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Asymptotic distributions of some scale estimators in nonlinear models with long memory errors having infinite variance

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Joint work with Hira L. Koul (Michigan State U.)

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Median of absolute residuals:

$$s_1 := \text{med}\Big\{|r_{ni}|;\ 1 \leq i \leq n\Big\} \tag{2}$$

• To have scale invariant M-estimators of regression parameters in regression models there is a need for having a robust scale invariant estimator of a scale parameter.

• For this, the regression residuals in the definition of M-estimator must be divided by scale estimator. See Koul (2002).

• Robust estimation of scale parameter is of interest by itself, especially in the context of infinite variance errors as in the present paper. (In this case, the usual scale estimation by standard deviation is inconsistent.)

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• The fact that each of these estimators estimates a different scale parameter is not a point of concern if our goal is only to use them in arriving at scale invariant robust estimators of $\boldsymbol{\beta}_0$.

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$$\omega(\alpha, u) := -\frac{\Gamma(2-\alpha)(c_+ + c_-)}{\alpha - 1} \cos(\pi\alpha/2)\Big(1 - \mathrm{i}\frac{c_+ - c_-}{c_+ + c_-}\mathrm{sgn}(u)\tan(\pi\alpha/2)\Big).$$

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• Assumption E(α, d) is satisfied by ARFIMA(p, d, q) with α-stable innovations (Kokoszka and Taqqu, 1995)

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(i) Let $f(\sigma_1) \neq f(-\sigma_1)$. Then, for every $x \in \mathbb{R}$,

$$\begin{aligned} & P(n^{1-d-1/\alpha}(s_1 - \sigma_1) \le x\sigma_1) \\ = \quad & P\Big(n^{1-d-1/\alpha}\Big(\bar{\varepsilon}_n + \big(\frac{1}{n}\sum_{i=1}^{n} \dot{g}(\boldsymbol{\beta}_0, z_{ni})\big)'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\Big) \ge -\frac{x\sigma_1 f_+(\sigma_1)}{f_-(\sigma_1)}\Big) + o(1). \end{aligned}$$

(ii) Let $f(\sigma_1) = f(-\sigma_1)$. Then, for every $x \in \mathbb{R}$,

$$P\big(n^{1-1/\alpha_*}(s_1 - \sigma_1) \le x\sigma_1\big) \quad \to \quad P\big(Z_1^* \le x\sigma_1 f_+(\sigma_1)\big),$$

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$$\sup_{x\in\mathbb{R}} n^{1/2-d}\left|F_n(x) - F(x) + f(x)\bar{\varepsilon}_n\right| = o_p(1) \tag{11}$$

where $\bar{\varepsilon}_n = n^{-1}\sum_{i=1}^n \varepsilon_i$ is the sample mean.

• $f(x)\bar{\varepsilon}_n$ can be regarded as the *first term* of the asymptotic expansion of F_n which may vanish for some nonlinear statistics and is insufficient for some applications

• Higher-order asymptotic expansions of the EP and noncentral limit theorems:

$$F_n(x) - F(x) = \sum_{1\le k\le[1/(1-2d)]} (-1)^k F^{(k)}(x)\varepsilon_n^{(k)} + n^{-1/2}Q_n(x)$$

• For LM Gaussian process $\{\varepsilon_i\}$ with memory parameter $d \in (0, 1/2)$ the EP tends to completely a different *degenerated* limit (Dehling and Taqqu, 1989):

$$n^{1/2-d}(F_n(x) - F(x)) \Longrightarrow_{D(\bar{\mathbb{R}})} f(x)Z \tag{10}$$

where $f(x) = F'(x)$ is (Gaussian) density and $Z \sim N(0, \sigma^2)$ is a normal r.v.

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Rosenblatt (1962), Taqqu (1975, 1979), Dobrushin and Major (1979), ..., Ho and Hsing (1996), ...

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The last result is a consequence of the following URP for the EP:

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- In contrast, asymptotic expansion of Ho and Hsing (1996) of EP under finite 4th moment of ζ_0 contains only integer derivatives $F^{(k)}(x), k = 1, 2, \cdots$, see (12)

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Thm 5 (URP II)

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$$\begin{aligned} & F_n(x) - F(x) + f(x)\bar{\varepsilon}_n - \mathcal{Z}_n(x) \\ = \ & n^{-1} \sum_{i=1}^n \big(I(\varepsilon_i \le x) - F(x) + f(x)\varepsilon_i\big) - \mathcal{Z}_n(x) \\ = \ & \mathcal{R}_{n1}(x) + \mathcal{R}_{n2}(x) \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_{n1}(x) \quad := \quad n^{-1} \sum_{i=1}^n \Big\{ & I(\varepsilon_i \le x) - F(x) + f(x)\varepsilon_i \\ & - \sum_{s \le i} E\Big[I(\varepsilon_i \le x) - F(x) + f(x)\varepsilon_i \Big| \zeta_s\Big]\Big\} \end{aligned}$$

Step 3. Show Step 2 and α-tails of ζ_s imply α_*-tails of $\eta(x;\zeta_s)$ and hence α_*-stable limit of $\sum_{s=1}^n \eta_{n,s}(x;\zeta_s)$ for x fixed (also convergence of finite-dimensional distributions)

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$$E|W_{n,s,v}(x,y)|^r \leq \mu(x,y) \sum_{i=1\vee s}^{n} |b_{i-s}|^r |b_{i-v}|^r$$

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