On the Divergence and Vorticity of Vector Ambit Fields

Orimar Sauri[1]

Department of Mathematics
Aarhus University

Second Conference on Ambit Fields
and Related Topics
Aarhus 2017

[1] Based on joint works with O.E. Barndorff-Nielsen and J. Schmiegel

Outline

Outline

Flux and Circulation

- Two fundamental quantities in fluid mechanics are the so-called flux and circulation:

$$\text{Circulation around } D_r(p) = \oint_{\partial D_r(p)} X \cdot n^{\perp} \mathrm{d}s; \text{ (unit of area/unit of time)}$$

$$\text{Flux through } D_r(p) = \oint_{\partial D_r(p)} X \cdot n \mathrm{d}s. \text{ (unit of area/unit of time)}$$

- Where:
  - X is a 2-dimensional velocity field;
  - $D_r(p)$ is a disk of radius $r > 0$ and center $p \in \mathbb{R}^2$;
  - n and $n^{\perp}$ are the outward and tangent unit vectors on $\partial D_r(p)$, respectively.

Flux and Circulation

- The quantities obtained by normalizing the circulation and the flux by πr^2 are termed as the mean flux and mean circulation:

$$\text{Mean Circulation around } D_r(p) = \frac{1}{\pi r^2} \oint_{\partial D_r(p)} X \cdot n^{\perp} \mathrm{d}s; \text{ (/unit of time)}$$

$$\text{Mean Flux through } D_r(p) = \frac{1}{\pi r^2} \oint_{\partial D_r(p)} X \cdot n \mathrm{d}s \text{ (/unit of time)}$$

- Where:
  - X is a 2-dimensional velocity field;
  - $D_r(p)$ is a disk of radius $r > 0$ and center $p \in \mathbb{R}^2$;
  - n and $n^{\perp}$ are the outward and tangent unit vectors on $\partial D_r(p)$, respectively.

Flux and Circulation

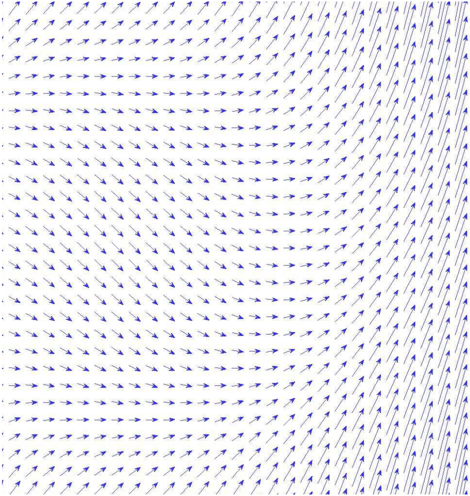
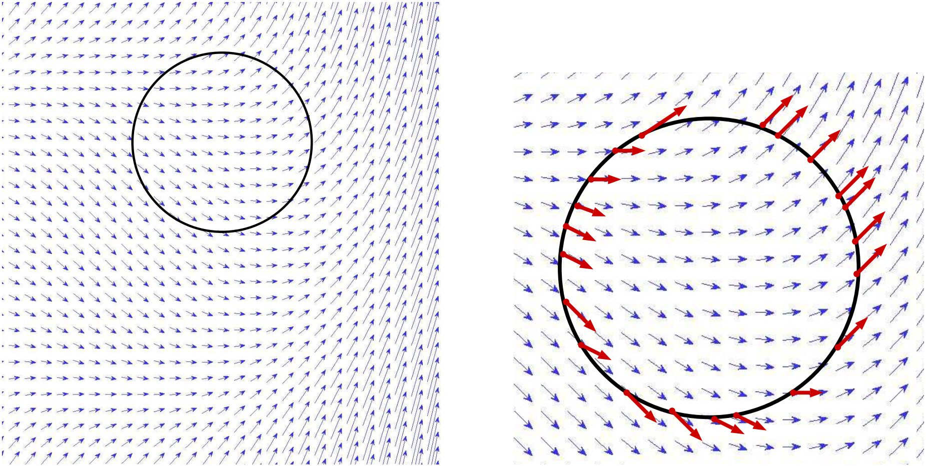


Figure: A vector field in $\mathbb{R}^2$.

Circulation = degree of rotation

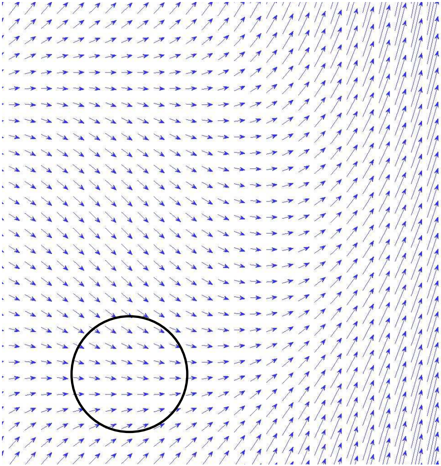
How a field rotates: The more the fluid is aligned to ∂D, the more the motion is of rotational type.



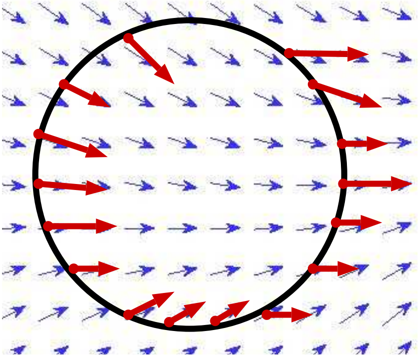
(a) (b)

flux = "amount" of fluid

Flux through a region: The larger the mean circulation, the more (less) fluid is entering D.



(a)



(b)

In real life, it looks like this

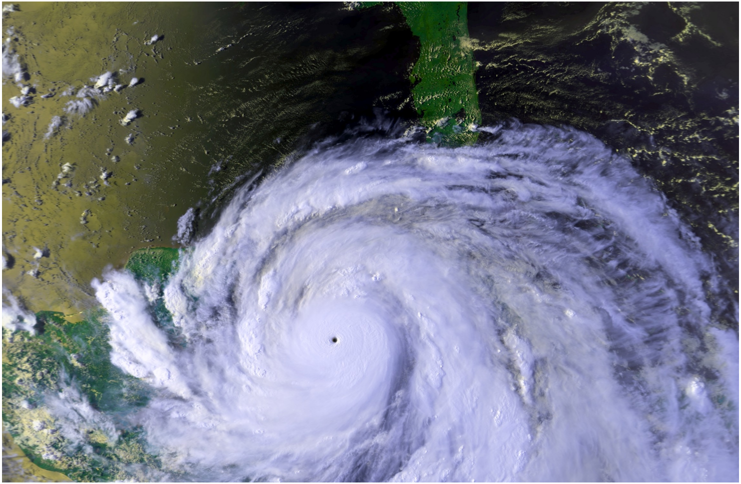


Figure: Gilbert Hurricane, Mexico, 1988. Source: commons.wikimedia.org

Rotation and incompressibility

- The concept of *incompressibility* of a fluid expresses the fact that the density of the fluid is constant: In terms of the mean flux

$$\lim_{r\downarrow 0}\frac{1}{\pi r^2}\oint_{\partial D_r(p)} X\cdot n\mathrm{d}s = 0,\ \forall p.$$

- *Rotation* and the related concept of *vortex* merging and stretching is believed to be the main dynamic process for 2-dimensional turbulent flows: Thus, if the fluid is turbulent, the mean circulation must satisfies that

$$\lim_{r\downarrow 0}\frac{1}{\pi r^2}\oint_{\partial D_r(p)} X\cdot n^{\perp}\mathrm{d}s \neq 0,\ \text{for some } p.$$

Divergence and Vorticity

- By Stokes' Theorem, when X is continuously differentiable

$$\frac{1}{\pi r^2}\oint_{\partial D_r(p)} X\cdot n\mathrm{d}s = \frac{1}{\pi r^2}\int_{D_r(p)}\nabla\cdot X(q)\mathrm{d}q \to \nabla\cdot X(p);$$

$$\frac{1}{\pi r^2}\oint_{\partial D_r(p)} X\cdot n^{\perp}\mathrm{d}s = \frac{1}{\pi r^2}\int_{D_r(p)}\nabla^{\perp}\cdot X(q)\mathrm{d}q \to \nabla^{\perp}\cdot X(p).$$

  with $\nabla := (\partial_x,\partial_y)',\nabla^{\perp} := (-\partial_y,\partial_x)'$.
- Incompressibility $\iff \nabla\cdot X\equiv 0$:Null Divergence.
- Rotation $\iff \nabla^{\perp}\cdot X\neq 0$: Non-vanishing Vorticity/Curl.

Divergence and Vorticity

- In this talk I will focus on the asymptotic behavior of the circulation and the flux of a random field X

$$\mathscr{C}_r(p;X) := \oint_{\partial D_r(p)} X \cdot n^{\perp} \mathrm{d}s, \; p \in \mathbb{R}^2, r > 0,$$

$$\mathscr{D}_r(p;X) := \oint_{\partial D_r(p)} X \cdot n \mathrm{d}s, \; p \in \mathbb{R}^2, r > 0.$$

- X is the (stationary) Infinitely Divisible field

$$X(p) := \int_{\mathscr{R}+p} F(p-q)L(\mathrm{d}q), \; p \in \mathbb{R}^2.$$

- $F : \mathbb{R}^2 \to \mathbb{R}^2$ continuously differentiable and $\mathscr{R}$ a compact set on $\mathbb{R}^2$.
- L a homogeneous Lévy basis.

Outline

Stokes' Theorem

- Stokes' Theorem in its standard form guarantees that

$$\oint_{\partial U} \alpha = \int_U \mathrm{d}\alpha,$$

whenever:
  - α is a smooth form, e.g. $\alpha = X \cdot n\mathrm{d}s$ and $\mathrm{d}\alpha = \nabla \cdot X(q)\mathrm{d}q$;
  - U is a smooth manifold.
- Generalizations of Stokes' Theorem:
  - Hsu (2002): U a path of a stochastic process;
  - Harrison (1999): U a *chainlet*, e.g. fractals and vector fields;
  - α is a smooth form.
- Züst (2011) considered non-smooth forms over Lipschitz manifolds. However there is **NO** Stokes' Theorem available in this setting.

Stokes' Theorem

- Stokes' Theorem in its standard form guarantees that

$$\oint_{\partial U} \alpha = \int_U \mathrm{d}\alpha,$$

  whenever:
    - α is a smooth form, e.g. $\alpha = X \cdot n \mathrm{d}s$ and $\mathrm{d}\alpha = \nabla \cdot X(q)\mathrm{d}q$;
    - U is a smooth manifold.
- Generalizations of Stokes' Theorem:
    - Hsu (2002): U a path of a stochastic process;
    - Harrison (1999): U a *chainlet*, e.g. fractals and vector fields;
    - α is a smooth form.
- Züst (2011) considered non-smooth forms over Lipschitz manifolds. However there is NO Stokes' Theorem available in this setting.

Stokes' Theorem

- Stokes' Theorem in its standard form guarantees that

$$\oint_{\partial U} \alpha = \int_U \mathrm{d}\alpha,$$

  whenever:
  - α is a smooth form, e.g. $\alpha = X \cdot n\mathrm{d}s$ and $\mathrm{d}\alpha = \nabla \cdot X(q)\mathrm{d}q$;
  - U is a smooth manifold.
- Generalizations of Stokes' Theorem:
  - Hsu (2002): U a path of a stochastic process;
  - Harrison (1999): U a *chainlet*, e.g. fractals and vector fields;
  - α is a smooth form.
- Züst (2011) considered non-smooth forms over Lipschitz manifolds. However there is **NO** Stokes' Theorem available in this setting.

What can we learn from the 1-dimensional case

- Let $X_t = \int_0^t f(t-s)\mathrm{d}L_s$.
- By definition

$$\mathscr{D}_r(t;X) = \oint_{\partial D_r(t)} X \cdot n\mathrm{d}s = X_{t+r} - X_{t-r}.$$

- We have that

$$X_t = f(0)L_t + \int_0^t [f(t-s) - f(0)]\mathrm{d}L_s =: \partial X_t + \mathring{X}_t,$$

  in such a way that

$$\mathscr{D}_r(t;X) = \mathscr{D}_r(t;\partial X) + \mathscr{D}_r(t;\mathring{X}).$$

- If f is continuously differentiable, then $\mathring{X}$ is absolutely continuous.
- $\mathscr{D}_r(t;\partial X)$ is proportional to the increments of a Lévy process. Thus it only depends on the interaction of L on $[0,t]$.

A key example

- Suppose that $\mathscr{R}$ is a disk of radius 1 and let $F(q) = G(\|q\|)$.
- X can be decomposed as

$$X(p) = \int_{\mathring{\mathscr{R}}+p} [G(\|p-q\|) - G(1)] L(\mathrm{d}q) + G(1) L(\mathscr{R}+p) =: \partial X(p) + \mathring{X}(p).$$

- It is easy to check that in this case

$$\frac{1}{\pi r^2} \oint_{\partial D_r(p)} \mathring{X} \cdot n ds = O_{\mathbb{P}}(r^2).$$

- By definition

$$\oint_{\partial D_r(p)} \partial X \cdot n ds = r \int_0^{2\pi} \langle f(1), u(\theta) \rangle L(\mathscr{R}(p) + ru(\theta)) \mathrm{d}\theta$$

$$\approx r \int_0^{2\pi} \langle f(1), u(\theta) \rangle L(\partial \mathscr{R}(p) + ru(\theta)) \mathrm{d}\theta.$$

A key example cont'd

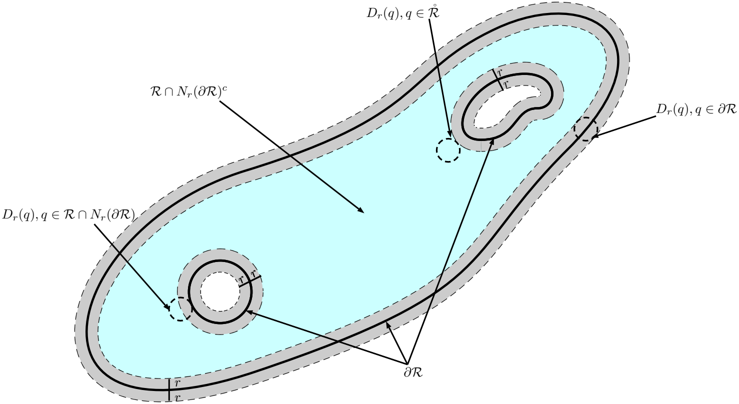


Figure: $\{\mathscr{R}(p) + ru(\theta)) : 0 \leq \theta \leq 2\pi\} \approx \mathscr{R}(p) \cup \{\partial\mathscr{R}(p) + ru(\theta)\}$.

A key example cont'd

- We conclude that in this example

$$\mathscr{D}_r(p;X) = \underbrace{\mathscr{D}_r(p;\partial X)}_{O_{\mathbb{P}}(r^2)} \quad + \quad \underbrace{\mathscr{D}_r(p;\mathring{X})}_{\text{Interaction of } F \text{ and } L \text{ around } \partial\mathscr{R}(\mathrm{p})} \quad .$$

Outline

Assumptions on the ambit set

Assumption

There are $C_1, \ldots, C_n$ disjoint regular smooth Jordan curves with non-null curvatures such that $\mathscr{R}$, the ambit set, can be written as

$$\mathscr{R} = (C_1 \cup \mathrm{Int}\, C_1) \setminus \bigcup_{i=2}^{n} \mathrm{Int}\, C_i.$$

Furthermore, it holds that $C_i \subset \mathrm{Int}\, C_1$ and $\mathrm{Int}\, C_i \cap \mathrm{Int}\, C_j = \emptyset$, for any $i, j = 2, \ldots, n$, $j \neq i$.

Typical ambit sets

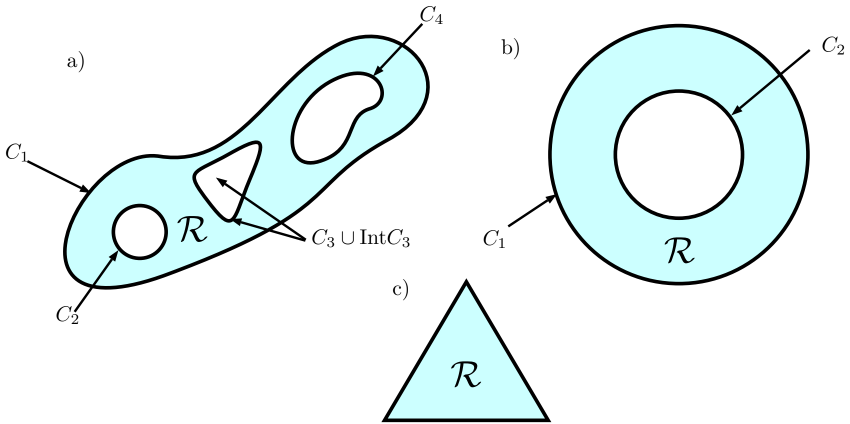


Figure: Regions exhibited in a) and b) are typical examples of the type of ambit sets considered in the previous assumption. Simple polygons as the one appearing in c) is a class of ambit sets that will not be considered in this talk.

Gaussian attractor

Theorem

Let $\mathscr{R} \subset \mathbb{R}^2$ be as in Assumption 1. Suppose that $F|_{-\partial\mathscr{R}} \neq 0$ and L has characteristic triplet (γ, b, ν) with $b > 0$. Then, as $r \downarrow 0$

$$\frac{1}{v_2 r^{1+1/2}} \mathscr{D}_r(p; X) \xrightarrow{\mathscr{F}\text{-fd}} \int_{\partial\mathscr{R}+p} \langle F(p-c), n_{\partial\mathscr{R}(p),O}(c) \rangle W_{\mathscr{H}^1}(\mathrm{d}c),$$

where $n_{\partial\mathscr{R}(p),O}$ is the outward unit vector to $\partial\mathscr{R}(p)$, $W_{\mathscr{H}^1}$ is a Gaussian Lévy basis defined on an extension of $(\Omega, \mathscr{F}, \mathbb{P})$ having the following properties:

- *Its cumulant function satisfies that*

$$C\{z \ddagger W_{\mathscr{H}^1}(A)\} = -\frac{1}{2} b^2 z^2 \mathscr{H}^1(A), \; \mathscr{H}^1(A) < \infty, z \in \mathbb{R}.$$

- *$\mathscr{H}^1$ is the 1-dimensional Hausdorff measure*
- *$W_{\mathscr{H}^1}$ is independent of L.*

Classical regime

Theorem

Let $\mathscr{R} \subset \mathbb{R}^2$ be as in Assumption 1. Then, as $r \downarrow 0$

$$\frac{1}{\pi r^2} \mathscr{D}_r(p;X) \stackrel{\mathbb{P}}{\rightarrow} \sigma(p), \; p \in \mathbb{R}^2,$$

if and only if one of the following (non-necessarily mutually exclusive) cases holds:

$$\mathbf{1}. b = 0 \text{ and } \int_{\mathbb{R}} (1 \wedge |x|)\nu(\mathrm{d}x) < \infty; \quad \mathbf{2}. F|_{-\partial \mathscr{R}} \equiv 0.$$

The limiting process is given by

$$\sigma(p) := \int_{\mathscr{R}+p} \nabla \cdot F(p-q)\tilde{L}(\mathrm{d}q),$$

with $\tilde{L} = L - \gamma_d Leb$, where $\gamma_d = \gamma - \int_{|x|\leq 1} x\nu(\mathrm{d}x)$ in 1. while $\gamma_d = \gamma$ in 2.

Stable attractor

Theorem

Now suppose that $F|_{-\partial\mathscr{R}} \neq 0$, $b = 0$ *and* $\int_{\mathbb{R}}(1 \wedge |x|)\nu(\mathrm{d}x) = +\infty$. *In addition, assume that there exists* $1 \leq \beta < 2$ *such that* $\nu(x,\infty) \sim \tilde{K}_+ x^{-\beta}$ *and* $\nu(-\infty,-x) \sim \tilde{K}_- x^{-\beta}$ *as* $x \downarrow 0$ *with* $\tilde{K}_+ + \tilde{K}_- > 0$. *Then*

1. *If* $1 < \beta < 2$, *then as* $r \downarrow 0$

$$\frac{1}{v_\beta r^{1+1/\beta}} \mathscr{D}_r(p;X) \xrightarrow{\mathscr{F}\text{-}fd} \int_{\partial\mathscr{R}+p} \langle F(p-c), n_{\partial\mathscr{R}(p),I}(c) \rangle M_{\mathscr{H}^1}(\mathrm{d}c).$$

2. *If* $\beta = 1$, *suppose that* $\tilde{K}_+ = \tilde{K}_-$ *and* $\mathrm{PV}\int_{-1}^{1} x\nu(\mathrm{d}x)$, *the Cauchy principal value, exists. Then, as* $r \downarrow 0$

$$\frac{1}{\pi r^2} \mathscr{D}_r(p;X) \xrightarrow{\mathscr{F}\text{-}fd} \sigma(p) + \int_{\partial\mathscr{R}+p} \langle F(p-c), n_{\partial\mathscr{R}(p),I}(c) \rangle M_{\mathscr{H}^1}(\mathrm{d}c).$$

Stable attractor cont'd

Theorem

Where

- $n_{\partial\mathscr{R}(p),l}$ *is the inward unit vector to* $\partial\mathscr{R}(p)$*;*
- $M_{\mathscr{H}^1}^{K_\pm,\beta,\hat{\gamma}}$ *is a Lévy basis (independent of L) defined on an extension of* $(\Omega,\mathscr{F},\mathbb{P})$ *whose cumulant function satisfies that*

$$C\{z \ddagger M_{\mathscr{H}^1}(A)\} = \mathscr{H}^1(A)\psi_{K_\pm,\beta,\hat{\gamma}}(z), \; \mathscr{H}^1(A) < \infty, z \in \mathbb{R},$$

with $\psi_{K_\pm,\beta,\hat{\gamma}}$ *the cumulant function of a strictly* β*-stable distribution whose parameters depend on* $K_\pm$ *and* $\hat{\gamma}$*.*

Remarks

- The limit cannot take place in probability: The convergence is stable and the limit is independent of the background driving Lévy basis.
- In general, the convergence cannot be strengthen to functional convergence: The limiting field might be a white noise.
- The dependence structure of the limiting field is entirely determined by the geometry of $\mathscr{R}$.
- The rates of convergence can be seen as an L^{β} norm of a certain parametrization of a disk: Put $g_{\beta}(s,\rho) := (1+\beta)\sqrt{1-s^2}\rho$ for $1 \leq \beta \leq 2$. Then $\pi r^2 = \|g_1\|_{L^1[-r,r]\times[-1,1]}$ and

$$r^{1+1/\beta} v_{\beta} = \left\|g_{\beta}\right\|_{L^{\beta}[-1,1]\times[-r,r]}.$$

- Different rates of convergence can be obtained. However, the limiting fields remain the same (Ivanovs (2016)).
- The classical Stokes' Theorem doesn't hold in this framework.

Open problems and generalizations

- Higher dimensions.
- $\mathscr{R}$ with non-smooth boundary, e.g. fractals.
- Line integrals over non-smooth manifolds, e.g. paths of stochastic process.
- More general stochastic forms.

Thank you!

References I

Harrison, J. (1999). Flux across nonsmooth boundaries and fractal Gauss/Green/Stokes' theorems. *Journal of Physics A: Mathematical and General 32*(28), 5317.

Hsu, E. P. (2002). *Stochastic Analysis on Manifolds*. Graduate Studies in Mathematics 38. American Mathematical Society.

Ivanovs, J. (2016). Zooming in on a Lévy process at its supremum. *ArXiv e-prints*.

Züst, R. (2011). Integration of Hölder forms and currents in snowflake spaces. *Calculus of Variations and Partial Differential Equations 40*(1), 99–124.