the intensity $\lambda_i(t+h)$ of events of type i at time $t+h$ by an amount of $k_{ji}(h)$.
  - A good candidate for modelling interactions between events (earthquake modelling, criminology, social networks, neurology).
- In LOB models (Large, 2007; Bacry et al., 2016), the marks carry information only on the event type, ignoring the the state of the LOB. However:
  - one might one to keep track of some salient state variables;
  - these state variables might actually influence the arrival intensities of buy and sell orders.

Motivation: Hawkes vs. Markov

- Continuous-time Markov chains: another prominent trend in the LOB modelling literature (Cont et al., 2010; Huang et al., 2015).
  - These models do capture the state of the LOB.
  - But the order flow dynamics can only depend on the state.
  - Interactions like in Hawkes processes are not possible.
- To summarise:
  - Hawkes processes have an **event viewpoint**.
  - Markov processes have a **state viewpoint**.
- Our goal is twofold:
  - propose a class of MPPs that allow for an **event–state viewpoint**;
  - find flexible conditions that ensure the existence and uniqueness of such MPPs.
- Morariu-Patrichi, M. and Pakkanen, M. S. (2017). Hybrid marked point processes: characterisation, existence and uniqueness. Preprint, available at: `http://arxiv.org/abs/1707.06970`.

Outline

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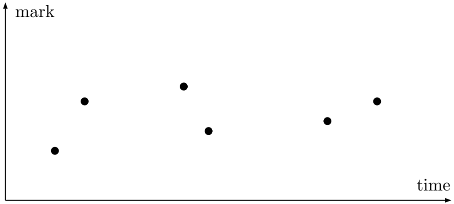
Spaces of integer-valued boundedly finite measures

- $\mathscr{M} = \{1, \dots, m\}$: our mark space (it could be any Polish space).
- $\mathcal{N}^{\#}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$: the space of integer-valued boundedly finite measures on $\mathbb{R}_{\geq 0} \times \mathscr{M}$. i.e., atomic and finite on all bounded sets.
  - The weak-hash metric $d^{\#}$ (Daley and Vere-Jones, 2003) makes $\mathcal{N}^{\#}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$ a complete separable metric space.
  - When $\mathscr{M}$ is locally compact, this coincides with the vague topology (Kallenberg, 1976).
  - A **non-explosive point process** (NEPP) is defined as random element in $(\mathcal{N}^{\#}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}, \mathcal{B}(\mathcal{N}^{\#}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}))$.
- $\mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$: all the $\xi \in \mathcal{N}^{\#}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$ such that $\xi(\{t\} \times \mathscr{M}) = 0$ or 1.
  - Hence $\mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0} \times \mathscr{M}} \subset \mathcal{N}^{\#}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$.
  - A **non-explosive marked point process** (NEMPP) is defined as a random element in $(\mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}, \mathcal{B}(\mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}))$.
  - A NEMPP is a MPP $(T_n, M_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} T_n = \infty$ a.s.

Histories

- For all $\xi \in \mathcal{N}^{\#}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$ and $t \in \mathbb{R}_{\geq 0}$, define $\xi_t \in \mathcal{N}^{\#}_{\mathbb{R}_{\leq 0} \times \mathscr{M}}$ by $\xi_t(A) := \xi((A + t) \cap [0, t] \times \mathscr{M})$, $A \in \mathcal{B}(\mathbb{R}_{\leq 0} \times \mathscr{M})$.
  - ξ_t is the measure ξ stopped at t and translated back to the origin.
  - Simply put, ξ_t is the measure ξ viewed from time t.



- Similarly, define $\xi_{t-}(\cdot) := \xi((\cdot + t) \cap [0, t) \times \mathscr{M})$.
- Given an NEPP N, its **internal history** is the filtration $\mathbb{F}^N = (\mathcal{F}^N_t)_{t \in \mathbb{R}_{\geq 0}}$ defined by $\mathcal{F}^N_t := \sigma(N_t)$.
- Given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_{\geq 0}}$, we say that N is $\mathbb{F}$**-adapted** if $\mathcal{F}^N_t \subset \mathcal{F}_t$ for all $t \geq 0$.

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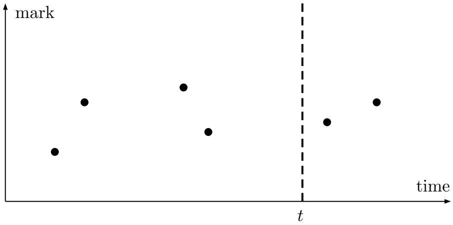

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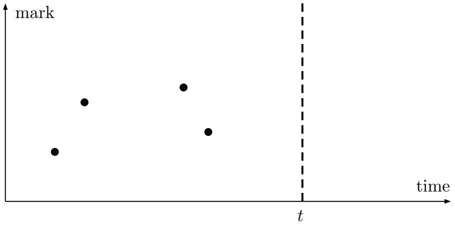

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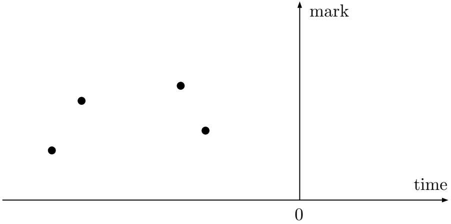

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Intensity process and functional

- Let N be an $\mathbb{F}$-adapted NEMPP on $\mathbb{R}_{\geq 0} \times \mathscr{M}$.
- Let $\lambda_i : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $i \in \mathscr{M}$, be a family of $\mathbb{F}$-predictable processes, that we denote by λ.
- We will use the notation $\lambda(t, m)$ to refer to $\lambda_m(t)$, $t \geq 0$, $m \in \mathscr{M}$.
- We say that λ is the $\mathbb{F}$-intensity of N if for every $0 \leq s < t$, $m \in \mathscr{M}$,

$$\mathbb{E}\left[N((s,t] \times \{m\}) \,|\, \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t \lambda(u,m)du \,\Big|\, \mathcal{F}_s\right].$$

Definition (Intensity functional)

Let $\psi : \mathscr{M} \times \mathcal{N}^{\#}_{\mathbb{R}_{\leq 0} \times \mathscr{M}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a measurable functional. We furthermore say that N admits ψ as its intensity functional if $\lambda(\omega, t, m) = \psi(m \,|\, N_{t-}(\omega))$ holds $\mathbb{P}(d\omega)dt$-a.e. for all $m \in \mathscr{M}$.

Outline

Mark space and state process

- Let $\mathscr{E} = \{1, \ldots, n_{\mathscr{E}}\}$, $n_{\mathscr{E}} \in \mathbb{N}$ and $\mathscr{X} = \{1, \ldots, n_{\mathscr{X}}\}$, $n_{\mathscr{X}} \in \mathbb{N}$.
  - Each $e \in \mathscr{E}$ is a type of event.
  - Each $x \in \mathscr{X}$ is a possible state of a system.
  - In the paper, $\mathscr{E}$ and $\mathscr{X}$ can be Polish spaces.
- We consider the mark space $\mathscr{M} := \mathscr{E} \times \mathscr{X}$.
  - Here, $\mathscr{M}$ is still of the form $\{1, \ldots, m\}$ with $m = n_{\mathscr{E}} n_{\mathscr{X}}$.
  - We refer to the elements of $\mathscr{M}$ using the notation (e, x), $e \in \mathscr{E}$, $x \in \mathscr{E}$.
- Given an NEMPP on $\mathbb{R}_{\geq 0} \times \mathscr{M}$, we define the **state process** X_t at time t as the component x of the most recent mark (e, x) up to time t, exlucluding t.
  - Denote by F the functional on $\mathcal{N}^{\#}_{\mathbb{R}_{\leq 0} \times \mathscr{M}}$ such that $X_t = F(N_{t-})$.
  - A mark $M_n = (e, x) \in \mathscr{M}$ can now be interpreted as an event of type e that moves the system to the state x.
  - If no events occurred up to time t, then $X_t = x_0$ for some arbitrary initial condition $x_0 \in \mathscr{X}$.

Example: state-dependent Hawkes process

Let $\phi : \mathscr{X} \times \mathscr{E} \times \mathscr{X} \to \mathbb{R}_{\geq 0}$ be a measurable function such that $\phi(\cdot \,|\, e, x)$ is a probability distribution on $\mathscr{X}$ for every $e \in \mathscr{E}$, $x \in \mathscr{X}$. Consider first a NEMPP N with an $\mathbb{F}^N$-intensity λ that satisfies

$$\lambda(t, e, x) = \phi(x \,|\, e, X_t)\left(\nu(e) + \int_0^t \int_{\mathscr{M}} k(t - s, m, e) N(dt, dm)\right).$$

- "Hawkes part": intensity of events
  - If we denote the arrival times and marks of N respectively by T_n and $M_n = (E_n, X_n)$, $n \in \mathbb{N}$, then the red term is the $\mathbb{F}^N$-intensity of the MPP $(T_n, E_n)_{n \in \mathbb{N}}$.
  - Events interact like in a Hawkes process but there is also a dependence on the state process.
- "Markov part": transition probabilities of the state process
  - The term $\phi(x \,|\, e, x')$ is the probability of transitioning from state x' to x when an event of type e occurs.
  - One transition matrix for each possible type of event.

Comparison with Hawkes processes

Why not using instead a Hawkes process on $\mathscr{M}$ with the same interpretation of the marks? Compare the previous intensity to the equation defining a linear Hawkes process:

$$\lambda(t,e,x) = \phi(x \mid e, X_t)\left(\nu(e) + \int_0^t \int_{\mathscr{M}} k(t-s,m,e)N(dt,dm)\right), \quad (1)$$

$$\lambda(t,e,x) = \nu(e,x) + \int_0^t \int_{\mathscr{M}} k(t-s,m,e,x)N(dt,dm).$$

- In a Hawkes process, if an event of type e occurs, the probability distribution of the new state depends on the entire history.
  - But in a LOB, this distribution depends mainly on the current state.
  - No such **event–state structure** with Hawkes processes.
  - The intensity in (1) does replicate such a structure however.

Generalisation to hybrid marked point processes

Definition (Hybrid marked point processes)

*Let $\eta : \mathscr{E} \times \mathcal{N}^{\#}_{\mathbb{R}\times\mathscr{M}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a measurable functional. A **hybrid MPP** with transition function ϕ and event functional η is a NEMPP $N : \Omega \to \mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0}\times\mathscr{M}}$ that admits an $\mathbb{F}^N$-intensity λ such that*

$$\lambda(\omega, t, e, x) = \phi(x \mid e, X_t)\eta(e \mid N_{t-}(\omega)), \quad \mathbb{P}(d\omega)dt\text{-a.e.}, (e, x) \in \mathscr{M}.$$

In other words, N admits ψ as its intensity functional, where

$$\psi(m \mid \xi) := \phi(x \mid e, F(\xi))\eta(e \mid \xi), \quad m = (e, x) \in \mathscr{M}, \xi \in \mathcal{N}^{\#}_{\mathbb{R}_{\leq 0}\times\mathscr{M}}.$$

Implied dynamics and characterisation

Theorem (Implied dynamics and characterisation)

Suppose that N is a NNEMP on $\mathbb{R}_{\geq 0} \times \mathscr{M}$ with an $\mathbb{F}^N$-intensity. Then, N is a hybrid MPP with transition function ϕ and event functional η if and only if the following two statements hold.

1. $N_{\mathscr{E}}(\cdot) := N(\cdot \times \mathscr{X})$ *is a NEMPP on* $\mathbb{R}_{\geq 0} \times \mathscr{E}$ *that admits an* $\mathbb{F}^N$*-intensity* $\lambda_{\mathscr{E}}$ *such that* $\lambda_{\mathscr{E}}(t, e) = \eta(e \,|\, N_{t-})$.
2. *Let* $t \in \mathbb{R}_{\geq 0}$ *and denote by* τ *the first event time after time t and by* $M = (E, X)$ *the corresponding mark. We have that*

$$\mathbb{P}\left(X = x \,|\, \sigma(E) \vee \mathcal{F}^N_{\tau-}\right) \mathbb{1}_{\{\tau<\infty\}} = \phi(x \,|\, E, X_t) \mathbb{1}_{\{\tau<\infty\}}, \quad a.s.$$

Outline

The existence and uniqueness problem

- Self-referential nature of the definition of a hybrid MPP: it is not clear that such a NEMPP exists.
- More generally, given a functional ψ, one can ask if there exists a NEMPP on $\mathbb{R}_{\geq 0} \times \mathscr{M}$ that admits ψ as its intensity functional.
- Massoulié (1998) tackles this question by reformulating the existence problem as a Poisson-driven SDE.
  - Strong existence and uniqueness is obtained by imposing the Lipschitz condition

    $$|\psi(m \,|\, \xi) - \psi(m \,|\, \xi')| \leq \iint_{\mathbb{R}_{<0} \times \mathscr{M}} \overline{k}(-t', m', m) |\xi - \xi'|(dt', dm'),$$

    where $m \in \mathscr{M}$, $\xi, \xi' \in \mathcal{N}^{\#}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$.
  - This can be applied to Hawkes processes.
  - But this condition is not satisfied by state-dependent Hawkes processes for example.
- Our goal: find a weaker condition that ensures strong existence and uniqueness.

The Poisson-driven SDE

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be given and complete.
- Let $M : \Omega \to \mathcal{N}^{\#}_{\mathbb{R}_{\geq 0} \times \mathscr{M} \times \mathbb{R}_{\geq 0}}$ be a Poisson PP on $\mathbb{R}_{\geq 0} \times \mathscr{M} \times \mathbb{R}_{\geq 0}$:
    - $M(A_1), \ldots, M(A_n)$ are mutually independent for disjoint and bounded sets $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}_{\geq 0} \times \mathscr{M} \times \mathbb{R}_{\geq 0})$;
    - $M(A \times \{m\} \times B)$ follows a Poisson distribution with parameter $Leb(A)Leb(B)$, where $A, B \in \mathcal{B}(\mathbb{R}_{\geq 0})$.
- Let $\mathbb{F}$ be the completion of the natural filtration $\mathbb{F}^M$.

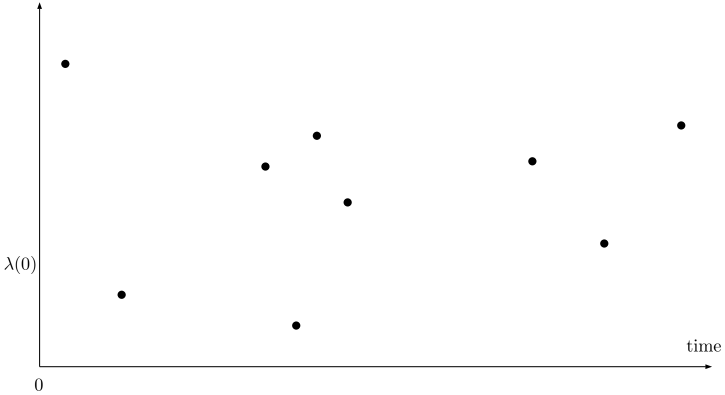
Definition (The Poisson-driven SDE)

By a solution to the Poisson-driven SDE, we mean an $\mathbb{F}$-adapted NEMPP $N : \Omega \to \mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$ that solves

$$\begin{cases} N(dt, dm) = M(dt, dm, (0, \lambda(t, m)]), & t \in \mathbb{R}_{\geq 0}, \ a.s., \\ \lambda(t, m) = \psi(m \,|\, \theta_t N^{<0}), & t \in \mathbb{R}_{\geq 0}, m \in \mathscr{M}, \omega \in \Omega. \end{cases}$$

Intuition: visualising the equation

- Take the simplest case when $\mathscr{M} = \{1\}$, i.e., no marks.
- For example, consider the intensity functional $\psi(\xi) = 1 + \int_{(-\infty,0)} e^{2t}\xi(dt)$, i.e., a Hawkes process.



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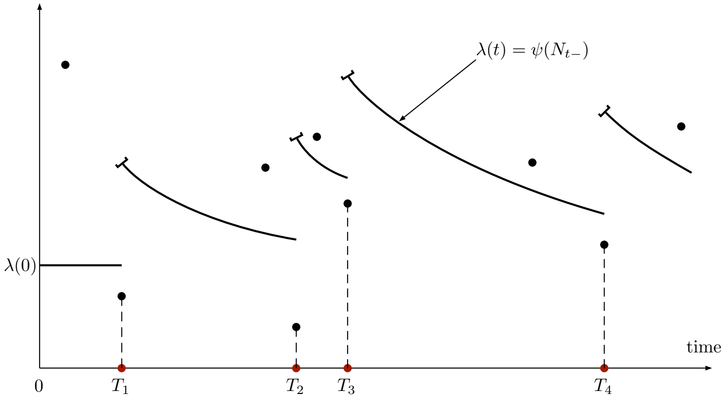
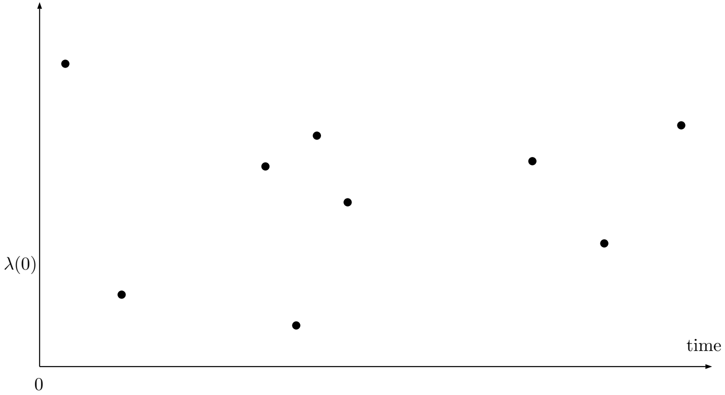


Intuition: pathwise construction

- Thanks to the discrete nature of M, at each event time, we know the intensity process until the next event time.
- For each $\omega \in \Omega$, it looks we can construct an increasing sequence $(T_n(\omega))_{n\in\mathbb{N}}$ such that $N(\omega) := \sum_{n\in\mathbb{N}} \delta_{T_1(\omega)} \mathbb{1}_{T_n(\omega)<\infty}$ is a solution.



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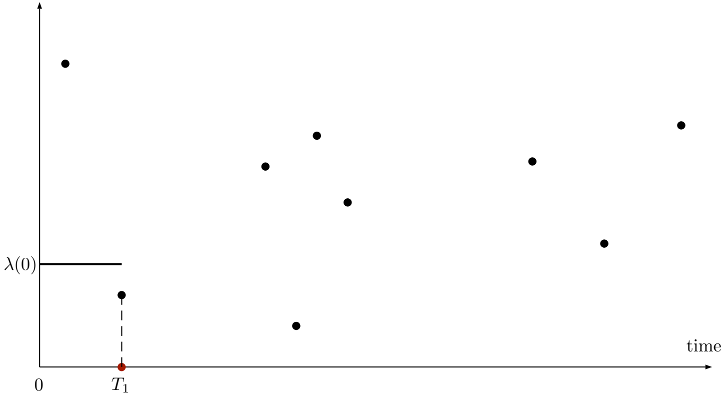

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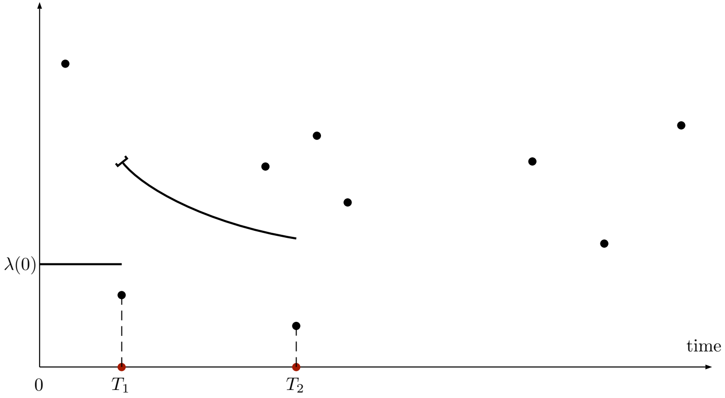

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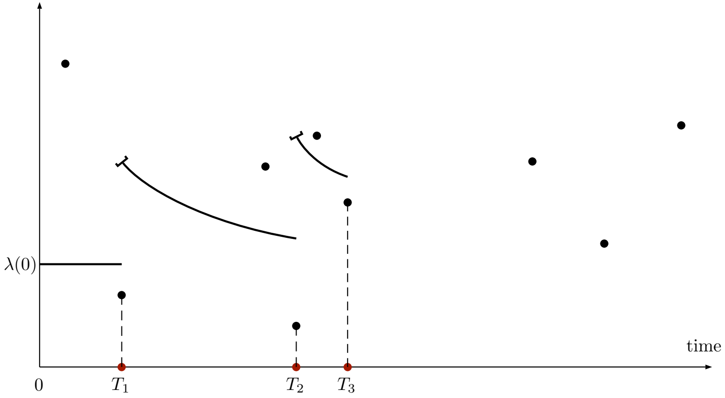

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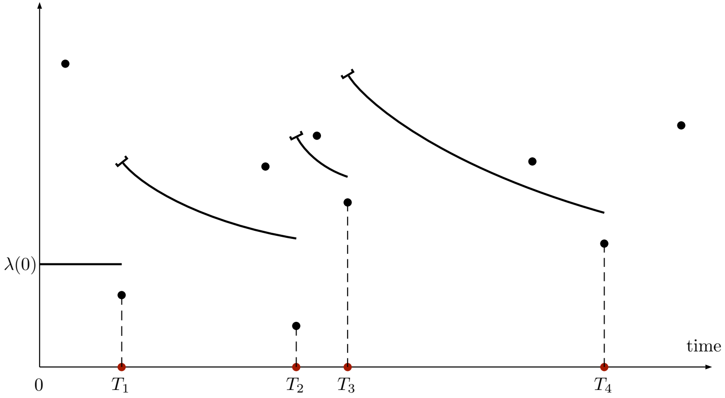
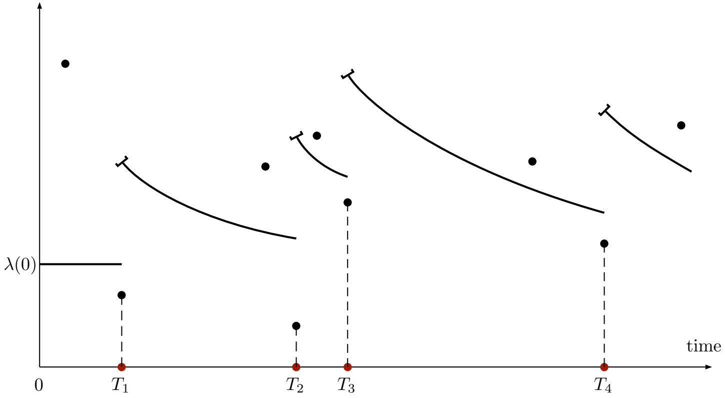


Intuition: pathwise construction

- Thanks to the discrete nature of M, at each event time, we know the intensity process until the next event time.
- For each $\omega \in \Omega$, it looks we can construct an increasing sequence $(T_n(\omega))_{n\in\mathbb{N}}$ such that $N(\omega) := \sum_{n\in\mathbb{N}} \delta_{T_1(\omega)} \mathbb{1}_{T_n(\omega)<\infty}$ is a solution.



Pathwise construction: how to be rigorous

To prove strong existence, we follow these steps.

- Write down formally this algorithm and prove that it is well-defined.
- Check that the constructed N solves the SDE up to each T_n.
- Show that each T_n is an $\mathbb{F}$-stopping time and that each "piece" of λ is $\mathbb{F}$-predictable.
- Dominate N by a NEPP $\overline{N}$ to show that $\lim_{n\to\infty} T_n = \infty$ a.s.
- Prove that $N \in \mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0}}$ a.s., i.e., $N(\{t\}) = 0$ or 1.
- Use a Poisson embedding lemma to conclude that a version of N is NEMPP that solves the Poisson-driven SDE and admits ψ as its intensity functional.

Assumptions

To follow these steps in the general case, we need some assumptions.

- The mark space $\mathscr{M}$ is bounded.
- There exists $\lambda_0 \in \mathbb{R}_{\geq 0}$ and a bounded measurable function $\overline{k} : \mathbb{R}_{>0} \times \mathscr{M} \times \mathscr{M} \to \mathbb{R}_{\geq 0}$ such that

$$\psi(m \,|\, \xi) \leq \lambda_0 + \iint_{(-\infty,0)\times\mathscr{M}} \overline{k}(-t', m', m)\xi(dt', dm') =: \overline{\psi}(m \,|\, \xi).$$

  - We dominate N by a Hawkes process with intensity functional $\overline{\psi}$.
- The kernel $\overline{k}$ satisfies $\sup_{m\in\mathscr{M}} \sum_{m'\in\mathscr{M}} \int_{(0,\infty)} \overline{k}(t', m', m)dt' < 1$.
  - This allows us to apply the results of Massoulié (1998) to $\overline{\psi}$.
- In the paper, we allow for a general initial condition on $(-\infty, 0]$.
  - Extra assumptions on the initial condition are required.

Remark. Here, there are no events before $t = 0$ and, thus, we could relax the above assumptions.

Existence and uniqueness

Theorem (Strong existence)

Under the previous assumptions, there exists a NEMPP $N : \Omega \to \mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$ that solves the Poisson-driven SDE. Any such N admits ψ as its intensity functional.

Since we restrict ourselves to NEMPPs, we can prove strong uniqueness without any specific assumptions.

Theorem (Strong uniqueness)

Let $N : \Omega \to \mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$ and $N' : \Omega \to \mathcal{N}^{\#g}_{\mathbb{R}_{\geq 0} \times \mathscr{M}}$ be two NEMPPs solving the Poisson-Driven SDE. Then $N = N'$ a.s.

Remark. By applying Theorem 3.4 in Jacod (1975), we also obtain weak uniqueness.

References

Bacry, E., Jaisson, T., and Muzy, J.-F. (2016). Estimation of slowly decreasing Hawkes kernels: application to high-frequency order book dynamics. *Quantitative Finance*, 16(8):1179–1201.

Cont, R., Stoikov, S., and Talreja, R. (2010). A stochastic model for order book dynamics. *Operations Research*, 58(3):549–563.

Daley, D. J. and Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes. Vol. I*. Springer, New York, second edition.

Huang, W., Lehalle, C.-A., and Rosenbaum, M. (2015). Simulating and analyzing order book data: The queue-reactive model. *Journal of the American Statistical Association*, 110(509):107–122.

Jacod, J. (1975). Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 31(3):235–253.

Kallenberg, O. (1976). *Random Measures*. Akademie-Verlag, Berlin.

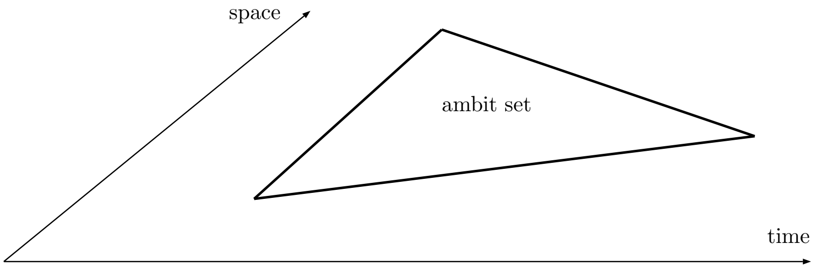
Large, J. (2007). Measuring the resiliency of an electronic limit order book. *Journal of Financial Markets*, 10(1):1–25.

Massoulié, L. (1998). Stability results for a general class of interacting point processes dynamics, and applications. *Stochastic Processes and their Applications*, 75(1):1–30.

Morariu-Patrichi, M. and Pakkanen, M. S. (2017). Hybrid marked point processes: characterisation, existence and uniqueness. Preprint, available at: `http://arxiv.org/abs/1707.06970`.

Conclusion: transporting the idea to ambit fields?

- The set $\{(t, m, z) \in \mathbb{R}_{\geq 0} \times \mathscr{M} \times \mathbb{R}_{\geq 0} \ : \ z \leq \lambda(t, m)\}$ can be interpreted as the "ambit set" of the process.
- This set **expands** randomly depending on the process.
- In ambit fields however, the ambit set is fixed and **translated.**
- Idea: combine both to obtain **endogenous intermittency**?



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