On the integration with respect to Volterra processes: fractional calculus and approximation

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Second Conference on Ambit Fields and Related Topics
Århus, 14-16 August 2017

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The scope

We consider Volterra processes

$$Y_t = \int_0^t g(t,s)\, dZ_s, \quad t \in [0,T],$$

where $g(t,s)$ is a given deterministic kernel, and Z is a Lévy process or a (square integrable) martingale process.

Goals:

- To define a concept of integral $\int_0^t X_s dY_s$ by means of fractional calculus
- Develop approximation techniques

Comments.

- The use of fractional calculus is bridging stochastic and deterministic methods.
- Other approaches to integration with respect to Volterra processes:
  - Bender and Marquardt (2008) propose a Skorokhod type integral based on S-transform originally developed for fractional Brownian motion;
  - Barndorff-Nielsen, Benth, Pedersen, Veraart (2014) propose another integration based on Malliavin calculus integration by parts rule. This extended by DiN, Vives (2017) to go beyond L_2-framework and by Barndorff-Nielsen, Benth, Szozda (2014) using white noise analysis in the case of Z Brownian;
  - In the case of Y semimartingale and X predictable, integration can be carried through by Itô type integration with respect to random measures. See e.g. Bichtler and Jacod (1983), Chong and Klüppelberg (2015).
- Besides the Itô type integration, also the fractional approach leads to a framework for modelling where it is easier to include information.

Outlines

1. Volterra processes
2. Approximations of the kernel
3. Integration with respect of Volterra processes
4. Approximation of the integral
5. Simulation of the integral

1. Volterra processes

Let Z be a Lévy process, then the Volterra process Y is well-defined within the framework of integration of Raijput and Rosinksi (1989).

For t fixed, consider the characteristic function

$$E\left[\exp\{iuZ_t\}\right] = \exp\{t\Psi(u)\}, \quad u \in \mathbb{R},$$

where

$$\Psi(u) = iau - \frac{bu^2}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iu\tau(x)\right) \pi(dx),$$

$a \in \mathbb{R}$, $b \geq 0$, π is a Lévy measure on $\mathbb{R}$, with

$$\int_{\mathbb{R}} \left(x^2 \wedge 1\right) \pi(dx) < \infty,$$

and $\pi(\{0\}) = 0$ and

$$\tau(z) := \begin{cases} z, & |z| \leq 1, \\ \frac{z}{|z|}, & |z| > 1. \end{cases}$$

Results [RR1989] Let t be fixed.

(*i*) The integral $Y_t := \int_0^t g(t,s)\,dZ_s$ is well defined for any $g(t,\cdot)$ Z-integrable function, i.e., there exists an approximating sequence of simple functions whose integrals converge in probability. The result does not depend on the approximating sequence.

(*ii*) Define

$$r(v) := bu^2 + \int_{\mathbb{R}} \left(|xv|^2 \wedge 1\right)\, \pi(dx) + \left| av + \int_{\mathbb{R}} \left(\tau(xv) - \tau(x)v\right) \pi(dx) \right|.$$

Then a measurable function $g(t,\cdot)\colon ([0,T],\mathcal{B}([0,T])) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ is Z-integrable if and only if $\int_{[0,T]} r(g(t,s))\,ds < \infty$.

(*iii*) If $g(t,\cdot)$ is Z-integrable, then the integral Y_t has infinitely divisible distribution and characteristic function with triplet

$$a_g = \int_{[0,T]} \left(ag(t,s) + \int_{\mathbb{R}} \left(\tau(xg(t,s)) - \tau(x)g(t,s)\right) \pi(dx) \right) ds,$$

$$b_g = \int_{[0,T]} bg^2(t,s)\,ds,$$

$$\pi_g(B) = \int_{[0,T]} \int_{\mathbb{R}} \mathbf{1}_{g(t,s)x \in B\setminus\{0\}}\, \pi(dx)\,ds, \quad B \in \mathcal{B}(\mathbb{R}).$$

Depending on Z , we obtain sufficient conditions for the Z-integrability of $g(t,\cdot)$, existence of moments for Y_t, and estimates.

Theorem Let t be fixed.

(A) For $p \in [1,2)$, consider Z such that $b = 0$ and π is symmetric with $\int_{\mathbb{R}} |x|^p \, \pi(dx) < \infty$. Then any $g(t,\cdot) \in L_p([0,t])$ is Z-integrable. Fix such a g, then the integral $Y_t \in L_p(\mathbf{P})$ and

$$\mathbf{E}\Big| \int_0^t g(t,s)\, dZ_s \Big|^p \le C \left(|a|^p \, \|g(t,\cdot)\|_{L_1}^p + \|g(t,\cdot)\|_{L_p}^p \int_{\mathbb{R}} |x|^p \, \pi(dx) \right) \le \tilde{C} \, \|g(t,\cdot)\|_{L_p}^p$$

(B) For $p \in [2,\infty)$, consider Z such that π is symmetric with $\int_{\mathbb{R}} |x|^p \, \pi(dx) < \infty$. Then any $g(t,\cdot) \in L_p([0,t])$ is Z-integrable. Fix such a g, then the integral $Y_t \in L_p(\mathbf{P})$ and

$$\mathbf{E}\left| \int_0^t g(t,s)\, dZ_s \right|^p \le C \Bigg(|a|^p \, \|g(t,\cdot)\|_{L_1}^p + b^{p/2} \, \|g(t,\cdot)\|_{L_2}^p + \|g(t,\cdot)\|_{L_p}^p \int_{\mathbb{R}} |x|^p \; \pi(dx) \Bigg).$$

Comments

- The assumption π symmetric is linked to the study of the estimates, which come from application of Orliz spaces and the Luxemburg norm obtained from a Young function.
- Particular study. We have studied in detailed the case when Z is a subordinated Brownian motion. In this case the conditions are expressed in terms of the characteristics of the subordinator.
- The case $p \geq 2$. In the conditions of the Theorem, plus the assumption $a = 0$, we obtain that Z is a square-integrable martingale. This case can be treated in larger generality considering a general square-integrable martingale M with $EM_t = 0$, instead of Z as driver.
  Then one would use the Burkholder-Davis-Gundy inequalities to obtain the estimates on the moments of the integral.
  Some further generalisations can also be obtained.

Semimartingale and not

The class of Volterra processes include both semimartingales and not. For a given driver Z this depends on the regularity of the kernel g. We shall consider the case:

$$Y_t = \int_0^t g(t-s)\, dZ_s, \quad t \in [0, T],$$

with Z Lévy process. We fix $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ to be the natural filtration generated by Z.

Theorem [Basse-Pedersen 2009].

- Let Z be of unbounded variation. Then Y is a $\mathbb{F}$-semimartingale if and only if g is absolutely continuous on $\mathbb{R}_+$ with a density g' which is locally square integrable when $b > 0$ and satisfies

$$\int_0^t \int_{[-1,1]} |xg'(s)|^2 \wedge |xg'(s)| \pi(dx) ds < \infty, \quad \forall\ t > 0,$$

  when $b = 0$.
- Let Z be of bounded variation. Then Y is a $\mathbb{F}$-semimartingale if and only if it is of bounded variation, which is equivalent to having g of bounded variation.

2. Approximations of the kernel

Let us consider some $g^{\varepsilon}(\cdot)$, $\varepsilon \in (0,1)$, approximating $g^{\varepsilon}(\cdot)$ in some sense. Consider the process

$$Y_t^{\varepsilon} = \int_0^t g^{\varepsilon}(t-s)\, dZ_s.$$

Proposition Let the Lévy process Z have symmetric Lévy measure π. Fix t and consider

(a) Z with characteristic triplet $(0,0,\pi)$. Then set $p \geq 1$.

(b) Z with characteristic triplet $(0,b,\pi)$. Then set $p \geq 2$.

Assume $\int_{\mathbb{R}} |z|^p \pi(dz) < \infty$ and $g(t-\cdot), g^{\varepsilon}(t-\cdot) \in L_p[0,t]$ such that

$$\|g^{\varepsilon}(t-\cdot) - g(t-\cdot)\|_{L_p} \to 0, \qquad \text{as } \varepsilon \to 0,$$

Then we have the convergence in $L_p(\Omega)$

$$\| Y_t^{\varepsilon} - Y_t \|_{L_p(\Omega)} \to 0, \qquad \text{as } \varepsilon \to 0.$$

Proof. Based on the moments estimates.

For illustration: Gamma-Volterra processes

We consider the process

$$Y_t := \int_0^t (t-s)^\beta e^{-\lambda(t-s)} dZ_s,$$

for $\beta \in (-1/2, 1/2)$, $\lambda \geq 0$. This appears explicitly in the modelling of turbulence and of environmental risk factor in energy finance (e.g. wind). See e.g. von Kármán (1948) and Barndorff-Nielssen (2012).

For illustration, consider Z with $(0, 0, \pi)$. Then Y is an $\mathbb{F}$-semimartingale if and only if one of the following is satisfied:

(i) $\beta > 1/2$,

(ii) $\beta = 1/2$ and $\int_{[-1,1]} z^2 |\log|z|| \pi(dz) < \infty$,

(iii) $\beta \in (0, 1/2)$ and $\int_{[-1,1]} z^{1/(1-\beta)} \pi(dz) < \infty$.

Proof. Based on the application of the result by Basse-Pedersen.

On the other side, for $\beta p + 1 > 0$ we have that $g(t - \cdot) \in L_p([0, t])$.

So we fix $p: \beta p + 1 > 0$, then Y can be both a semimartingale or not.

Also we onsider

$$g^{\varepsilon}(t-s) := g^{\varepsilon}(t-s+\varepsilon), \quad 0 \le s \le t.$$

Then $g^{\varepsilon}(t-\cdot) \in L_p([0,t])$ since it is bounded on $[0,t]$ $\forall t$ (and also of bounded variation), yielding Y^{ε} to be a semimartingale.

We can show the estimate

$$\|g^{\varepsilon}(t-\cdot) - g(t-\cdot)\|_{L_p} \le \varepsilon^{p} C(\lambda, \beta, p, T) \longrightarrow 0, \quad \varepsilon \to 0.$$

Note that this estimate is uniform for $t \in [0, T]$. This guarantees that the assumption in the Proposition are fulfilled and we have convergence of the integrals.

Comments. Thao (2003), Thao and Nguyen (2003) have studied the approximation of $\int_0^t (t-s)^{\beta} dW_s$, for $Z = W$ Brownian motion.

Example. Take $\lambda = 0$, $\beta = 1/8$, Z be a symmetric α-stable Lévy process with measure $\pi(dz) = c|z|^{-\alpha_Z - 1} dz$ and $\alpha_Z = 13/84$. Fix $p = 7/6$. Then

$$\int_{[-1,1]} |z|^{7/6} \pi(dz) = c \int_{[-1,1]} |z|^{1/6-\alpha_Z} dz < \infty,$$

and

$$\int_{[-1,1]} |z|^{8/7} \pi(dz) = c \int_{[-1,1]} |z|^{1/7-\alpha_Z} dz = \infty.$$

3. Integration with respect of Volterra processes

Definition. For two stochastic processes X and Y the generalised Lebesgue-Stieltjes integral is given by

$$\int_0^t X_s\, dY_s := \int_0^t \left(\mathcal{D}_{0+}^{\alpha} X\right)(s) \left(\mathcal{D}_{t-}^{1-\alpha} Y_{t-}\right)(s)\, ds$$

if the right-side exists with probability 1 for some $\alpha \in (0,1)$.
We say that (X, Y) are *fractionally α-connected*.

Here the left- and right-sided fractional derivatives of order $\alpha \in (0,1)$ are given by the Riemann-Liouville fractional derivatives, which admit the *Weyl representation*

$$\mathcal{D}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}\, dy \right) \mathbf{1}_{(a,b)}(x),$$

$$\mathcal{D}_{b-}^{\alpha} f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}}\, dy \right) \mathbf{1}_{(a,b)}(x),$$

where the convergence of the integrals holds pointwise for a. a. $x \in (a,b)$ for $p=1$ and in $L_p(a,b)$ for $p>1$.

See e.g. Zähle (1998,1999,2001).

Results [Z '98,99] (X, Y) are fractionally α-connected if

$$X \in D_q^+(\alpha, T), \quad Y \in D_p^-(\alpha, T) \qquad \frac{1}{p} + \frac{1}{q} = 1$$

where

$$\mathcal{D}_q^+(\alpha, T) := \left\{ X : \int_0^T |\left(\mathcal{D}_{0+}^\alpha X\right)(s)|^q ds < \infty \; a.s. \right\}$$

$$\mathcal{D}_\infty^+(\alpha, T) := \left\{ X : \sup_{0 \le s \le T} |\left(\mathcal{D}_{0+}^\alpha X\right)(s)| < \infty \; a.s. \right\}$$

$$\mathcal{D}_p^-(\alpha, T) := \left\{ Y : \int_0^t |\left(\mathcal{D}_{t-}^{1-\alpha} Y_{t-}\right)(s)|^p ds < \infty \; a.s., \; t \in [0, T] \right\}$$

$$\mathcal{D}_\infty^-(\alpha, T) := \left\{ Y : \sup_{0 \le s \le t} |\left(\mathcal{D}_{t-}^{1-\alpha} Y_{t-}\right)(s)| < \infty \; a.s., \; t \in [0, T] \right\}$$

In view of the estimates on Y found earlier we define

$$\mathbf{E}\mathcal{D}_p^-(\alpha, T) := \left\{ Y : \int_0^t \mathbf{E}|\left(\mathcal{D}_{t-}^{1-\alpha} Y_{t-}\right)(s)|^p ds < \infty \,, \; t \in [0, T] \right\} \subset \mathcal{D}_p^-(\alpha, T)$$

$$\mathbf{E}\mathcal{D}_\infty^-(\alpha, T) := \left\{ Y : \sup_{0 \le s \le t} \mathbf{E}|\left(\mathcal{D}_{t-}^{1-\alpha} Y_{t-}\right)(s)| < \infty \,, \; t \in [0, T] \right\} \subset \mathcal{D}_\infty^-(\alpha, T)$$

Then we can study the integrators of the type $Y_t = \int_0^t g(t,s)dZ_s$, $t \in [0, T]$, to integrate the largest class of integrands. Fix t.

Theorem.
For $p \in [1,2)$ consider the driver Z with $(0,0,\pi)$.
For $p \in (2,\infty)$ consider the driver Z with $(0,b,\pi)$.
Assume π symmetric with $\int_{\mathbb{R}} |z|^p \pi(dz) < \infty$.
Consider $g(t,\cdot) \in L_p([0,t])$. If, for some $\alpha \in (0,1)$, we have

(i) $\int_0^t (t-s)^{\alpha p - p} \left(\int_s^t |g(t-v)|^p dv \right) ds < \infty$,

(ii) $\int_0^t (t-s)^{\alpha p - p} \left(\int_0^s |g(t-v) - g(s-v)|^p dv \right) ds < \infty$,

(iii) $\int_0^t \int_s^t (u-s)^{\alpha p - 2p} \left(\int_s^u |g(u-v)|^p dv \right) duds < \infty$,

(iv) $\int_0^t \int_s^t (u-s)^{\alpha p - 2p} \left(\int_0^s |g(u-v) - g(s-v)|^p dv \right) duds < \infty$,

then $Y \in \mathbf{E}\mathcal{D}_p^-(\alpha, T)$ and it is an appropriate (p,α)-integrator for any $X \in \mathcal{D}_q^+(\alpha, T)$.

Theorem.
For $p = 2$ consider the driver $Z = M$ square integrable càdlàg martingale with $\langle M \rangle$ càdlàg non-decreasing process. Set $\mu_t := \mathbf{E}\langle M \rangle$.
Let $g(t, \cdot)$ such that

$$\mathbf{E} \int_0^t g^2(t,s) d\langle M \rangle_s = \int_0^t g^2(t,s) d\mu_s < \infty, \quad t \geq 0.$$

Assume

(i) $\int_0^t (t-s)^{2\alpha-2} \int_s^t g(t-v)^2 d\mu_v ds < \infty$,

(ii) $\int_0^t (t-s)^{2\alpha-2} \int_0^s \left(g(t-v) - g(s-v)\right)^2 d\mu_v ds < \infty$,

(iii) $\int_0^t \int_s^t \left(\int_v^t \frac{g(u-v)}{(u-s)^{2-\alpha}} du \right)^2 d\mu_v ds < \infty$,

(iv) $\int_0^t \int_0^s \left(\int_s^t \frac{g(u-v)-g(s-v)}{(u-s)^{2-\alpha}} du \right)^2 d\mu_v ds < \infty$.

Then $Y \in \mathbf{E}\mathcal{D}_2^-(\alpha, T)$ and it is a $(2, \alpha)$-integrator for any $X \in \mathcal{D}_2^+(\alpha, T)$.

N.B. A result also for $p = \infty$ is given for the case $Z = M$ continuous square-integrable martingale. Hölder continuity is necessary to have $Y \in D_\infty^-(\alpha, T)$.

Comments

- The results are detailed for the case of Y driven by a subordinated Brownian motion.
- We study kernel functions of the form:

$$g(t,s) = g(j(\cdot),t,s) = c_H s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-3/2} j(u)\, du,$$

where $H \in \left(\frac{1}{2}, 1\right)$, j is a measurable bounded function, $|j(u)| \leq G$, where $G > 0$ is some constant.
We consider $Z = M$ square-integrable martingale with $\mu_t = \mathbf{E}\langle M\rangle_t = \int_0^t \sigma_s^2 ds$ and σ bounded.
We obtain that Y is an appropriate $(2,\alpha)$-integrator for any $1 - H < \alpha < 1$.

This study includes the fractional Brownian motion (Molchan-Golosov kernel).

Illustration: Gamma-Volterra integrators

We return to the process

$$Y_t := \int_0^t (t-s)^\beta e^{-\lambda(t-s)} dZ_s,$$

for $\beta \in (-1/2, 1/2)$, $\lambda \geq 0$, with driver Z such that $(0, 0, \pi)$ and π symmetric and $\int_{\mathbb{R}} |z|^p \pi(dz) < \infty$ for some p.

To see if the process is a proper integrator for some parameters, we need to verify the conditions before.
The job is long and tedious and it results with the the following positive outcome for parameters p, α, β, satisfying the conditions:

$$1 + \beta p > 0, \quad \min(1 + (\alpha - 1)p, 2 + (\alpha + \beta - 2)p) > 0.$$

4. Approximation of the integral

Form the approximations of the integrators, we proceed to study the corresponding approximations of the integrals in terms of $L_1(\mathbf{P})$-convergence. Here we focus on kernels of type:

$$g(t,s) = g(t-s), \quad 0 \leq s \leq t.$$

We consider a family $g^\varepsilon(t-s), \quad 0 \leq s \leq t$ for $\varepsilon \in (0,1)$ and the corresponding Y^ε.

Proposition.

(a) Consider Z with $(0,0,\pi)$, and then let $p \geq 1$, or

(b) consider Z with $(0,b,\nu)$, and let $p \geq 2$.

Assume π symmetric and $\int_{\mathbb{R}} |z|^p \pi(dz) < \infty$.
Let Y and Y^ε be in $\mathbf{E}\mathcal{D}_p^-(\alpha, T)$ for some $\alpha \in (0,1)$ and assume $X \in \mathbf{E}\mathcal{D}_q^+(\alpha, T)$ for $p^{-1} + q^{-1} = 1$.

Assume the following as $\varepsilon \to 0$:

$$\int_0^T \int_s^T \frac{|g^\varepsilon(T-v) - g(T-v)|^p}{(T-s)^{p-\alpha p}} dvds \to 0$$
$$\int_0^T \int_0^s \frac{|(g^\varepsilon(T-v) - g(T-v)) - (g^\varepsilon(s-v) - g(s-v))|^p}{(T-s)^{p-\alpha p}} dvds \to 0$$
$$\int_0^T \int_s^T \int_s^u \frac{|g^\varepsilon(u-v) - g(u-v)|^p}{(u-s)^{2p-\alpha p}} dvduds \to 0$$
$$\int_0^T \int_s^T \int_0^s \frac{|(g^\varepsilon(u-v) - g(u-v)) - (g^\varepsilon(s-v) - g(s-v))|^p}{(u-s)^{2p-\alpha p}} dvduds \to 0$$

Then:

$$\int_0^T X(t) dY_t^\varepsilon \xrightarrow{L_1(\mathbf{P})} \int_0^T X(t) dY_t, \quad \varepsilon \to 0.$$

Illustration: Gamma-Volterra processes

Finally, we consider the two integrators in the same conditions given before

$$Y_t := \int_0^t (t-s)^\beta e^{-\lambda(t-s)} dZ_s, \qquad Y_t^\varepsilon := \int_0^t (t-s+\varepsilon)^\beta e^{-\lambda(t-s+\varepsilon)} dZ_s,$$

for $\beta \in (-1/2, 1/2)$, $\lambda = 0$, with driver Z such that $(0,0,\pi)$ and π symmetric and $\int_{\mathbb{R}} |z|^p \pi(dz) < \infty$ for some p.

The analysis of the conditions in the theorem implies a set of requirements:

$$\begin{array}{ll} 1+(\beta-1)p>0 & 2+(\alpha+\beta-2)p>0 \\ 2+(\beta-2)p>0 & \\ 2+(\alpha+\beta-3)p>0 & \\ 1+\alpha p - p > 0 & 2+(\beta-2)p>0 \end{array}$$

which are all implied by $2+(\alpha+\beta-3)p>0$. This is a strong restriction.

For example, we can consider $\beta \leq 1/2$ so that Y is not a semimartingale. Then we have that:

$$p \leq \frac{4}{5-\alpha} < \frac{4}{3}.$$

On the other side, if we have $p = 1$, then this is possible when

$$\beta > 1 - \alpha > 0,$$

so $\beta \in (0, 1/2]$.

Example. Take $\lambda = 0$, $\beta = 1/8$, Z be a symmetric α-stable Lévy process with measure $\pi(dz) = c|z|^{-\alpha_Z - 1} dz$ and $\alpha_Z = 13/84$. Fix $p = 7/6$. Then

$$\int_{[-1,1]} |z|^{7/6} \pi(dz) = c \int_{[-1,1]} |z|^{1/6 - \alpha_Z} dz < \infty,$$

and

$$\int_{[-1,1]} |z|^{8/7} \pi(dz) = c \int_{[-1,1]} |z|^{1/7 - \alpha_Z} dz = \infty.$$

For these we can see that both the convergence of the integrators and of the integral is satisfied.

5. Simulation of the integral

- With a clever choice of approximating kernel we can approximate a non-semimartingale Y with semimartingales Y^{ε}.
- For the simulation of Y^{ε}, we can rely on connection between Volterra processes and mild solutions to Hyperbolic SPDEs, see Benth and Eyjolfsson (2016).
- For X càglàd and adapted we have can exploit the relationship between the pathwise integral and the Itô integral in the same lines as in Russo and Valois (1995).
- In this case we can rely on finite difference scheme to deal with the integral $\int_0^T X_t dY_t$.

Possible approach:
If the integrand X is not a predictable process but a Volterra type process again, then we can possibly proceed by yet another approximation X^{ε} to deal with $\int_0^T X_t^{\varepsilon} dY_t^{\varepsilon}$.

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